Karlsruhe Institute of Technology

A Comparison of Ball and Weighted Embeddings

Bachelor's Thesis of

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(Malik Tomasz Weddig)

Abstract

Geometric graph models assign geometric structures to vertices making them useful for analysis of real-world graph behavior and to obtain low-dimensional representations of graphs. A commonly used model is the graph class of Geometric Inhomogeneous Random Graphs (GIRGs). In the non-random variant, called weighted graphs, vertices are mapped to points and weights such that two vertices are adjacent if the distance between the points is less or equal to the product of their weights. Another heavily studied model is the intersection graph of balls with radius 1. A variant called ball graphs allows for arbitrary radii, i.e., is a mapping from vertices to points and radii such that two vertices are adjacent if the distance between the points is less or equal to the sum of their radii. We can analyze how expressive a model is by using the minimal dimension for which there exists a representation for a given graph.

We seek to compare ball- and weighted graphs and give a complete comparison in the one-dimensional case. Furthermore, we show that the minimal dimension for ball graphs (ball dimension) can at most be one larger than the minimal dimension for weighted graphs (weighted dimension), but also that the weighted dimension can exceed the ball dimension by an arbitrary amount. To prove this we introduce the model of grounded ball graphs, where every ball must touch a plane. We also show that the ball dimension for the complete bipartite graph $K_{3,n}$ and the weighted dimension for the $K_{2,n}$ are in $\Omega(\log n)$. Furthermore, we prove that the minimal dimensions can decrease by an arbitrary amount when removing a vertex from the graph.

Zusammenfassung

Geometrische Graphmodelle weisen Knoten geometrische Strukturen zu und sind deshalb nützlich um das Verhalten von Graphen in der echten Welt zu analysieren und um niedrigdimensionale Repräsentationen von Graphen zu erhalten. Ein viel benutztes Modell ist die Graphklasse der Geometric Inhomogeneous Random Graphen (GIRGs). In der deterministischen Variante, genannt gewichtete Graphen, werden Knoten auf Punkte und Gewichte abgebildet, sodass zwei Knoten benachbart sind, wenn die Distanz zwischen den Punkten kleiner gleich dem Produkt der Gewichte ist. Ein anderes, viel untersuchtes Modell ist der Schnittgraph von Bällen mit Radius 1. Eine Variante davon sind Ball Graphen, also eine Abbildung von Knoten auf Punkte und Radien, sodass zwei Knoten benachbart sind, wenn die Distanz zwischen den Punkten kleiner gleich der Summe der Radien ist. Wir können analysieren wie ausdrucksstark ein Modell ist, in dem wir die minimale Dimension, für die eine Repräsentation für einen gegebenen Graphen existiert, betrachten.

Wir möchten Ball- und gewichtete Graphen vergleichen und geben einen kompletten Vergleich im 1-dimensionalen Fall. Außerdem zeigen wir, dass die minimale Dimension von Ballgraphen (Balldimension) maximal 1 größer als die minimale Dimension von gewichteten Graphen (gewichtete Dimension) sein kann, aber umgekehrt die Balldimension die gewichtete um einen beliebigen Wert übersteigen kann. Um das zu zeigen, führen wir das Modell der Grounded Ball Graphs ein, bei dem alle Bälle eine Ebene berühren müssen. Außerdem beweisen wir, dass die Balldimension vom vollständig bipartiten Graphen $K_{3,n}$ und die gewichtete Dimension vom $K_{2,n}$ in $\Omega(\log n)$ liegen. Des Weiteren, zeigen wir, dass beide Dimensionen um einen beliebigen Wert kleiner werden können, wenn man einen Knoten aus dem Graphen entfernt.

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1 Introduction

Real world graphs are often based on an underlying geometric structure or exhibit *clustering*, i.e., vertices that share a common neighbor are more likely to be connected. For example two radio towers might only be able to communicate if the distance between them is small. Another example would be in navigation. It is more likely that a road exists between two places, if both places are close to each other. Clustering can even happen in places where a geometry is not as apparent e.g., in social networks [Fel81].

Still, clustering can often be modelled using geometric graph models which has lead to the study of various geometric graph representations. These representations map every vertex to a geometric "structure" e.g., points in a space such that two vertices are adjacent if and only if their corresponding structures are *close*. Especially graph models that are more *expressive* are really useful, i.e., models where the minimal dimension for which we can find a representation for a given graph is low.

There are a variety of applications for these representations. Representations often lend itself to obvious visualizations for 2 dimensions, but visualizations are also viable in 3 dimensions [KW01]. Graph representations also allow for numerical representations of graphs which is necessary for machine learning approaches [GL16]. Since high-dimensional inputs are more likely to result in overfitting, the minimal dimension is also of interest here [Haw04]. Another area of study has been compression where instead of storing an adjacency matrix, we can just store numerical representations of the geometric structures. An adjacency matrix is always an $n \times n$ matrix of ones and zeroes, but for n representations in d dimensions we only need (roughly) d many numbers, where d is the dimension of our space [Spi03].[1] Furthermore, graph representations are useful for observing behavior of algorithms on them in the context of graph generation. This is done using random sampling and random or empirical analysis, i.e., through an average case analysis or by generating graphs and running algorithms on them. For example some algorithms might be slow on arbitrary graphs but fast in practice [Kat23 | BF24]. It is therefore useful to find representations and distributions which closely resemble example graphs as seen in practice to gain insight into how and why this happens.

Since there are a lot of applications of graph representations, a lot of research has been done on different variants. Maehara heavily studied unit ball graphs [Mae84]. In this representation we map vertices to unit balls, i.e., balls with radius 1, such that two vertices are adjacent if and only if their balls intersect. He also introduced the *sphericity of a graph*. This is the minimal dimension for which such a representation exists. It has been shown that the sphericity is always finite, but can get large even for simple graphs like stars where the sphericity is in $\Omega(\log n)$ [Mae84 | Mae86]. It has furthermore been shown that determining the sphericity of an arbitrary graph is NP-hard and even $\exists\mathbb{R}$-complete[KM12 | Sch10].

A variant of this model are *ball graphs* where we consider balls with arbitrary radii, i.e., consider a mapping of vertices to points p and radii r such that two vertices $u \neq v$ are adjacent if and only if $\|p_u - p_v\| \leq r_u + r_v$. The minimal dimension for which a ball representation

[1] Sadly for some representations, you might need an exponential number of bits of precision [KM12].

exists is obviously always less or equal to the sphericity of a graph. The most work in this area has been done in the 2-dimensional case where ball graphs are also called *disk graphs*. Notable results in 2-dimensions include that all planar graphs are also disk graphs [Koe36][2]. Another notable result in the 2-dimensional case is that it is NP-hard to decide whether a graph is a 2-dimensional ball graph [HK01]. In fact by the same principle as for sphericity it is $\exists\mathbb{R}$-complete [Sch10]. Especially the 2-dimensional case (or variants of it) have been useful to model radio networks [HS95 | HK01].

A second model that has been researched extensively is the *Dot Product Representation* of a graph [RRŠ89]. In this representation we map vertices to vectors such that two vertices $u \neq v$ are adjacent if and only if their corresponding vectors have a dot product ≥ 1. Similar to the sphericity, the *dot product dimension of a graph* is the smallest dimension which admits a dot product representation for a given graph. It has been shown that obtaining the dot product dimension of a graph is also NP-hard [KM12]. Structural results about this dimension include that the dot product dimension of a graph can at most decrease by 1 when removing a vertex [FSTZ98]. In other words this means that when adding a vertex to a graph, the dot product dimension can increase by at most 1. This is a really powerful statement for finding a somewhat low dimensional embedding as it allows splitting a graph along a small separator into easily representable disconnected subgraphs and combining them afterwards without increasing the dimension too much. Besides the minimal dimension, Dot Product graphs have often been studied in the context of machine learning applications [al17].

Another recent focus of study are *Geometric Inhomogeneous Random Graphs (GIRG)* which have been introduced by Bringmann et al. [al19]. These have mostly been studied in the context of network analysis and generation [al22 | al19]. In this representation we generate a graph by randomly sampling positions and weights for every vertex. When stripping away the randomness and ignoring some technicalities two vertices $u \neq v$ are adjacent if the product of their weights exceeds the distance between the corresponding positions, i.e., $\|p_u - p_v\| \leq w_u \cdot w_v$. This non-random variant is called a *weighted graph*. By using weights $\sqrt{2}$ it is also obvious that sphericity must always be greater or equal to the minimal dimension for which a graph admits a weighted representation.

While a lot of different representations have been studied, there have not been many comparisons between them. These comparisons are interesting though as having a more expressive model, i.e., a model with lower minimal dimensions, is often preferable. We will focus on the non-random case here. This case can differ a lot from the random one. For example in the random case the choice of our norm which induces a distance function is often irrelevant, since all norms in the $\mathbb{R}^d$ are equivalent [al19]. In contrast, in the deterministic case it has been shown that using the euclidean distance can lead to a different minimal dimension than the taxicab distance for sphericity [Mae86].

Goal of this thesis. We seek to compare *ball graphs* to *weighted graph*, i.e., we want to compare the intersection graph of disks with arbitrary radii to the non-random variant of GIRGs. Since there are not a lot of results about ball graphs in higher dimensions or weighted graphs we furthermore want to find out how expressive these models are in general. In other words we would also like to get some general results on their respective minimal dimensions.

When comparing minimal dimensions it is useful to think about transformations of weighted representations to ball representations and vice versa such that both representations correspond to the same graph. Such a transformation can be quite complicated. It can for example

[2] This has been independently proven by multiple authors; for a complete history see [Sac94].

depend on the minimal distance between two points or any information, which we can obtain from a finite set of vectors. This is not preferable since it makes future analysis harder e.g., in the random case. It is preferable to have transformations that always map the same ball to the same point-weight pair regardless of the surrounding balls, i.e., a transformation that only uses *local* information. Such a transformation we call *strong*, i.e., it is applied separately for every ball or point-weight pair. Only using the position and weight or radius as input also allows us to understand structural similarities more easily.

This leads to the following questions we seek to answer.

- Which model is more expressive, i.e., can we transform every d-dimensional ball graph into a d'-dimensional weighted graph or vice versa?
- If we find such a transformation, can we find one that is *strong*, i.e., only uses local information?
- How much can the minimal dimensions decrease if we remove a vertex from the graph?
- Can we find a family of graphs for which the dimension is not bounded by a constant value?

Obviously the first and second question are strongly related.

Contribution. In this thesis we answer these questions through the following results. Regarding the first and second question we show that we can transform every weighted graph into a ball graph while only increasing the dimension by 1. This is done by introducing the concept of *grounded ball graphs*, i.e., ball graphs that lie on a plane and proving that $(d+1)$-dimensional grounded ball graphs are exactly d-dimensional weighted graphs. Furthermore, this transformation only uses local information, i.e., it always transforms the same point and weight to the same ball regardless of the rest of the representation. In the other direction we show that the complete bipartite graph $K_{2,n}$ has a minimal ball dimension of 2, but the minimal weighted dimension of the $K_{2,n}$ is in $\Omega(\log(n))$. This means that we can not find a general transformation that maps ≥ 2 dimensional ball graphs to weighted graphs regardless of the dimension. Furthermore, we show that the minimal ball dimension of $K_{3,n}$ is also in $\Omega(\log n)$, answering question 4. We conclude that in both the weighted and the ball case the dimension can decrease by an arbitrary amount when removing a vertex. These results answer question 2 and 3.

As a special case of questions 1 and 2 we consider 1-dimensional ball graphs and show that they are a proper subset of 1-dimensional weighted graphs. Furthermore, we can show that we cannot find a strong transformation between them that is also continuous. A collection of our results regarding transformations can be found in Figure 1.1. For our results regarding set inclusion see Figure 1.2.

1.1 Outline

In Chapter 2 we begin by formally defining the various graph models we use. We also go into a lot of detail about the previously mentioned transformations and how we can formalize only using local information. Continuing onwards in Chapter 3 we find a simple transformation from weighted graphs to grounded ball graphs. In Chapter 3 we find a family of graphs, i.e., the $K_{2,n}$, for which we can easily find a 2-dimensional ball representation. We prove that the

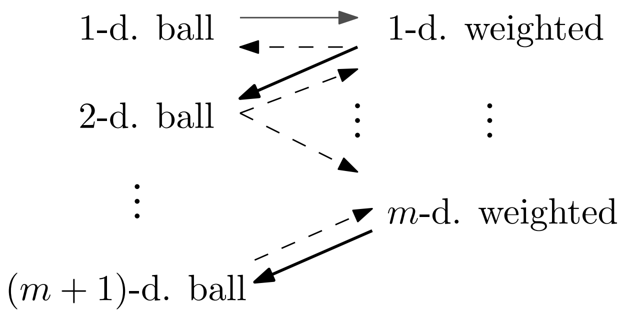


Figure 1.1: Summary of which transformations we can find from ball graphs to weighted graphs and vice versa. Dotted arrows correspond to the non-existence of transformations. Thick black arrows are strong transformations, while thinner gray arrows are non-strong transformations and denote that no strong continuous transformation can exist.

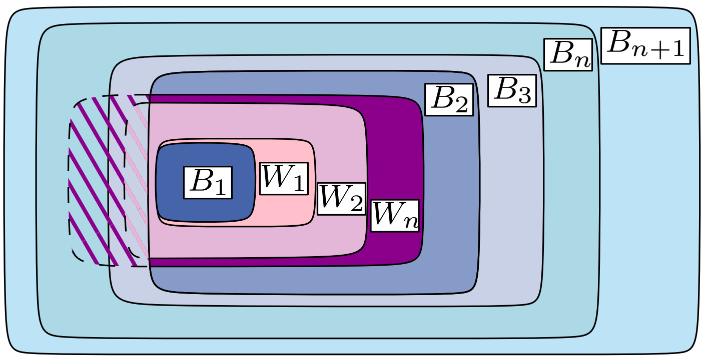


Figure 1.2: The knowledge we have of the relationship between ball graphs and weighted graphs after our analysis. B_d is the set of all d-dimensional ball graphs and W_d the set of all d-dimensional weighted graphs. The striped areas are areas where we do not exactly know if they exist, i.e., we only know $W_d \subseteq B_{d+1}$ and that $B_2 \nsubseteq W_d$ but do not know whether $W_d \subseteq B_{d'}$ for some $d' \leq d$ except for $d = 1$.

minimal grounded ball dimension is in $\Omega(\log n)$ and obtain some useful corollaries like that the minimal weighted dimension can decrease by an arbitrary amount when removing a vertex. We show a similar result for the $K_{3,n}$ and ball graphs in Chapter 5. Our investigation ends in Chapter 6 by considering the case of 1-dimensional ball graphs and (strong) transformations between them and 1-dimensional weighted graphs. We summarize our findings in Chapter 7 and mention further questions.

2 Preliminaries

We begin by defining our various graph representations and specifying our notations. Furthermore, we formalize different variants of transformations.

Vectors and Sets. We define the set $[n] = \{1, \dots, n\}$ for all $n \in \mathbb{N}^+$.

A *vector* is written as $v = (v_1, \dots, v_n)^\top \in \mathbb{R}^n$ and a specific entry of it as v_i. We write $v_{[d]} = (v_1, \dots, v_d)^\top \in \mathbb{R}^d$ to obtain a vector using the first $d \leq n$ coordinates. The *Euclidean norm* is denoted by $\|\cdot\|$, i.e., $\|v\| = \sqrt{\sum_{i=1}^n v_i^2}$. Furthermore, we denote the *Euclidean distance* $d_\mathbb{E}$ as $d_\mathbb{E} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_0^+, d_\mathbb{E}(u, v) = \|p_u - p_v\|$. We refrain from specifying the input dimension of the norm or the euclidean distance as it should always be clear from the concrete input. The *triangle* $\triangle ABC$ is defined by three pairwise distinct vectors $A, B, C \in \mathbb{R}^n$. Unlike usual definitions we do not require the vectors to be *collinear*, i.e., $B - A$ and $C - A$ may be linearly dependent. We denote the *angle in a triangle* $\triangle ABC$ *at vector* A as $\angle BAC$. If the vectors are on a line we take the angle at the vector in the middle to be 180° and all other angles to be 0°.

2.1 Graphs.

An (undirected) *graph* $G = (V, E)$ consists of a set of *vertices* V and a set of *edges* between vertices $E \subseteq \{\{u, v\} \mid u \neq v \in V\}$, see Figure 2.3a. We call two vertices $u, v \in V$ *adjacent* if they share an edge, i.e., $\{u, v\} \in E$ which we write as $uv = vu \in E$.

Special Graph Classes. For $n, m \in \mathbb{N}^+$ we define the *complete bipartite graph* $K_{n,m}$ as the graph with $n+m$ vertices $V = \{x_1, \dots, x_n\} \cup \{y_1, \dots, y_m\}$ and edges $x_i y_j$ for all $i \in [n], j \in [m]$. This is a graph with two disjoint sets of vertices such that two vertices are adjacent if and only if they are not in the same set, see Figure 2.1

A graph G is called an *interval graph* if for every vertex v there exists an interval $S_v = [l_v, r_v] \subset \mathbb{R}$ such that two vertices $u \neq v$ are adjacent if and only if the corresponding intervals are non-disjoint, i.e., $S_u \cap S_v \neq \emptyset$, see Figure 2.2.

Representations. A *d-dimensional ball* is given by its center point $p \in \mathbb{R}^d$ and radius $r \in \mathbb{R}^+$ and consists of all points that have distance less than r to the center p. For the sake of brevity we write (p, r) to define a ball, although balls technically are a set.

We define the *d-dimensional ball space* $\mathbb{B}_d$ as the tuple $(\mathbb{R}^d \times \mathbb{R}^+, d_\mathbb{B})$ and write $\mathbb{B}$ if the dimension is clear. We denote elements in $\mathbb{R}^d \times \mathbb{R}^+$ as $b = (p, r)$, i.e., balls and write $(p, r) \in \mathbb{B}$. *The ball distance* $d_\mathbb{B}$ given by $d_\mathbb{B} : \mathbb{B} \times \mathbb{B} \to \mathbb{R}_0^+, d_\mathbb{B}(u, v) = \frac{d_\mathbb{E}(p_u, p_v)}{r_u + r_v}$. Two balls $u, v \in \mathbb{B}$ *intersect* if $d_\mathbb{B}(u, v) \leq 1$. Notably this differs from the *distance in between two non-intersecting balls*, see Figure 2.5. A graph G is called a *d-dimensional ball graph* if there exists a mapping that assigns every vertex v to a ball $b_v = (p_v, r_v) \in \mathbb{B}$ such that $u \neq v \in V$ are adjacent if and only if their corresponding balls b_u, b_v intersect. See Figure 2.3b for an example. Such a mapping is called a *ball representation of* G. Given a ball representation of G we call G

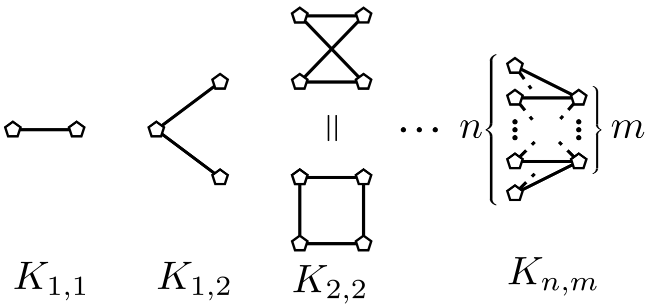


Figure 2.1: Examples of the $K_{n,m}$ for varying $n, m \in \mathbb{N}^+$. A special example is the $K_{2,2}$ as it is just a cycle of 4 vertices.

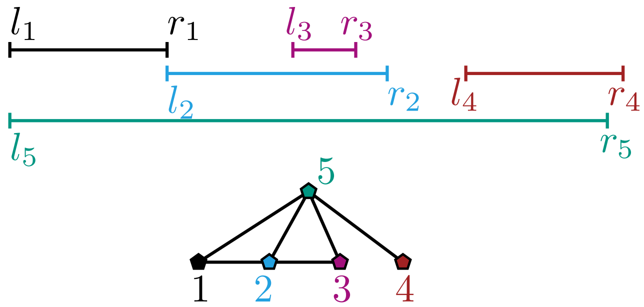
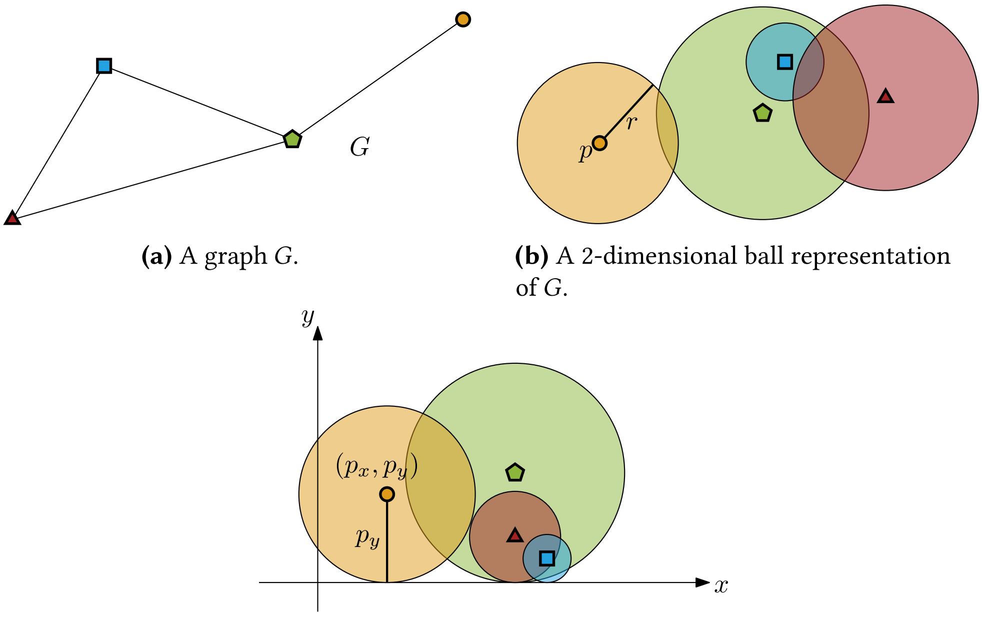


Figure 2.2: Example of an interval graph. Every vertex v has a corresponding interval $[l_v, r_v]$ such that two intervals intersect if and only if the corresponding vertices are adjacent. The different heights of the intervals are only to provide clarity. In reality, they are all subsets of the real numbers.

the *corresponding* graph. The set of all ball representations of a fixed dimension d is called $\mathcal{B}(d)$. We define the minimal dimension for which such a representation exists for a given graph G as the *ball dimension of* G and denote it by $\dim_{\mathcal{B}}(G)$. This is well-defined since it is obviously bounded by sphericity, which is finite [Mae84]. We can also visualize a ball graph in low dimensions using *adjacency regions*. Consider a fixed ball $b = (p, r) \in \mathbb{B}$ and the set $\mathrm{Adj}_{\mathbb{B}}(b) = \{\tilde{b} \in \mathbb{B} \mid b \text{ and } \tilde{b} \text{ intersect}\}$, which we call *ball adjacency region of* b. This means $\tilde{b} = (\tilde{p}, \tilde{r})$ is in $\mathrm{Adj}_{\mathbb{B}}(b)$ if and only if $\tilde{r} \geq \|p - \tilde{p}\| - r$. In other words the adjacency region is a cone with slope 1 that has been shifted down by r and cut off at the bottom, see Figure 2.4a for the 1-dimensional case.

A ball $(p, r) \in \mathbb{B}$ is a *grounded ball* if $p_d = r$. Geometrically this means that the ball touches the upper plane orthogonal to the x_d-axis, see Figure 2.3c. The *d-dimensional grounded ball space* $\mathbb{GB}_d$ is a subspace of the ball space $\mathbb{B}_d$ that only consists of grounded balls. We use the same distance function and the same intersection definition as before, just constrained to grounded balls and write $\mathbb{GB}$ if the dimension is clear. A graph G is called a *d-dimensional grounded ball graph* if there exists a d-dimensional ball representation of G such that all balls are grounded balls, i.e., all balls are in $\mathbb{GB}$. The set of all grounded ball representations of a fixed dimension d is written as $\mathcal{GB}(d)$. We define the minimal dimension for which such a representation exists as the *grounded ball dimension of* G, which we denote by $\dim_{\mathcal{GB}}(G)$. We prove in Chapter 3 that this is always finite.



(a) A graph G.

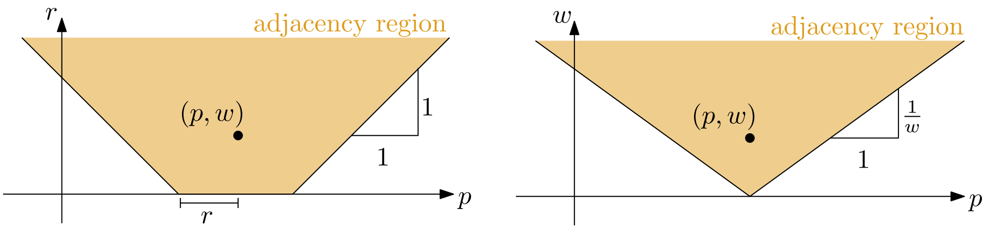
(b) A 2-dimensional ball representation of G.

(c) A 2-dimensional grounded ball representation of G.

Figure 2.3: A 2-dimensional (grounded) ball graph with both representations.

A *d-dimensional weighted point* is given by a point $p \in \mathbb{R}^d$ and a weight $w \in \mathbb{R}^+$. Similarly to before we define the *d-dimensional weighted space* $\mathbb{W}_d = \mathbb{W}$ as the tuple $(\mathbb{R}^d \times \mathbb{R}^+, d_\mathbb{W})$ and write $\mathbb{W}$ if the dimension is clear. We denote elements in $\mathbb{R}^d \times \mathbb{R}^+$ as $g = (p, w)$, i.e., as weighted points and write $g \in \mathbb{W}$. The *weighted distance* $d_\mathbb{W}$ between two weighted points is defined as $d_\mathbb{W} : \mathbb{W} \times \mathbb{W} \to \mathbb{R}, d_\mathbb{W}(u, v) = \frac{d_\mathbb{E}(p_u, p_v)}{w_u \cdot w_v}$. Two weighted points $a, b \in \mathbb{W}$ *intersect* if $d_\mathbb{W}(a, b) \leq 1$. A graph G is called a *d-dimensional weighted graph* if there exists a mapping from vertices v to weighted points $g_v = (p_v, w_v) \in \mathbb{W}$ such that $u \neq v$ are adjacent if and only if g_v and g_u intersect. We call such a mapping a *weighted representation of* G. Given a weighted representation of G we call G the *corresponding* graph. The minimal dimension for which such a representation exists is called the *weighted dimension of* G and written as $\dim_\mathcal{W}(G)$. This is well-defined since it is bounded by sphericity, which is finite [Mae84]. The set of all weighted representations of a fixed dimension d is written as $\mathcal{W}(d)$. Unlike with balls, it is not obvious how to visualize this as an intersection graph, although later results allow us to do it more easily, see Chapter 3. This is why we mostly use adjacency regions in the 1-dimensional case. Given a fixed weighted point $g = (p, w) \in \mathbb{W}$ the *weighted adjacency region of* g is given by $\mathrm{Adj}_\mathbb{W}(g) = \{\tilde{g} \in \mathbb{W} \mid g \text{ and } \tilde{g} \text{ intersect}\}$. A weighted point $\tilde{g} = (\tilde{p}, \tilde{w}) \in \mathbb{W}$ is in $\mathrm{Adj}_\mathbb{W}(g)$ if and only if $\tilde{w} \geq \frac{1}{w} \|p - \tilde{p}\|$. Like for the ball adjacency region this is a cone but with slope $\frac{1}{w}$ and without being shifted down, see Figure 2.4b.

It is not completely trivial but fairly easy to show that these definitions admit the same graphs as definitions where weight 0 or radius 0 are allowed.



(a) The ball adjacency region is a cone of slope 1 that has been cut of at the bottom at $r = 0$.

(b) The weighted adjacency region is a cone of slope $\frac{1}{w}$.

Figure 2.4: The different adjacency regions for 1-dimensional ball graphs and 1-dimensional weighted graphs.

2.2 Transformers

We are interested in whether we can always transform a ball representation into a weighted representation without (or while barely) increasing the dimension d. More formally we are interested if for every ball representation $(p_v, r_v)_{v \in V}$ of dimension d we can find a weighted representation $(\tilde{p}_v, w_v)_{v \in V}$ of dimension d' such that they describe the same graph with $d' \approx d$ and vice versa from weighted to ball representations. These transformations can get quite complicated e.g., they can use the minimal distance between balls or in general consider information of multiple balls at once to construct a new weighted point/ball. This makes analysis on them harder. We would prefer a "simple" mapping between representations where balls are always mapped to the same weighted point regardless of the rest of the representation. This means only using *local* information and no *global* information. An additional benefit of such a mapping is its value to random analysis. Note that if $X : \Omega \to \mathbb{R}^d$ is a (real) random variable then $h(X)$ is a random variable as well if h is a Borel function, i.e., measurable. A general transformation would allow a ball to be mapped to different weighted points depending on the rest of the representation. If we want to apply the function on random variables as easily as h before we would want the "simple" transformation. This gives rise to the next definition.

An *(n-to-m) Ball-Weight transformer* is a function $\Psi : \mathcal{B}(n) \to \mathcal{W}(m)$ such that for every ball representation $B \in \mathcal{B}(n)$ the corresponding ball graph is the same as the corresponding weighted graph of $\Psi(B)$. We call a Ball-Weight-transformer *strong* if there exists a function $\Phi : \mathbb{B}_m \to \mathbb{W}_n$ such that $\Psi(B)$ maps a vertex v to the weighted point $\Phi(p_v, r_v)$ where (p_v, r_v) is the ball given by the representation B. A more compact way to state this is $\Psi(W) = \Phi \circ W$ since W is a mapping itself. Informally the mapping of a single ball only depends on the ball itself. We usually write $\Phi = (\Phi_p, \Phi_w)$ and call Φ the *underlying transformer* which *induces* Ψ. Note that a function $\Phi : \mathbb{B}_m \to \mathbb{W}_n$ is an underlying transformer if and only if for all $u, v \in \mathbb{B}_m$ it holds that $d_{\mathbb{B}}(u, v) \leq 1 \iff d_{\mathbb{W}}(\Phi(u, v)) \leq 1$.

(Strong) (n-to-m) Weight-Ball, Ball-Ball-, Weight-Weight transformers and *Grounded Ball-Grounded Ball transformers* are defined analogously e.g., a Weight-Weight transformer is a function $\Psi : \mathcal{W}(n) \to \mathcal{W}(m)$ such that the corresponding graphs stay the same.

Example Transformers. We now define some simple example transformers, also listed in Table 2.1, to give some intuition on our definitions. Besides gaining intuition, we also use them to transform our representations into forms that are easier to manage for our later

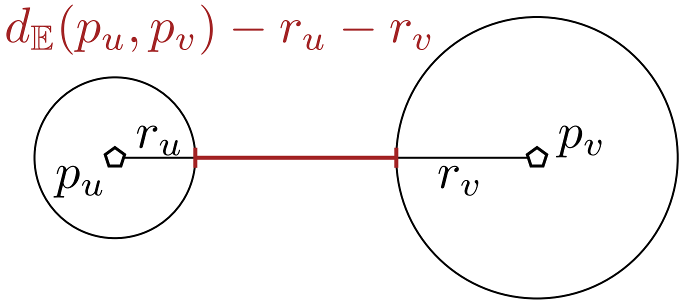


Figure 2.5: The distance between two non-intersecting balls (not the ball distance).

proofs. While we provide a definition of all of them we only sketch out the correctness of the first transformer and the less obvious ones. The rest can be easily verified by just plugging the output into the corresponding distance function.

We define $\mathrm{Iso}_f((p_v, r_v)_{v \in V}) = (f(p_v), x_v)_{v \in V}$ for isometries f of $\mathbb{R}^d$. This is a strong transformer since it is induced by $I : \mathbb{B} \to \mathbb{B}, (p, r) \mapsto (f(p), r)$ and because the distances stay the same, i.e., $d_{\mathbb{B}}(u, v) = \frac{d_{\mathbb{E}}(p_u, p_v)}{r_u + r_v} \overset{f \text{ Isometry}}{=} \frac{d_{\mathbb{E}}(f(p_u), f(p_v))}{r_u + r_v} = d_{\mathbb{B}}(f(u), f(v))$. For weighted representations Iso_f is defined analogously. This transformer allows us to apply rotations, translations and mirroring to the center points of the balls. We can define Iso_f as a Grounded Ball-Grounded Ball transformer as well by only allowing isometries f that map grounded balls to grounded balls. This means we cannot affect the last coordinate, i.e., for all $x \in \mathbb{R}^d$ it must hold that $f(x)_d = x_d$. This allows us to use rotations around the x_d-axis and mirroring across the plane orthogonal to the x_i-axis for $i < d$. Furthermore, we can use translations, which do not change the last coordinate, i.e., translations along a vector v with $v_d = 0$.

We now give strong transformers that allow us to scale our representations, i.e., balls or weighted points, by a factor $\lambda > 0$. For (grounded) balls this can be done with the strong (Grounded) Ball-(Grounded) Ball transformer $\textsc{Scale-Ball}_\lambda((p_v, r_v)_{v \in V}) = (\lambda \cdot p_v, \lambda \cdot r_v)_{v \in V}$. We also define $\textsc{Scale-Weight}_\lambda((p_v, w_v)_{v \in V}) = (\lambda \cdot p_v, \sqrt{\lambda} \cdot r_v)_{v \in V}$ for $\lambda > 0$. The square root appears since $\|\lambda p - \lambda \tilde{p}\| = \lambda \|p - \tilde{p}\|$ but $\lambda w \cdot \lambda \tilde{w} = \lambda^2 w \tilde{w}$. This transformer is a strong Weight-Weight transformer.

We have not considered non-strong transformers yet which we do now. Being non-strong means we use information of the whole representation or at least more than just a singular ball at once. We define $\mu = \mu(b_{v \in V}) = \min\{d_{\mathbb{E}}(p_u, p_v) - r_u - r_v \mid u, v \in V \text{ not intersecting}\}$, i.e., as the minimal distance in between two non-intersecting balls of our representation, see Figure 2.5. The (Grounded) Ball-(Grounded) Ball transformer $\textsc{PushApart}$ is defined as $\textsc{PushApart}(b_{v \in V}) = \textsc{Scale-Ball}_{\frac{1}{\mu}}(b_{v \in V})$. Since $\mu > 0$ the correctness of $\textsc{PushApart}$ follows from the correctness of $\textsc{Scale-Ball}$. The resulting representation after applying this mapping is one where all non-intersecting balls have a distance in between each other of at least 1.

A property of strong transformers. A nice property of underlying transformers is that they are injective as we quickly sketch out. We omit small details such as worrying about every type of transformer one by one.

Lemma 2.1: *Let Φ be an underlying transformer. Then Φ is injective. This is correct regardless of the type of transformer.*

Table 2.1: A list of transformers. All transformers listed are d-to-d transformers. The definitions of the transformers can be found in Section 2.2. The first row contains the name we reference the transformer by. The second row denotes the input and output of the transformer. In the third row we provide a short description of the transformer. In the last column we denote whether the transformer is strong.

Name	Type	Description	strong
Iso_f for an Isometry f	$\mathcal{B}(d)$-$\mathcal{B}(d)$, $\mathcal{W}(d)$-$\mathcal{W}(d)$	rotations, mirroring and translations of the points p_v.	✓
Iso_f for an Isometry f with $f(x)_d = x_d$	$\mathcal{GB}(d)$-$\mathcal{GB}(d)$	rotations around the x_d-axis, mirroring across the plane orthogonal to the x_i-axis for $i < d$ and translations along a vector v with $v_d = 0$.	✓
Scale-Ball$_\lambda$ for $\lambda > 0$	$\mathcal{B}(d)$-$\mathcal{B}(d)$, $\mathcal{GB}(d)$-$\mathcal{GB}(d)$	scaling of elements in $\mathbb{B}$, i.e., balls, by factor λ.	✓
Scale-Weight$_\lambda$ for $\lambda > 0$	$\mathcal{W}(d)$-$\mathcal{W}(d)$	scaling of elements in $\mathbb{W}$ by factor λ.	✓
PushApart	$\mathcal{B}(d)$-$\mathcal{B}(d)$, $\mathcal{GB}(d)$-$\mathcal{GB}(d)$	results in non-intersecting balls to have at least a distance of 1 between each other.	✗

Proof Sketch. Assume that $(p, x) \neq (p', x')$ exist with $\Phi(p, x) = \Phi(p', x')$. Notice that since (p, x) and (p', x') are not the same, their adjacency regions must be non-identical. W.l.o.g. there exists a $(\tilde{p}, \tilde{x})$ in the adjacency region of (p, x) but not the adjacency region of (p', x'). But since the adjacency regions of $\Phi(p, x)$ and $\Phi(p', x')$ are the same, $\Phi(\tilde{p}, \tilde{x})$ must lie in both of them, which is a contradiction. ▢

This is useful as even if we do not get surjectivity, we can still consider the image of our underlying transformer and argue about this substructure.

3 Transforming Weighted Graphs

An n-to-m-Weight-Ball transformer exists if and only if every n-dimensional weighted graph is also an m-dimensional ball graph. We want to answer whether weighted graphs are ball graphs. Therefore, we consider the existence of transformers. We show that there exists a bijective strong transformer between d-dimensional weighted graphs and $(d+1)$-dimensional grounded ball graphs. Since grounded ball graphs are also ball graphs, this proves that d-dimensional weighted graphs are a subset of $(d+1)$-dimensional ball graphs.

Theorem 3.1:

The d-dimensional weighted graphs are exactly $d+1$-dimensional grounded ball graphs.

Proof. Consider the mapping $\Phi(p, w) = (\tilde{p}, r)$ with $r = \frac{1}{2}w^2$ and $\tilde{p} = (p, r)^\top$. Notice that for all $x \in \mathbb{R}^{d+1}$ it holds that $\|x\|^2 = \sum_{i=1}^{d+1} x_i^2 = \sum_{i=1}^{d} x_i^2 + x_{d+1}^2 = \left\|x_{[d]}\right\|^2 + x_{d+1}^2$. Using this fact we obtain

$$\begin{aligned}
&\|\tilde{p}_u - \tilde{p}_v\| &&\leq r_u + r_v \\
\iff &\|\tilde{p}_u - \tilde{p}_v\|^2 = \|p_u - p_v\|^2 + r_u^2 + r_v^2 - 2r_u r_v &&\leq r_u^2 + r_v^2 + 2r_u r_v \\
\iff &\|p_u - p_v\|^2 &&\leq 4 \cdot r_u r_v = w_u^2 \cdot w_v^2 \\
\iff &\|p_u - p_v\| &&\leq w_u \cdot w_v.
\end{aligned}$$

Therefore, weighted points $g, h \in \mathbb{W}_d$ intersect exactly when $\Phi(g), \Phi(h) \in \mathbb{GB}$ intersect and Φ induces a strong Weight-Grounded-Ball-transformer, which is obviously bijective. ∎

An example of the application of this transformer is given in Figure 3.1. We can see that the resulting structure is one where every ball touches the space orthogonal to the x_{d+1}-axis, i.e., is a grounded ball representation. As previously stated we can use the image of strong transformers as a substructure to argue reason about instead of the original space of representations. This is something we do extensively in Chapter 4. The only reason we are interested in grounded balls in the first place is that the image of the previous transformer consists exactly of all grounded ball representations.

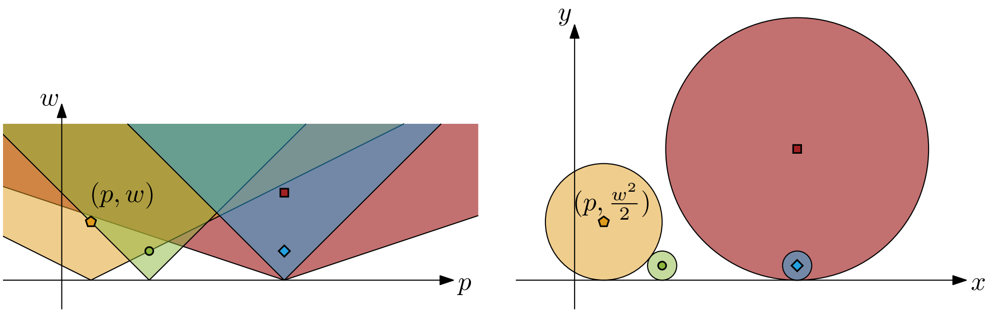
Corollary 3.2: *Every d-dimensional weighted graph is a $(d+1)$-dimensional ball graph. There exists a strong Weight-Ball transformer.*

Furthermore, we can see that the grounded ball dimension is actually well-defined, i.e., is always finite.

Corollary 3.3: *For all graphs G with* $\dim_{\mathcal{GB}}(G) > 1$ *it holds* $\dim_{\mathcal{GB}}(G) = \dim_{\mathcal{W}}(G) + 1$.

We also get a result on the ball dimension.

Corollary 3.4: *For all graphs G it holds* $\dim_{\mathcal{B}}(G) \leq \dim_{\mathcal{W}}(G) + 1$.



(a) A weighted representation as input visualized as adjacency regions.

(b) The output after applying the transformer visualized by intersections of balls is a grounded ball representation.

Figure 3.1: The effect of a 1-to-2 Weight-Ball-transformer as described in Theorem 3.1.

4 Ball graphs are not weighted graphs

We already know that weighted graphs are almost a subset of ball graphs, if we increase the dimension by 1. In this chapter we show that the reverse does not hold for dimensions ≥ 2. In order to do this, we find a counter example, i.e., a family of ball graphs that can be embedded in dimension 2. We show that for every dimension d there exists a graph in this family that cannot be embedded as a d-dimensional weighted graph. Since by Theorem 3.1 d-dimensional weighted graphs are the same as $d+1$-dimensional grounded ball graphs, we argue about them instead, allowing us to apply a more geometric reasoning. This is formulated in the following theorem.

Theorem 4.1: *For all $d \in \mathbb{N}^+$ there exists a 2-dimensional ball graph that is not a d-dimensional grounded ball graph.*

We consider the complete bipartite graph $K_{2,n}$ with vertices $\{A, B, 1, \ldots, n\}$ and edges $\forall i \in [n] : iA, iB \in E$, see Figure 4.1 and prove that this is a 2-dimensional ball graph. In order to prove that this is not a d-dimensional grounded ball for all n, we show that the radii r_i in a grounded representation are bounded by $\frac{1}{4}\min(r_A, r_B)$. This is done by giving a lower bound depending on the choice of the center of the ball p_i and then finding the minimum using the derivative. We continue by showing that there is a limited amount of balls (even in the non-grounded setting) that can intersect a ball C, if the original balls do not intersect and all have a radius larger than a fixed fraction of the radius of C. Since all the balls mapped to by $i \in V$ may not intersect, this allows us to find an upper bound on n for which a d-dimensional ball representation of the $K_{2,n}$ can exist.

Lemma 4.2: *The $K_{2,n}$ is a 2-dimensional ball graph.*

Proof. As a ball becomes larger and larger the boundary of the ball approaches a straight line at point on its boundary as seen in Figure 4.2a. Using this we can place parallel "lines" for the vertices A and B. In between those lines we can just put small balls for the i-vertices, see Figure 4.2b. There are multiple possible representations that arise from this idea, but we only present one. The inclined reader may think of different ones.

We make sure that the A and B balls have a distance in between them of 1 and give the small balls radius 1 with distances of at least 4 in between them. We have to overcompensate a bit since we do not get perfectly straight lines. Let $p_i = \left(4 \cdot (i - \frac{n}{2}), 0\right)^\top$ with radius $r_i = 1$. Furthermore, we define $p_A = \left(0, 4n^2 + 0.5\right)^\top$ and $p_B = -p_A$ with radii $r_A = r_B = 4n^2$. It remains to prove that (p_A, r_A) and (p_B, r_A) do not intersect but all intersect with (p_i, r_i) which in turn do not intersect with themselves. We verify that $\|p_A - p_B\| = 8n^2 + 1 > 8n^2 = r_A + r_B$ and for $i \neq j \in [n]$ that $\left\|p_i - p_j\right\| = 4 \cdot |i - j| \geq 4 > 1 + 1 = r_i + r_j$. By using symmetry it remains to show that

$$\|p_A - p_i\|^2 = (4n^2 + 0.5)^2 + 16 \underbrace{\left(i - \frac{n}{2}\right)^2}_{\leq (\frac{n}{2})^2} \leq 16n^4 + 8n^2 + 0.25 < (4n^2 + 1)^2 = (r_A + r_i)^2$$

Therefore $K_{2,n}$ is a 2-dimensional ball graph. ∎

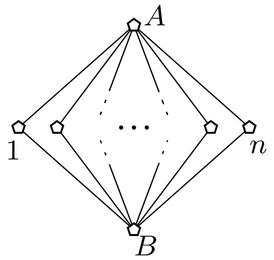
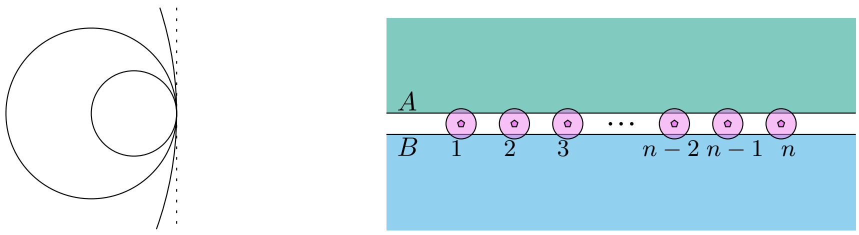


Figure 4.1: The complete bipartite graph $K_{2,n}$.



(a) Moving a ball away and increasing its radius causes its boundary to approach a line.

(b) Idea of $K_{2,n}$ being a 2-dimensional ball graph.

Figure 4.2: Ideas used in the proof of Lemma 4.2

Lemma 4.3: *Let $d \geq 2$ and $n \in \mathbb{N}^+$. If there exists a d-dimensional grounded ball representation of $K_{2,n}$, i.e., $(p_v, r_v)_{v \in V}$, then $r_i = (p_i)_d \geq \frac{1}{4}\min(r_A, r_B)$.*

Proof. W.l.o.g. we assume that $r_B \leq r_A$. Since we can apply specific isometries (see Table 2.1 for more details) we can use translations and scaling to also assume that $p_A = e_d = (0, \dots, 0, 1)^\top$. Furthermore, we can apply a rotation around the x_d-axis to ensure that $p_B = (b_1, 0, \dots, 0, b_d)^\top$ with $0 < b_d \leq 1$. For this lemma we are allowed to use scaling since we scale the ball (p_B, r_B) by the same factor as all balls (p_i, r_i), which cancels out in the statement we want to show.

We prove three inequalities, which we combine afterwards to obtain the desired lower bound. All of them rely on the fact that the radii are exactly the last coordinate, i.e., $(p_u)_d = r_u$ which in the following we always use directly. Also, we use the following calculations for each of them. For $(x, x_d), (y, y_d) \in \mathbb{GB}$ it holds that $\|x - y\|^2 = \left\|(x - y)_{[d-1]}\right\|^2 + x_d^2 - 2x_d y_d + y_d^2$ and $(x_d + y_d)^2 = x_d^2 + 2x_d y_d + y_d^2$. In the following let $a = p_A, b = p_B$ and $p_i = p$.

We know that A and B are not adjacent meaning their corresponding balls do not intersect. But i and A and respectively i and B are adjacent, therefore their corresponding balls must intersect.

$$\|a - b\|^2 = b_1^2 + b_d^2 - 2b_d + 1 > 1 + 2b_d + b_d^2 \iff b_1^2 > 4b_d \tag{4.1}$$

$$\|p - a\|^2 = \left\|p_{[d-1]}\right\|^2 + p_d^2 - 2p_d + 1 \leq p_d^2 + 2p_d + 1 \iff p_d \geq \frac{1}{4}\left\|p_{[d-1]}\right\|^2 \tag{4.2}$$

$$\left\|(p - b)_{[d-1]}\right\|^2 + p_d^2 - 2b_d p_d + b_d^2 \leq p_d^2 + 2b_d p_d + b_d^2 \iff p_d \geq \frac{1}{4b_d}\left\|(p - b)_{[d-1]}\right\|^2 \tag{4.3}$$

The lower bounds 4.2 and 4.3 allow us to formulate the following lower bound by using $\max(a, b) \geq \frac{a+b}{2}$ or by just adding both inequalities to each other.

$$8p_d \geq \frac{1}{b_d}\left\|(p - b)_{[d-1]}\right\|^2 + \left\|p_{[d-1]}\right\|^2 \tag{4.4}$$

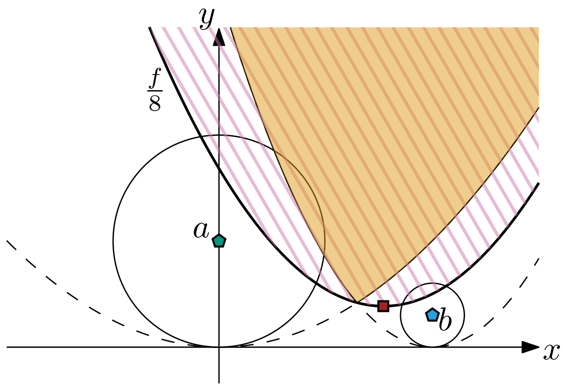


Figure 4.3: The combined lower bounds in the proof of Lemma 4.3. $p_i = (x, y)$ must lie in the filled orange region which corresponds to $p_d \geq \max\left(\frac{1}{4}\left\|p_{[d-1]}\right\|^2, \frac{1}{4b_d}\left\|(p-b)_{[d-1]}\right\|^2\right)$. The lower bound 4.4 is shaded in the striped pink pattern. The red square is our (scaled) minimal point.

We define the function $f : \mathbb{R}^{d-1} \to \mathbb{R}, f(\tilde{p}) = \frac{1}{b_d}\left\|\tilde{p} - b_{[d-1]}\right\|^2 + \|\tilde{p}\|^2$. This is a differentiable function with derivative

$$f'(\tilde{p}) = \frac{2}{b_d}(\tilde{p} - b_{[d-1]}) + 2\tilde{p} = 2 \cdot (1 + \frac{1}{b_d})\tilde{p} - \frac{2}{b_d}b_{[d-1]}.$$

Setting $f'(\tilde{p}^*) \stackrel{!}{=} 0$ yields the only solution $\tilde{p}^* = \frac{b_{[d-1]}}{b_d+1}$. The hessian matrix $H_f = 2\cdot(1+\frac{1}{b_d})\mathbb{1}_{d-1}$, with $\mathbb{1}_{d-1}$ denoting the identity matrix, is positive definite. Therefore, we know that $\tilde{p}^*$ is a minimal point with value

$$\begin{aligned} f(\tilde{p}) \geq f(\tilde{p}^*) =& \frac{1}{b_d}\left\|\left(\frac{b}{b_d+1} - b\right)_{[d-1]}\right\|^2 + \left\|\frac{b_{[d-1]}}{b_d+1}\right\|^2 \\ =& \frac{1}{b_d}b_1^2\left(1 - \frac{1}{b_d+1}\right)^2 + \left(\frac{1}{b_d+1}\right)^2 b_1^2 \\ =& b_1^2\left(\frac{1}{b_d}\left(\frac{b_d}{b_d+1}\right)^2 + \frac{1}{(b_d+1)^2}\right) \\ =& b_1^2\frac{1}{1+b_d} \stackrel{4.1}{>} \frac{4b_d}{b_d+1} \stackrel{b_d \leq 1}{>} 2b_d. \end{aligned} \tag{4.5}$$

The relationships between these lower bounds are also visualized in Figure 4.3.

The statement $r_i = p_d \geq \frac{1}{4}b_d = \min(r_A, r_B)$ follows directly from the lower bounds 4.4 and 4.5. ∎

The previous lemma is the only point where we actually use the fact that we have grounded balls. In the following lemma we consider balls in general even if they are not grounded.

Lemma 4.4: *For all $\lambda > 0$ and dimensions d there exists an $N \in \mathbb{N}^+$ such that any d-dimensional ball $C \in \mathbb{B}$ of radius r can have no more than N balls of radius $\geq \lambda r$ that intersect with C while not intersecting with each other.*

(a) The structure which we assume without loss of generality. The D_i are given by the orange balls and have a radius of at least λr while all intersecting with C (the green ball). All balls are contained by B, i.e., the ball with striped filling.

(b) The reason we can assume the structure in Figure 4.4a. The blue ball (diamond center) is our original D_i. The orange (hexagon center) ball is the one we obtain as our new $\tilde{D}_i$.

Figure 4.4: Ideas used in the proof of Lemma 4.2

Proof. The general idea of this proof is calculating the volume of C and arguing that the balls D_i must cover a certain part of C.

Let $C = (p, r) \in \mathbb{B}$ and consider the ball $B = (p, r \cdot (1 + 2\lambda)) \in \mathbb{B}$, i.e., the ball that has the same center as C and has a radius of $r \cdot (1 + 2\lambda)$. Let $D_1, \ldots, D_n \in \mathbb{B}$ be $n \in \mathbb{N}^+$ balls of radius at least λr that do not intersect with each other, but all intersect with C. We can assume the D_i are completely contained in B, which results in the structure as seen in Figure 4.4a.

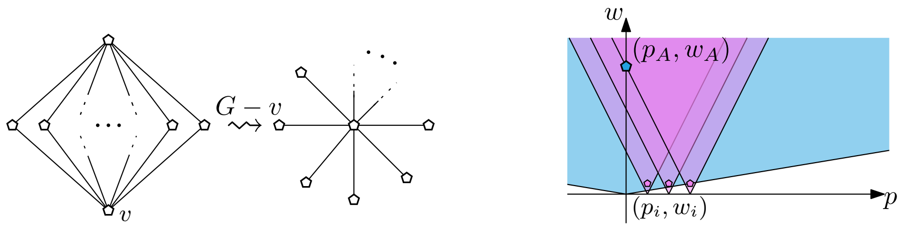
If any Ball D_i is not completely contained in B we follow the consideration of Figure 4.4b to find a ball that is completely contained in D_i and B while intersecting with C and having radius $\geq \lambda r$. We do that by considering the ray from the center of $D_i = (d, r)$ to its furthest intersection with C, which we denote by $p_{\max}$. Consider the first intersection of this ray with B which we call $p_{\min}$. The line between $p_{\min}, p_{\max}$ must be fully contained inside B as balls are convex. Consider the new ball $\tilde{D}_i$ that is defined by its center $\tilde{d} = \frac{p_{\min}+p_{\max}}{2}$ and radius $\tilde{r} = \frac{\|p_{\min}-p_{\max}\|}{2}$, i.e., is defined by the segment between $p_{\min}, p_{\max}$ being its diameter. This new ball is completely contained in D_i as $\left\| d - \tilde{d} \right\| = r - \tilde{r}$ which is the case if one ball is inside the other (and their boundaries touch). Also note that $\tilde{r} \geq \lambda r$.

We now consider the volumes V of the balls D_i and B. The volume of a d-dimensional ball with radius R is given by the formula $V = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{n}{2}+1)} R^d$, where Γ denotes the gamma function, i.e., an extension of the factorial for real numbers. It holds that $V_B \geq \sum_{i=1}^{n} V_{D_i}$ since D_i, D_j are disjoint for all $i \neq j$ and completely contained in B. Plugging in our radii, we obtain $V_B = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{n}{2}+1)} (r(1 + 2\lambda))^d$ and $V_{D_i} \geq \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{n}{2}+1)} (\lambda r)^d$. This yields the upper bound:

$$(r(1 + 2\lambda))^d \geq n \cdot (\lambda r)^d \iff n \leq \left(\frac{1 + 2\lambda}{\lambda} \right)^d$$

Taking $N = \left\lfloor \left(\frac{1+2\lambda}{\lambda} \right)^d \right\rfloor$ yields the desired result which is notably independent of the radius r. ∎

Note that for our intents and purposes this upper bound for N is pretty rough as we completely abandon the fact that our balls must be grounded, but it still allows us to prove Theorem 4.1.



(a) $G = K_{2,n}$ and $G - v$ with $v = B$ becomes a star.

(b) Weighted representation of a star.

Figure 4.5: Removing a B from the $K_{2,n}$ yields a star, which is a 1-dimensional weighted graph.

Proof of Theorem 4.1. We know by Lemma 4.2 that the $K_{2,n}$ is a 2-dimensional ball graph.

W.l.o.g. we can assume $d > 1$ as for $d = 1$ we can choose that we show not to be a 2-dimensional grounded ball graph. By Lemma 4.4 there exists an $N \in \mathbb{N}^+$, such that any d-dimensional ball of radius r can have no more that N balls with radius at least $\frac{1}{4}r$. Note that this N does not depend on the specific radius. We now assume for the sake of contradiction that the $K_{2,N+1}$ is a grounded ball graph. By Lemma 4.3 we know that $r_i = (p_i)_d \geq \frac{1}{4}\min(r_A, r_B)$. Let $C \in \{p_A, p_B\}$ be the ball that has minimal radius $\min(r_A, r_B)$. But all grounded balls induced by the p_i do not intersect with each other but all intersect with C. This is a contradiction to our choice of N. ∎

Theorem 4.1 also yields some interesting corollaries. We can restate the theorem in terms of transformers.

Corollary 4.5: *For all $d \geq 2$ and $d' \in \mathbb{N}^+$ there exists no d-to-d' Ball-Weight transformer.*

Furthermore, it tells us that the dimension of a weighted graph may increase by an unbounded amount if we add a singular vertex. Notice that $\dim_{\mathcal{W}}(K_{1,n}) = 1$ by defining A to have a large weight and all i's to have a tiny weight. An example of this is shown in Figure 4.5b. Since $K_{2,n} - B = K_{1,n}$ the dimension can decrease by an arbitrary amount when removing a vertex.

Corollary 4.6: *For all $k \in \mathbb{N}^+$ there exists a graph G and vertex v such that* $\dim_{\mathcal{W}}(G) > \dim_{\mathcal{W}}(G - v) + k$.

We can also state this in terms of grounded ball graphs.

Corollary 4.7: *For all $k \in \mathbb{N}^+$ there exists a graph G and vertex v such that* $\dim_{\mathcal{GB}}(G) > \dim_{\mathcal{GB}}(G - v) + k$.

We also get a lower bound on the weighted dimension of $K_{2,n}$ from Lemma 4.4 and the value of $\lambda = \frac{1}{4}$ from the proof.

Corollary 4.8: $\dim_{\mathcal{W}}(K_{2,n}) \geq \log_6(n) - 1$.

Proof. Notice that the grounded ball dimension is exactly 1 if the graph is a clique which the $K_{2,n}$ never is. Therefore, we can apply Corollary 3.3.

Let $d = \dim_{\mathcal{GB}}(K_{2,n})$ and by Corollary 3.3 also $\dim_{\mathcal{W}}(K_{2,n}) = d - 1$. Combining Lemma 4.3 and Lemma 4.4 yields:

$$n \leq \left(\frac{1 + 2\frac{1}{4}}{\frac{1}{4}}\right)^d = 6^d \iff \log_6(n) \leq d.$$

∎

It is unclear though how good this lower bound is. We initiate further discussions of this in Section 7.1. It is also obvious that for $n, m \geq 2$ the $K_{n,m}$ must have at least the dimension of the $K_{2,m}$, since removing vertices can never increase the dimension. Therefore, we obtain the last corollary.

Corollary 4.9: *For* $n, m \geq 2$ *it holds* $\dim_{\mathcal{W}}(K_{n,m}) \geq \log_6(\max(n, m)) - 1$

5 Ball Dimension of the $K_{3,n}$

In Chapter 4 we were able to find a family of graphs, for which the weighted dimension cannot be bounded by a constant value. We now show that the $K_{3,n}$ also has an ball dimension in $\omega(1)$ and prove the following theorem.

Theorem 5.1: *For all n it holds* $\dim_{\mathcal{B}}(K_{3,n}) \geq \log_5(n)$.

We use a similar notation to the previous chapter and consider the $K_{3,n}$ with vertices $\{A, B, C, 1, \ldots, n\}$ and edges $\forall i \in [n] : iA, iB, iC \in E$, see Figure 5.1. We follow similar steps as in the proof of Theorem 4.1 and even use Lemma 4.4. This means we first show that the radii r_i are bigger than a fraction of $\min(r_A, r_B, r_C)$ and then apply Lemma 4.4. The general idea comes from noticing that p_A, p_B, p_C must lie on a plane which means we can reduce to 2 dimensions. We give a lower bound for the minimal radius by considering the triangle $\triangle p_A, p_B, p_C$ and minimizing the maximal distance of a point P to the vertices of the triangle, i.e., p_A, p_B, p_C. Afterwards we rely on the concept of the *Fermat point*[1] to do some of the heavy lifting for us. This point allows us to give a lower bound on the minimal radius a ball must have in order to intersect with three other balls. We begin by defining some notation and the Fermat point. After this we restate some results about this point.

Definition 5.2: *Given three fixed points $A, B, C \in \mathbb{R}^d$ we define their distance to a point $P \in \mathbb{R}^d$ as*

$$S_{ABC}(P) = S(P) = \|P - A\| + \|P - B\| + \|P - C\| .$$

For a visualization of $S(P)$ see Figure 5.2.

Definition 5.3: *Given a triangle $\triangle ABC$ in the plane, the point $P_F \in \mathbb{R}^2$ is defined as the point that minimizes the sum of the distances to the vertices, i.e.,*

$$P_F = \operatorname{argmin}_{P \in \mathbb{R}^2} S_{ABC}(P).$$

The point P_F is called the Fermat point *of $\triangle ABC$.*

When constructing the Fermat point, there are two separate cases. We give the explicit construction for the easier case and only consider the value of $S(P_F)$ for the more difficult case.

Theorem 5.4: *[KV97] If one of the angles of a triangle is $\geq 120°$ then the Fermat point of said triangle lies on the vertex at this angle, see Figure 5.3a.*

This also includes the case where all points lie on a line if we consider the point in the middle to have an angle of 180°. In the more general case with all angles being less than 120° a solution for the Fermat point can also be constructed. This was first done by Torricelli, see Figure 5.3b. Giving explicit (Cartesian) coordinates leads to a somewhat ugly result, which we do not need. We are more interested in the following result.

[1] This is also known as the Fermat-Torricelli point, the Fermat-Steiner point or the first isogonic center.

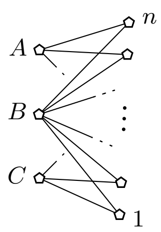


Figure 5.1: The complete bipartite graph $K_{3,n}$.

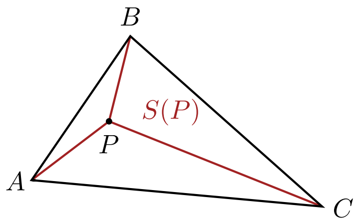


Figure 5.2: $S_{ABC}(P)$ is the sum of all distances from A, B, C to P.

Theorem 5.5: *[htt16] If a triangle $\triangle ABC$ has no angle $\geq 120^\circ$ than it holds that*

$$S(P)^2 \geq S(P_F)^2 = \frac{1}{2}\left(\|B-C\|^2 + \|B-A\|^2 + \|C-A\|^2\right) + 2\sqrt{3}\mathcal{A}(\triangle ABC)$$

where $\mathcal{A}(\triangle ABC)$ is the area of the triangle.

We begin by stating the necessary lemmata to prove our theorem.

Lemma 5.6: *If there exists a d-dimensional Ball representation of $K_{3,n}$, i.e. $(p_v, r_v)_{v \in V}$, then $r_i = (p_i)_d \geq \frac{1}{3}\min(r_A, r_B, r_C)$.*

While proving this we end up with a rather large term for which we would like a lower bound. We prove a lower bound for this separately since it is not as interesting and purely algebraic while the proof of Lemma 5.6 is guided more by geometry.

Lemma 5.7: *Let $x, y, z > 0$ with $x + y + z = 1$. Let $m = \min(x, y, z)$. Then*

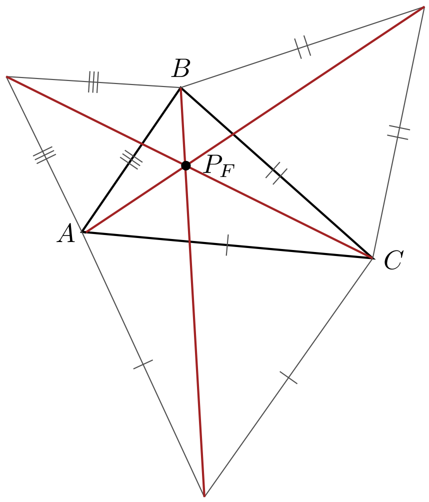
$$\frac{1}{2}\left((x+y)^2 + (x+z)^2 + (y+z)^2\right) + 3 \cdot (x+z)(x+y) \geq (m+1)^2.$$

We now prove things in reverse order starting with Theorem 5.1 under the assumption that Lemma 5.6 holds. Then we show Lemma 5.6 under the assumption that Lemma 5.7 holds which we prove at the end.

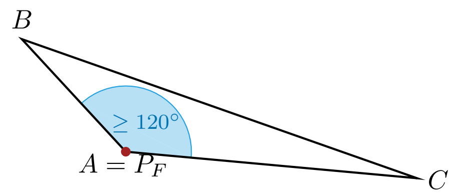
Proof of Theorem 5.1. The idea of this proof is pretty much the same as for Theorem 4.1 and Corollary 4.8. Let $d = \dim_{\mathcal{B}}(K_{3,n})$ be the ball dimension of $K_{3,n}$. Then there exists a d-dimensional ball representation of the $K_{3,n}$ with balls b_i, b_A, b_B, b_C. Applying Lemma 5.6 yields $r_i \geq \frac{1}{3}\min(r_A, r_B, r_C)$. Let C be a ball in $\{(b_A, b_B, b_C)\}$ such that it has the smallest radius. Since all b_i balls intersect with C, but do not intersect with each other, we can apply Lemma 4.4 to obtain

$$n \leq \left(\frac{1 + 2 \cdot \frac{1}{3}}{\frac{1}{3}}\right)^d = 5^d \iff \log_5(n) \leq d.$$

∎



(a) If all angles are $< 120°$, we can construct P_F drawing equilateral triangles on two sides of the triangle. By drawing lines from the new vertices to the original vertex opposite of the extended side, we obtain the Fermat point P_F as the intersection of those lines. This is the original solution as given by Torricelli [KV97].



(b) If one angle of the triangle is $\geq 120°$ then the Fermat point P_F lies on the vertex at that angle.

Figure 5.3: Construction of the Fermat point P_F, i.e., the point which minimizes the sum of distances to the corners of a triangle.

Proof of Lemma 5.6. We define $a = p_A, b = p_B, c = p_C$ and fix an arbitrary ball corresponding to the i vertices, i.e., $(p_i, r_i) = (p, r) \in \mathbb{B}$. W.l.o.g. assume the following of our representation. The biggest angle of the triangle $\triangle abc$ lies at a, i.e., $\angle cab \geq \angle abc, \angle bca$. Furthermore, by scaling we assume $r_A + r_B + r_C = 1$. By applying rotations and translations we can also assume that $a = 0, b = (b_1, 0, \ldots, 0)^\top, c = (c_1, c_2, 0, \ldots, 0)^\top$, see Figure 5.4. If $d = 1$, i.e., the one dimensional case, we define $c_2 = 0$. This is not completely in line with our previous notation, but allows us to avoid having to consider this case separately.

Similarly to Lemma 4.3 we obtain some inequalities by noticing that $(a, r_A), (b, r_B), (c, r_C)$ are all pairwise non-intersecting.

$$\|b\| > r_A + r_B \tag{5.1}$$
$$\|c\| > r_A + r_C \tag{5.2}$$
$$\|b - c\| > r_B + r_C \tag{5.3}$$

We also know that (p, r) intersects with all of them, therefore:

$$\begin{aligned} \|p\| - r_A &\leq r \\ \|p - b\| - r_B &\leq r \\ \|p - c\| - r_C &\leq r \end{aligned} \tag{5.4}$$

We can combine inequalities (5.4) and our assumption $r_A + r_B + r_C = 1$ into a singular inequality.

$$3r \geq \|p\| + \|p - b\| + \|p - c\| - (r_A + r_B + r_C) = S_{abc}(P) - 1$$

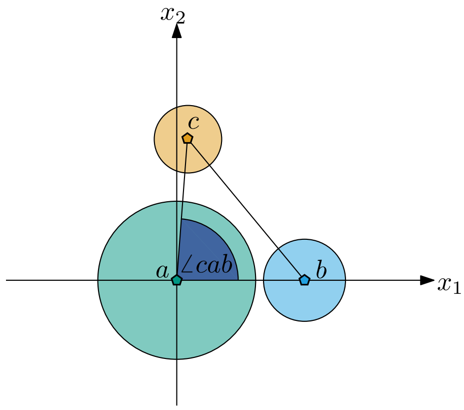


Figure 5.4: Our assumptions of the construction of $K_{3,n}$. The biggest angle is $\angle cab$, a is in the origin, b lies on the x_1-axis and c in the x_1, x_2 plane.

Consider the projection $\pi : \mathbb{R}^d \to \mathbb{R}^2, x \mapsto \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, where for $d = 1$ we define $x_2 = 0$. We notice that $S_{abc}(p) \geq S_{\pi(a),\pi(b),\pi(c)}(\pi(p))$ since a, b, c have zero entries everywhere but the first two coordinates. We can therefore in the following only consider the 2-dimensional case, i.e., only consider $S(p) = S_{\pi(a),\pi(b),\pi(c)}(\pi(p))$. For the sake of decluttering our notation we do not distinguish the original points and their projections in the following. By the definition of the Fermat point P_F we obtain

$$3r \geq S(P_F) - 1. \tag{5.5}$$

If $\angle cab \geq 120°$ (this case also includes 180°, i.e., if all lie on a line and especially includes the case $d = 1$), then by 5.4 we know that $P_F = a$ and by the definition of S we get

$$3r \geq \|c\| + \|b\| - 1 \overset{5.1,5.2}{>} (r_A + r_B) + (r_B + r_C) - 1 \overset{r_A + r_B + r_C = 1}{=} r_B \geq \min(r_A, r_B, r_C).$$

If $\angle cab < 120°$ we can apply Theorem 5.5. We know that the area of the triangle is given by $\mathcal{A}(\triangle abc) = \frac{1}{2}\sin(\angle cab)\,\|C\| \cdot \|B\|$.

Since $\angle cab$ is the biggest angle and all angles sum up to 180° we also get $\angle cab \geq 60°$. Notice that $\sin(60°) = \sin(120°) = \frac{\sqrt{3}}{2}$ and also that sin is concave on the interval $[0, \pi]$. Therefore, $\sin(\angle cab) \geq \frac{\sqrt{3}}{2}$. We obtain

$$\mathcal{A}(\triangle abc) \geq \frac{\sqrt{3}}{2}\|C\| \cdot \|B\| \overset{5.1,\,5.2}{>} \frac{\sqrt{3}}{2} \cdot (r_A + r_C) \cdot (r_A + r_B).$$

By plugging this into Theorem 5.5 and by using Equations (5.1) to (5.3) we get

$$(S(P_F))^2 \geq \frac{1}{2}\left((r_B + r_C)^2 + (r_A + r_C)^2 + (r_A + r_B)^2\right) + 3 \cdot (r_A + r_C) \cdot (r_A + r_B).$$

This is of the form as specified in Lemma 5.7 and yields

$$S(P_F) \geq \min(r_A, r_B, r_C) + 1.$$

We can finally plug this into Equation (5.5) to obtain our desired result

$$3r \geq \min(r_A, r_B, r_C)$$

■

It remains to prove Lemma 5.7. This basically boils down to multiplying everything out and collecting terms.

Proof of Lemma 5.7. We refrain from doing every calculation step by step. Unless explicitly mentioned each equality is obvious if you multiply both sides of the equality out.

$$\begin{aligned}
&\frac{1}{2}\left((x+y)^2 + (x+z)^2 + (y+z)^2\right) + 3\cdot(x+z)(x+y) \\
=&4x^2 + (y+z)^2 + 4x\cdot(y+z) + 2yz \\
=&4x^2 + (1-x)^2 + 4x(1-x) + 2yz && (x+y+z=1) \\
=&(x+1)^2 + 2yz \\
\geq&(m+1)^2 && (x \geq m, y, z \geq 0)
\end{aligned}$$

∎

Like in Chapter 4 we also get a couple of corollaries. Since the $K_{2,n}$ is 2-dimensional and $K_{3,n} - C = K_{2,n}$ we obtain a similar corollary to Corollary 4.6 and Corollary 4.7.

Corollary 5.8: *For all $k \in \mathbb{N}^+$ there exists a graph G and vertex v such that* $\dim_{\mathcal{B}}(G) > \dim_{\mathcal{B}}(G-v) + k$.

Furthermore, we get a lower bound for the $K_{n,m}$ for $n, m \geq 3$.

Corollary 5.9: *For $n, m \geq 3$ it holds* $\dim_{\mathcal{W}}(K_{n,m}) \geq \log_5(\max(n,m))$

6 1-dimensional Ball Graphs

We now know that d-dimensional weighted graphs are also $(d+1)$-dimensional ball graphs and that ≥ 2 dimensional ball graphs are not always d-dimensional ball graphs but the case of 1-dimensional ball graphs has not been considered yet. In this chapter we show that 1-dimensional ball graphs are a proper subset of 1-dimensional weighted graphs, but that no strong transformer with a continuous underlying transformer exists between them. We begin in Section 6.1 by showing that 1-dimensional weighted graphs are not a subset of 1-dimensional ball graphs. In Section 6.2 we continue by analyzing a special subset of strong transformers. We use our analysis to show that these transformers cannot exist, but also to motivate a 1-to-1 Ball-Weight transformer in Section 6.3.

6.1 Interval graphs

Dimension 1 is a special dimension as 1-dimensional graphs are exactly interval graphs. The equivalence of representations becomes apparent by noticing that 1-dimensional balls are exactly intervals, see Figure 6.1. Since interval graphs have been extensively studied, we already know a lot about them. For example the cycle $C_4 = v_0 v_1 v_2 v_3$ with edges $v_i v_{i+1} \mod 3$ is not an interval graph [LB62]. We restate this in terms of complete bipartite graphs since the $K_{2,2}$ is just the C_4, see Figure 2.1. This is done to highlight the difference to Chapter 4.

Lemma 6.1: *The complete bipartite graph $K_{2,2} = C_4$ is not a 1-dimensional ball graph.*

But we can actually find a 1-dimensional weighted representation of the $K_{2,n}$. This tells us that 1-dimensional ball graphs can either be a proper subset or incomparable to 1-dimensional weighted graphs.

Lemma 6.2: *The complete bipartite graph $K_{2,2}$ is a 1-dimensional weighted graph.*

Proof Sketch. Geometrically this becomes more obvious by thinking about grounded ball graphs. A 2-dimensional grounded ball representation of the $K_{2,2} = C_4$ is given in Figure 6.2. ▢

Corollary 6.3: *1-dimensional weighted graphs are not a subset of 1-dimensional ball graphs.*

6.2 Continuous underlying transformers

It remains to answer whether 1-dimensional ball graphs are a subset of 1-dimensional weighted graphs or if they are incomparable. Another way to state this is whether a 1-to-1 transformer exists or not. We start by arguing about a specific subset of transformers namely strong transformers with underlying continuous transformers. Our goal is to understand this simpler case first. We use the continuity to reconstruct Φ_w purely from Φ_p under the assumption that such a continuous underlying transformer exists. While we would of course like to find such a transformer in the end we show that this is impossible by using this reconstruction. We go

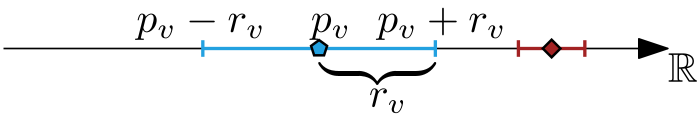


Figure 6.1: The 1-dimensional ball graphs are interval graphs. Every ball p, r is an interval $[p - r, p + r]$.

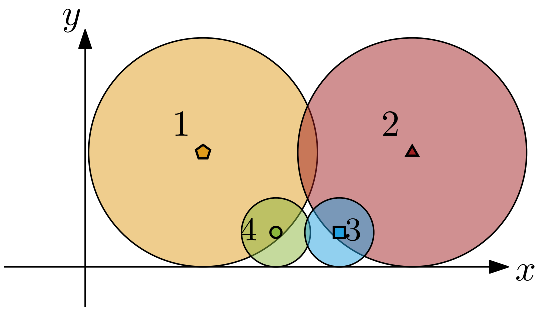


Figure 6.2: A 2-dimensional grounded ball representation of the $K_{2,2}$, i.e., a cycle with four vertices.

into a lot of detail why such a transformer does not exist and notice that the main problem is at the boundary of adjacency regions. This allows us to use our reconstruction from before in order to "try" different Φ_p's. We pick the one that works the best, i.e., works everywhere but the boundary. By applying a non-strong ball-ball transformer we will be able to ignore this boundary though.

A hard thing about analyzing (strong) transformers is that we always have to argue about inequalities. It is a lot easier to argue about transformers if you know the exact mapping at certain points, which we can do in the continuous case. Intuitively if a ball is on the boundary of a ball adjacency region of a different ball, i.e., they touch, then they are on the boundary of the weighted adjacency region after applying the transformer.

Lemma 6.4: *If the underlying transformer Φ of a Ball-Weight-transformer is continuous in (p, r) then for any p', r' with $\|p - p'\| = r + r'$ it holds that $\Phi_w(p, r) = \frac{\|\Phi_w(p,r) - \Phi_w(p',r')\|}{\Phi_w(p',r')}$*

Proof. We prove this by using a sequence of balls that do not intersect with $(p, r) \in \mathbb{B}$ but approaches the ball $(p', r') \in \mathbb{B}$ that just touches (p, r). By using facts about limits we get the desired equality.

Consider any sequence $(p_n, r_n)_{n \in \mathbb{N}^+}$ such that for all $n \in \mathbb{N}^+$ it holds that $\|p_n - p'\| > r_n + r'$ and $(p_n, r_n) \overset{n\to\infty}{\longrightarrow} (p, r)$. Since Φ is an underling transformer it holds that

$$\left\|\Phi_p(p', r') - \Phi_p(p_n, r_n)\right\| > \Phi_w(p, r) \cdot \Phi_w(p_n, r_n). \tag{6.1}$$

Since Φ is continuous in (p, r) and $\|\Phi(p', r') - \Phi(p, r)\| \leq \Phi_w(p, r) \cdot \Phi_w(p', r')$ we know that the limit of Equation (6.1) holds with equality. Rearranging the equation yields the desired formula. ∎

Let $R > 0$ be an arbitrary radius. In Section 6.3 we get more hand-wavy in the motivation and use $R = 0$ to make our formulas cleaner which is why we keep an arbitrary radius for the moment instead of for example fixing radius 1. Applying Lemma 6.4 allows us to gain more insights into Φ. Let us fix the concrete ball $b = (0, R)$ with center at the origin and radius R. If we know the value of $\Phi(0, R)$ then we can reconstruct the value of $\Phi_w(p', r')$ from $\Phi_p(p', r')$ for all balls p, r with $\|p\| = R$ by using Lemma 6.4. This means for those balls we only have to define the mapping to points instead of defining the whole mapping to weighted points. We can actually take this further and reconstruct Φ_w from Φ_p for all balls by looking at intermediary balls. This quickly becomes tedious though which is why we only consider the subcase of $\|p\| > r + R$, i.e. when both balls do not intersect, in detail. The other cases are not actually necessary for our further arguments on the existence of (continuous underlying) 1-dimensional transformers which still is our goal! In the following we assume $\Phi(0, R) = (0, 1)$ by scaling and translating, see Table 2.1.

Lemma 6.5: *Let Φ be an underlying continuous ball-weight transformer with $\Phi(0, R) = (0, 1)$. Then for all balls (p, r) with $\|p\| > R + r$ it holds that*

$$\Phi_w(p, r) = \frac{\left\|\Phi_p(p, r) - \Phi_p(\tilde{p}, \tilde{r})\right\|}{\left\|\Phi_p(\tilde{p}, \tilde{r})\right\|} \text{ with } \tilde{r} = \frac{\|p\| - r - R}{2} \text{ and } \tilde{p} = \frac{\tilde{r} + R}{\|p\|} \cdot p.$$

Proof. Notice that $\|p\| > r + R$ happens if $(0, R) \in \mathbb{B}$ and $(p, r) \in \mathbb{B}$ do not intersect. We construct an intermediary ball in the space between both balls that exactly touches both balls and use that to apply Lemma 6.4 twice, see Figure 6.3 for a visualization of our construction. In the one dimensional case there exists exactly one ball that fulfills this requirement, which is why we use the corresponding ball in higher dimensions even though there might be multiple options there.

We define the intermediary ball $(\tilde{p}, \tilde{r}) \in \mathbb{B}$ with radius $\tilde{r} = \frac{\|p\| - r - R}{2}$ and center point $\tilde{p} = \frac{p}{\|p\|} \cdot (\tilde{r} + R) = \frac{p}{2\|p\|} \cdot (\|p\| - r + R)$. It holds that $\|p\| \neq 0$, i.e., this is well-defined, since $\|p\| > r + R > 0$. We show now that this ball exactly touches both (p, r) and $(0, R)$. Notice that

$$\|\tilde{p}\| = \frac{\|p\|}{\|p\|} |\tilde{r} + R| = \tilde{r} + R.$$

In other words the $(\tilde{p}, \tilde{r})$ ball touches the $(0, R)$ ball. Furthermore, observe the same for the (p, r) ball:

$$\|p - \tilde{p}\| = \left\| \frac{2\,\|p\| \cdot p}{2\,\|p\|} - \tilde{p} \right\| = \frac{1}{2} \cdot \left| 2\,\|p\| - (\|p\| - r + R) \right| = \frac{\|p\| - r - R}{2} + r = \tilde{r} + r.$$

Therefore, we can use Lemma 6.4 to obtain

$$\Phi_w(p, r) = \frac{\left\|\Phi_p(p, r) - \Phi_p(\tilde{p}, \tilde{r})\right\|}{\Phi_w(\tilde{p}, \tilde{r})} = \frac{\left\|\Phi_p(p, r) - \Phi_p(\tilde{p}, \tilde{r})\right\|}{\left\|\Phi_p(\tilde{p}, \tilde{r})\right\|}.$$

∎

This lemma allows us to do two things. Quickly checking some possible functions Φ_p for plausibility and also showing that there can be no strong 1-to-1 Ball-Weight transformer with a continuous underlying transformer.

The main problem that arises is at the boundary of the adjacency regions, i.e., where balls almost touch. The contradiction arises by considering the case where a ball gets closer and closer to the $(0, R)$ ball. We consider what value Φ_p approaches for balls with radius approaching zero. We do not take the existence of such a value for granted and argue that it exists. The actual contradictions arise by considering a family of balls that have the same

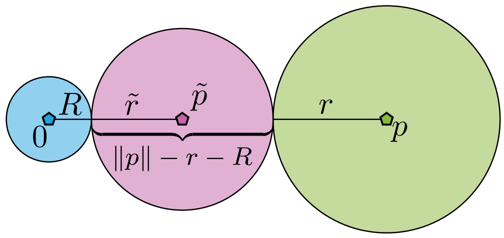


Figure 6.3: We construct $\Phi_w(p,r)$ using the red ball using Lemma 6.4 since it touches both the (p,r) and the $(0,R)$ ball.

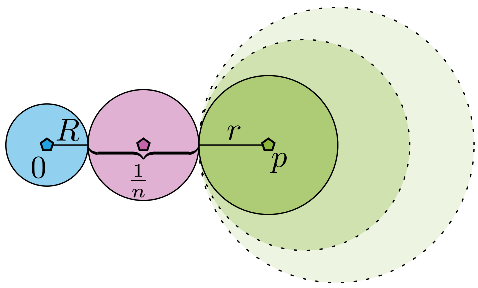


Figure 6.4: The choice of our family of balls for a constant n can be seen in black. The center moves to the right as r increases, but the ball keeps the same distance in between itself and the $(0,R)$ ball.

intermediary ball from Lemma 6.5. We move every ball of this family closer to $(0,R)$ by the same amount such that for every amount we move them by they still share the same intermediary ball. Using this we find multiple balls that map to the same value which contradicts the fact that strong transformers are injective as shown in Lemma 2.1.

Theorem 6.6: *There exists no continuous underlying* 1*-to-*1 *ball-weight transformer.*

Proof. Assume that such an underlying transformer Φ exists. W.l.o.g. $\Phi(0,R) = (0,1)$ by translating and scaling.

We define the family of sequences $p_{n,r} = r + R + \frac{1}{n} \overset{n\to\infty}{\longrightarrow} r + R = p_{\infty,r}$ for $r > 0$ and consider the balls $(p_{n,r}, r)$. For all $n \in \mathbb{N}^+$ it holds that $\left\|p_{n,r}\right\| = r + R + \frac{1}{n} > r + R$. Therefore, we can apply Lemma 6.5 to the ball $(p_{n,r}, r)$.

$$\Phi_w(p_{n,r}, r) = \frac{\left|\Phi_p(p_{n,r}, r) - \Phi_p(\frac{1}{2n} + R, \frac{1}{2n})\right|}{\left|\Phi_p(\frac{1}{2n} + R, \frac{1}{2n})\right|} = \left|\frac{\Phi_p(p_{n,r}, r)}{\Phi_p(\frac{1}{2n} + R, \frac{1}{2n})} - 1\right| \overset{n\to\infty}{\longrightarrow} \left|\Phi_p(p_{\infty,r}, r)\right|. \quad (6.2)$$

Notice that the ball $(\frac{1}{2n} + R, \frac{1}{n})$ does not depend on r. This situation is highlighted in Figure 6.4.

We first show that $\frac{1}{\Phi_p(\frac{1}{2n}+R,\frac{1}{2n})}$ converges by reducing this to the fact that all the other components in the above equation converge. The only caveat here is the absolute value. Therefore, we have to make sure that

$$w_r(n) = \frac{\Phi_p(p_{n,r}, r)}{\Phi_p(\frac{1}{2n} + R, \frac{1}{2n})} - 1$$

never switches sign.

Assume that there are $n, n' \in \mathbb{N}^+$ such that $w_r(n) > 0$ and $w_r(n') < 0$. Consider the path between both n and n' given by $g(t) = t \cdot n + (1-t) \cdot n'$. Notice that $w_r(g(t))$ is continuous in the interval $[0,1]$. Since $w_r(g(0)) = w_r(n') < 0$ and $w_r(g(1)) = w_r(n) > 0$ we can use the intermediate value theorem to obtain a $t \in (0,1)$ such that $w_r(g(t)) = 0$. But since $|w_r(g(t))| = \Phi_w(r + R + g(t), r)$ we get a contradiction since we map to non-zero weights[1].

We can now argue about the rest of the terms in Equation (6.2) to obtain the convergence of $\frac{1}{\Phi_p(\frac{1}{2n}+R, \frac{1}{2n})}$. Notice that $\Phi_p(p_{\infty,r}, r) \neq 0$ since otherwise the weight would be zero. By the previous observations we can conclude that $\frac{1}{\Phi_p(\frac{1}{2n}+R, \frac{1}{2n})}$ converges for $n \longrightarrow \infty$ as we can add (or subtract depending on the sign) 1 from $w_r(n)$ and divide by $\Phi_p(p_{n,r}, r)$ (for almost all n)[2].

We consider the value the sequence approaches, i.e., $\frac{1}{\Phi_p(\frac{1}{2n}+R, \frac{1}{2n})} \overset{n\to\infty}{\longrightarrow} c$. We use this value to find two balls that map to the weighted point. Let $\Phi_p(p_{\infty,r}, r) = x_r$. Notice that since $|x_r| = \Phi_w(p_{\infty,r}, r)$ it must hold that $x_r \neq x_{\tilde{r}}$ for $r \neq \tilde{r} > 0$ as Φ is injective by Lemma 2.1. We use the fact here that $(p_{\infty,r}, r) \neq (p_{\infty,\tilde{r}}, \tilde{r})$.

Combining our results we notice that

$$\begin{aligned} |x_r| &\overset{6.4}{=} \Phi_w(p_{\infty,r}, r) = \lim_{n\to\infty} \Phi_w(p_{n,r}, r) = |x_r \cdot c - 1| \\ \implies x_r^2 &= |\Phi_p(p_{\infty,r}, r)|^2 = \Phi_w(p_{\infty,r}, r)^2 = (x_r \cdot c - 1)^2. \end{aligned}$$

This is a polynomial equation in x_r of either degree 1 or 2. Therefore, at most 2 values for x_r can exist which contradicts the injectivity of Φ. ∎

6.3 1-to-1 Ball Weight Transformer

We now want to show that although no strong 1-to-1 Ball-Weight transformers with underlying continuous transformer may exist, we can still find a non-strong transformer, i.e., we want to show the following theorem.

Theorem 6.7: *1-dimensional ball-graphs are a subset of 1-dimensional weighted graphs.*

Since the fundamental problem of strong continuous transformers is when non-intersecting balls are to close, we know that we can fix this by using PushApart from Table 2.1. This non-strong Ball-Ball transformer results in representations where all non-intersecting balls have at least a distance of 1 in between them, see Figure 2.5. This means we can now abuse Lemma 6.5 by just trying out different Φ_p and finding one that works well enough, i.e., is correct everywhere but in the area around the boundary. We are less concerned with actually meeting the requirements of the lemma properly, as we only want to gain an intuition experimentally into which functions could work. Furthermore, we get quite hand-wavy at certain points. This part is only intended as a motivation for the transformer which we later find and rigorously prove. We even take $R = 0$ to make our formulas cleaner and reduce the amount of balls that differ from the case where we are actually allowed to use Lemma 6.5. We always consider a version of

$$\Phi_w(p, r) = \frac{\left|\Phi_p(p, r) - \Phi_p(\tilde{p}, \tilde{r})\right|}{\left|\Phi_p(\tilde{p}, \tilde{r})\right|} \text{ with } \tilde{r} = \frac{|p| - r - 0}{2} \text{ and } \tilde{p} = \frac{\tilde{r} + 0}{|p|} \cdot p.$$

[1] Even if we allowed weights being zero in our definition of weighted graphs, it is easy to show that a strong Ball-Weight transformer may never map to a weight of 0.

[2] You can actually prove that $\Phi_p(p_{n,r}, r) \neq 0$ holds in general, but this is unnecessary here.

even if $\|p\| \leq r + 0$. Furthermore, we do not care about fulfilling $\Phi_p(0, R) = (0, 1)$. We can replace $\frac{p}{|p|} = \operatorname{sign}(p)$ to make this formula work for $p = 0$ as well.

We "measure" how good a function Φ_p works by looking at a fixed adjacency region of a ball (p, r) and considering the *transformed adjacency region*. A ball p', r' lies inside this region if and only if $\left\|\Phi_p(p, r) - \Phi_p(p', r')\right\| \leq \Phi_w(p, r) \cdot \Phi_w(p', r')$, i.e., the corresponding vertices are adjacent according to the transformation. To do this quickly we use plots as seen in Figure 6.5a. The script used to create them can be found in GitLab[3]. We can observe two things. The first is that $\Phi_p(p, r) = e^p$ looks quite good. This makes a lot of sense as the exponential function translates between sums and products. Secondly we notice that $p < 0$ appears to be problematic. One thing one might notice is that in our construction of Φ_w we use $\|p\| = |p|$ quite frequently. Since our Φ_p's appear to work better for the case where $p = |p|$, i.e. $p \geq 0$ we try replacing all $|p|$ with p. This yields

$$\tilde{\Phi}_w(p, r) = \left|\frac{\Phi_p(p, r)}{\Phi_p(\frac{p-r}{2}, \frac{p-r}{2})} - 1\right|.$$

As we can see in Figure 6.5b this looks a lot better than our previous variant. Furthermore, to make our resulting formulas easier, we are inclined to try removing the -1. Not only do we get a cleaner formula, we also get a better result, see Figure 6.5c. This yields the final $\Phi_p(p, r) = e^p$ and $\Phi_w(p, r) = |e^{p - \frac{1}{2}(p-r)}| = e^{\frac{p+r}{2}}$.

We use this Φ in conjunction with PushApart to prove Theorem 6.7. The general idea is to use the fact that we can ignore the absolute value for $|e^p - e^{\tilde{p}}| \leq e^{\frac{p+r}{2}} \cdot e^{\frac{\tilde{p}+\tilde{r}}{2}}$ since we are in the one-dimensional case and can just take the bigger one as the minuend, i.e. the value we subtract from. Applying the logarithm yields the equivalent statement $\ln(e^p - e^{\tilde{p}}) \leq \frac{1}{2}(p + \tilde{p} + r + \tilde{r})$. The right side of this inequality already looks quite good as it only consists of sums. To fix the left side we first show a good approximation $\ln(e^p - e^{\tilde{p}})$ in terms of p and $\tilde{p}$. Afterwards we can apply a scaled version of PushApart to mitigate the error we get from our approximation.

Lemma 6.8: *For all $x > y \in \mathbb{R}$ there exists an $R \in \left(\frac{1}{e^{x-y}}, \frac{1}{e^{x-y}-1}\right)$ such that* $\ln(e^x - e^y) = x - R$

Proof. We will use a Taylor expansion to prove this result. Consider $\ln(e^z - 1)$ for $z > 0$. Let $f(t) = \ln(e^z - t)$. Using a Taylor Expansion of degree 0 at $t_0 = 0$ we obtain that for all $t > 0$ there exists a $\xi \in (0, t)$ such that $f(t) = f(0) + \frac{f'(\xi)}{1} \cdot t = z - \frac{1}{e^z + \xi} \cdot t$. Therefore, for $t = 1$ there exists an $R \in \left(\frac{1}{e^z}, \frac{1}{e^z-1}\right)$ such that $\ln(e^x - 1) = x - R$.

Since $\ln(e^x - e^y) = \ln(e^y \cdot (e^{x-y} - 1)) = y + \ln(e^{x-y} - 1)$ and $x - y > 0$ there exists an $R \in \left(\frac{1}{e^{x-y}}, \frac{1}{e^{x-y}-1}\right)$ such that $\ln(e^x - e^y) = y + (x - y) - R = x - R$. ∎

We can now use this approximation to prove our actual theorem with the transformer $\Psi = \Phi \circ$ Scale-Ball$_2$ $\circ$ Push-Apart. As a reminder Scale-Ball$_2$ scales all balls by the factor 2. In conjunction with Push-Apart this results in a representation where non-intersecting balls have at least a distance of 2 in between them. We discuss our choice of the factor 2 at the end.

Proof of Theorem 6.7. We define the transformer $\Psi = \Phi \circ$ Scale-Ball$_2$ $\circ$ Push-Apart where $\Phi_p(p, r) = e^p$ and $\Phi_w(p, r) = e^{\frac{p+r}{2}}$. In the following we show Ψ to indeed be a transformer. Let $(p_v, r_v)_{v \in V}$ be an arbitrary 1-dimensional ball representation and consider two vertices $u \neq v \in V$. Define $(\tilde{p}_v, \tilde{r}_v)_{v \in V}$ = Scale-Ball$_2$ $\circ$ Push-Apart$((p_v, r_v)_{v \in V})$. We have to prove

[3] *https://gitlab.kit.edu/udvin/reconstruction-of-1-dimensional-ball-graphs*

that $\left\|\Phi_p(\tilde{p}_u,\tilde{r}_u)-\Phi_p(\tilde{p}_u,\tilde{r}_u)\right\| \leq \Phi_w(\tilde{p}_u,\tilde{r}_u)\cdot\Phi_w(\tilde{p}_u,\tilde{r}_u)$ if and only if $\|\tilde{p}_u-\tilde{p}_v\| \leq \tilde{r}_u+\tilde{r}_v$. Since we can only apply Lemma 6.8 when $\tilde{p}_u \neq \tilde{p}_v$ we have to consider the case where they are equal separately.

Case $p_u = p_v$: Then u and v are adjacent, and it holds $\tilde{p}_u = \tilde{p}_v$ and $\Phi_p(\tilde{p}_u,\tilde{r}_u)=\Phi_p(\tilde{p}_v,\tilde{r}_v)$. Therefore, $\left\|\Phi_p(\tilde{p}_u,\tilde{r}_u)-\Phi_p(\tilde{p}_v,\tilde{r}_v)\right\| = 0 \leq \Phi_w(\tilde{p}_u,\tilde{r}_u)\cdot\Phi_w(\tilde{p}_v,\tilde{r}_v)$ or in other words the weighted points $\Phi(p_u,r_u)\in\mathbb{W}$ and $\Phi(p_v,r_v)\in\mathbb{W}$ intersect.

Case $p_u \neq p_v$: W.l.o.g. $p_u > p_v$ and therefore $\tilde{p}_u > \tilde{p}_v$ and $e^{\tilde{p}_u} > e^{\tilde{p}_v}$. Consider the following equivalent statements.

$$\begin{aligned}
\left\|\Phi_p(\tilde{p}_u,\tilde{r}_u)-\Phi_p(\tilde{p}_v,\tilde{r}_v)\right\| = e^{\tilde{p}_u}-e^{\tilde{p}_v} \quad &\leq \Phi_w(\tilde{p}_u,\tilde{r}_u)\cdot\Phi_w(\tilde{p}_v,\tilde{r}_v) = e^{\frac{\tilde{p}_u+\tilde{r}_u}{2}}\cdot e^{\frac{\tilde{p}_v+\tilde{r}_v}{2}}\\
\iff \ln(e^{\tilde{p}_u}-e^{\tilde{p}_v}) \quad &\leq \frac{1}{2}(\tilde{p}_u+\tilde{p}_v+\tilde{r}_u+\tilde{r}_v)
\end{aligned}$$

By Lemma 6.8 there exists an $R \in \left(\frac{1}{e^{\tilde{p}_u-\tilde{p}_v}}, \frac{1}{e^{\tilde{p}_u-\tilde{p}_v}-1}\right)$ such that $\ln(e^{\tilde{p}_u-\tilde{p}_v}) = \tilde{p}_u - R$. We continue with rearranging the equations.

$$\begin{aligned}
\tilde{p}_u - R &\leq \frac{1}{2}(\tilde{p}_u+\tilde{p}_v+\tilde{r}_u+\tilde{r}_v)\\
\iff \frac{1}{2}(\tilde{p}_u-\tilde{p}_v) - R &\leq \frac{1}{2}(\tilde{r}_u+\tilde{r}_v)\\
\iff \tilde{p}_u-\tilde{p}_v-2R &\leq \tilde{r}_u+\tilde{r}_v\\
\iff \|\tilde{p}_u-\tilde{p}_v\| - r_u - r_v &\leq 2R
\end{aligned}$$

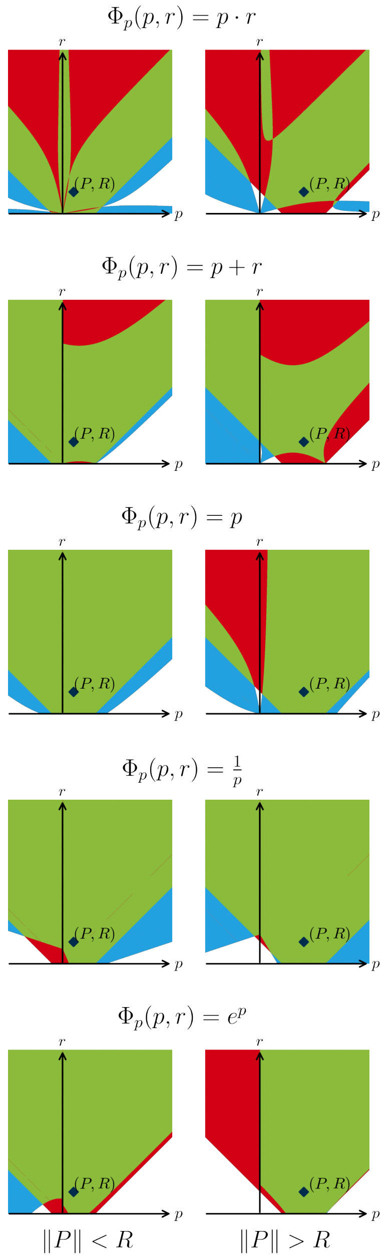
If both balls intersect, i.e., $\|p_u-p_v\| \leq r_u+r_v$ then $\|\tilde{p}_u-\tilde{p}_v\| - 2\cdot R \leq \|\tilde{p}_u-\tilde{p}_v\| \leq \tilde{r}_u+\tilde{r}_v$.

Otherwise, if they do not intersect it holds $\|p_u-p_v\| > r_u+r_v$. Since we are using Scale-Ball$_2$ $\circ$ Push-Apart it follows that $\|\tilde{p}_u-\tilde{p}_v\| \geq \|\tilde{p}_u-\tilde{p}_v\| - \tilde{r}_u - \tilde{r}_v \geq 2$, i.e., the distance in between both balls is at least 2. Since

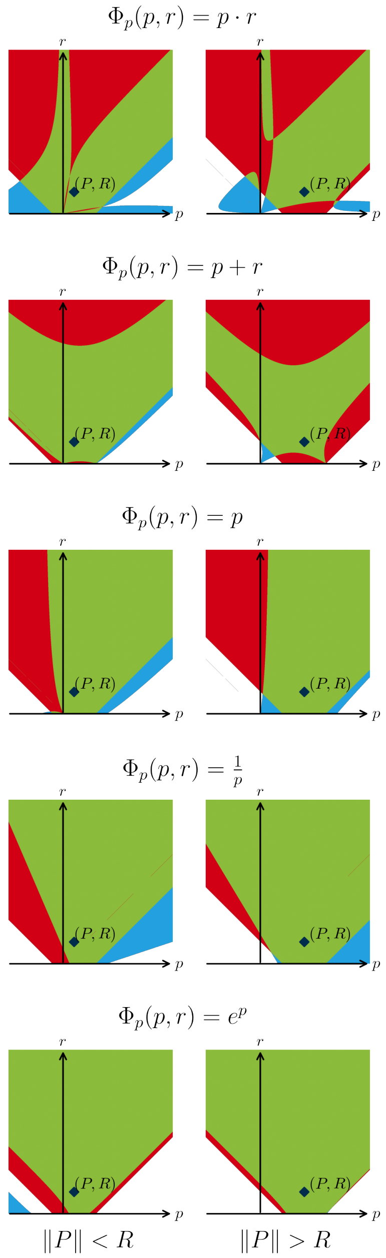
$$2R < \frac{2}{e^{\tilde{p}_u-\tilde{p}_v}-1} < \frac{2}{e^2-1} \leq 2$$

it follows that $\tilde{p}_u-\tilde{p}_v-\tilde{r}_u-\tilde{r}_v > 2R$ or equivalently the weighted points $\Phi(\tilde{p}_u,\tilde{r}_u)$ and $\Phi(\tilde{p}_v,\tilde{r}_v)$ do not intersect. ∎

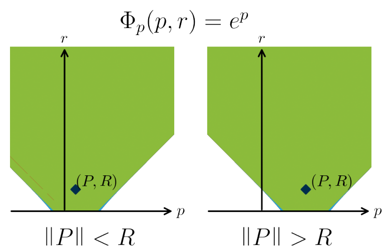
Obviously we can easily combine Φ with Scale-Ball$_2$ into one formula. We refrained from doing so since we wanted to be explicit that not all constant factors for scaling are valid here. This argument only works if we scale by λ such that $\frac{2}{e^\lambda-1} \leq \lambda$, which happens for $\lambda \geq 1.0601$.



(a) Reconstruction Φ_w according to Lemma 6.5.



(b) Reconstruction of Φ_w using our "fix".



(c) Reconstruction of Φ_w using our "fix" and ignoring the unnecessary -1.

Figure 6.5: Multiple variations of Φ_p with a reconstructed Φ_p. The green regions correspond to where points are both in the transformed adjacency region and the ball adjacency region. The points in red are points that are only inside the ball adjacency region. We color points in cyan if they are only in the transformed adjacency region.

7 Conclusion

We have introduced the models weighted, ball graphs and grounded ball graphs and managed to compare them quite definitively. We have shown the following relationships between the respective dimensions for every graph G:

$$\begin{aligned} \dim_{\mathcal{B}}(G) &\leq \dim_{\mathcal{W}}(G) + 1 \\ \dim_{\mathcal{GB}}(G) > 1 &\implies \dim_{\mathcal{GB}}(G) = \dim_{\mathcal{W}}(G) + 1 \\ \dim_{\mathcal{B}}(G) = 1 &\implies \dim_{\mathcal{W}}(G) = 1 \end{aligned}$$

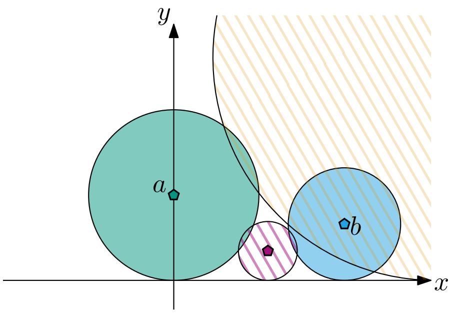
While proving the first two relationships, we gave a strong weight-grounded ball transformer. This allowed us to establish a different visualization of weighted graphs as the intersection of grounded balls. We were able to construct a 1-to-1 Ball-Weight transformer, but were also able to show that this transformer can never be a strong transformer with a continuous underlying transformer. We established that the weighted dimension can in fact be 1 less than the ball dimension in the example of the circle $C_4 = K_{2,2}$, which we used to conclude that 1-dimensional ball graphs are a proper subset of 1-dimensional weighted graphs. In general, we were able to show that regardless of the dimension d the weighted graphs are never a superset of (≥ 2)-dimensional ball graphs, but always a subset of $(d + 1)$-dimensional ball graphs. Furthermore, we found two new lower bounds for the ball and weight dimensions of families of graphs. By our analysis it holds that $\dim_{\mathcal{W}}(K_{2,n}), \dim_{\mathcal{GB}}(K_{2,n}) \in \Omega(\log(n))$ and also $\dim_{\mathcal{B}}(K_{3,n}) \in \Omega(\log(n))$. In both of these cases this lead us to show that for all $k \in \mathbb{N}^+$ there exists a graph G such that $\dim_{\mathcal{W}}(G - v) + k < \dim_{\mathcal{W}}(G)$ and similarly that $\dim_{\mathcal{B}}(G - v) + k < \dim_{\mathcal{B}}(G)$ or in other words that the dimension can increase by an arbitrary amount when adding a vertex.

7.1 Future Work

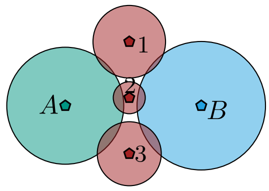
Last but not least we want to highlight areas of future research.

Tight Bounds. While we have shown that $\dim_{\mathcal{W}}(K_{2,n})$ and $\dim_{\mathcal{B}}(K_{3,n})$ grows at least logarithmically, we would like to know how tight this bound is asymptotically. At least for small n we are certain that our exact lower bound is not tight. We give a (very) rough sketch of the argument for $d = 2$. Our calculations would say that $K_{2,n}$ can be a 2-dimensional grounded ball graph for at most $n \leq 6^2 = 36$. But looking at the 2-dimensional case it quickly becomes apparent that we can find a representation for at most $n = 2$. This is due to the fact that at most one grounded ball can lie between two grounded balls and only one extra one can lie outside while touching both, see Figure 7.1a.

We also believe that $\dim_{\mathcal{B}}(K_{3,3}) > 2$, since intuitively when placing three balls between p_A and p_B it forces the third ball p_C to be too large and intersect with p_A or p_B, see Figure 7.1b.



(a) There can be at most two disjoint grounded balls that intersect with two disjoint grounded balls. We can at most have one ball between both centers (when ignoring the last coordinate). Also, at most one ball can lie not between both centers. This is due to the fact that its radius becomes really large.



(b) We can find a 2-dimensional ball representation of the $K_{2,3}$ but if we try to put a third ball C that intersects with 1,2 and 3 we were only able to find examples where this new ball intersects with A or B as well.

Figure 7.1: Intuitions for our belief that the given lower bounds of $\dim_{\mathcal{W}}(K_{2,n})$ and $\dim_{\mathcal{B}}(K_{3,n})$ are not tight for small n.

Further subset relationships. As seen in Figure 1.2 there are some inclusion relationships we do not know yet. We would like to know whether d-dimensional weighted graphs are a subset of d'-dimensional ball graphs for $d' \leq d$. For $d = 1$ this has been shown not to be the case, but we do not know anything about other dimensions.

Random Graphs Random analysis differs from a deterministic analysis. For example while the choice of norms are relevant in the deterministic case, they do not matter in the random case [Mae86 | al19]. While random weighted graphs have been considered, i.e., GIRGs[1], random ball graphs have not been considered as much. We would like to know how both models relate in the random case.

[1] The definition of GIRGs involves more technicalities than our definition and as such is not directly the comparable.

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