Degree Independent Weights for Weighted Embeddings

Bachelor's Thesis of

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I declare that I have developed and written the enclosed thesis completely by myself. I have not used any other than the aids that I have mentioned. I have marked all parts of the thesis that I have included from referenced literature, either in their original wording or paraphrasing their contents. I have followed the by-laws to implement scientific integrity at KIT.

Karlsruhe, March 2025

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Abstract

Geometric representations of discrete structures, such as graphs, can be useful for many information processing tasks. In particular, hyperbolic and weighted graph embeddings have gained significant attention in recent years, as they allow particularly good representations of scale-free networks. A *weighted embedding* of a graph assigns each vertex a position in $\mathbb{R}^d$ and a weight in $\mathbb{R}_+$. We require that for any two vertices, they are adjacent iff. they are assigned similar positions, where the required similarity of positions decreases as the weights of the vertices increase.

While weights are usually assigned based on degree, this approach fails to embed complete trees of sufficient height. Thus, we examine weight assignments based on more nuanced centrality measures. For that reason, we consider sufficient and necessary conditions of weights for high-quality weighted embeddings, in particular for trees and grids. Based on those conditions, we propose choosing weights according to the k-hop centrality of a vertex. We show that such weight assignments yield high-quality embeddings of complete trees. Furthermore, we show that k-hop based weights perform comparably to degree-based weights on grids. However, we also observe shortcomings of k-hop based weights on caterpillars, some induced subgraphs of grids and random geometric graphs.

Zusammenfassung

Geometrische Repräsentationen diskreter Strukturen, wie beispielsweise von Graphen, können zur Informationsverarbeitung nützlich sein. Insbesondere hyperbolische und gewichtete Einbettungen von Graphen haben in letzter Zeit erhebliche Aufmerksamkeit erhalten, da diese gute Repräsentationen, insbesondere für skalenfreie Netzwerke, ermöglichen. Eine *gewichtete Einbettung* ordnet jedem Knoten eine Position in $\mathbb{R}^d$ sowie ein Gewicht in $\mathbb{R}_+$ zu. Dabei fordern wir, dass zwei Knoten genau dann adjazent sind, wenn sie ähnliche Positionen haben, wobei die erforderliche Ähnlichkeit der Positionen umso geringer sein darf, je größer die Gewichte der Knoten sind.

Während Gewichte üblicherweise basierend auf dem Knotengrad zugewiesen werden, scheitert dieser Ansatz daran, vollständige Bäume mit hinreichend großer Tiefe einzubetten. Daher untersuchen wir Gewichtszuweisungen, die auf differenzierteren Zentralitätsmaßen basieren. Zu diesem Zweck betrachten wir hinreichende und notwendige Kriterien von gewichteten Einbettungen von Bäumen und Gittern. Ausgehend von diesen Kriterien schlagen wir vor, Gewichte gemäß der k-Hop-Zentralität eines Knotens zu wählen. Wir zeigen, dass diese Gewichtszuweisung hochwertige Einbettungen von vollständigen Bäumen ermöglicht. Zudem zeigen wir, dass Gewichte auf Basis der k-Hop-Zentralität und des Knotengrads Einbettungen in vergleichbarer Qualität ermöglichen. Allerdings beobachten wir auch Schwächen der vorgeschlagenen Gewichtszuweisung bei allgemeinen Bäumen, induzierten Teilgraphen von Gittern und zufälligen geometrischen Graphen.

Contents

1. Introduction

Graphs are a fundamental tool for modeling discrete relationships in real-world systems, providing a natural representation for networks, social interactions, and various structured data. Meanwhile, many information processing tasks, as for instance many machine learning tasks, can only process continues information. This highlights the need for a geometric representation of graphs, which can close the gap between discrete and continues data.

A popular way of solving this problem is using embeddings of graphs. A d-dimensional *Euclidean embedding* of a graph assigns each vertex v of a graph, a position $p_v \in \mathbb{R}^d$ in Euclidean space such that two vertices have similar positions if and only if they are adjacent [GF18]. In Euclidean space two points $x, y \in \mathbb{R}^d$ can be considered similar, if they have a Euclidean distance $\|x, y\| \leq 1$. By that definition, a suitable (Euclidean) embedding of a graph G requires that

$$\|p_u - p_v\| \leq 1 \iff \{u, v\} \in E(G) \tag{1.1}$$

for as many vertices $u, v \in V(G)$ as possible.

The existence of such embeddings that satisfy this condition for all pairs of vertices $u, v \in V(G)$ has already been studied under the term sphericity [Mae84] [Fis83] and graph dimension [RRS89]. Those concept concern the smallest dimension sufficient to represent the graph perfectly. Graphs with sphericity 2 are also referred to as unit disk graphs and are well-studied class in literature [CCJ90] [HHZ24] [An+24].

Maehara showed that it is always possible to find suitable embeddings of arbitrary graphs in n dimensions [Mae84]. However, as n can become very big for a lot of real-world networks, an embedding in such a high-dimensional space becomes unpractical. Furthermore, high-dimensional embeddings have a high risk of overfitting. Hence, finding low-dimensional embeddings of graphs is an important task.

Another important motivation for finding low-dimensional embeddings of a graph is their potential to accelerate algorithms for classical graph problems. For instance, given a 2-dimensional embedding that satisfies Condition 1.1 for all vertices u, v, the maximum clique can be found in polynomial time, whereas this problem remains NP-hard on general graphs [CCJ90].

However, not all graphs are best represented by Euclidean geometry. Findings from the network science community indicate that scale-free graphs, i.e., graphs with a degree distribution that follows a power law, are better represented by hyperbolic geometry [PKBV10]. This is particularly relevant, as most real-world graphs, for example the internet graph [BPK10], tend to be scale-free.

Hyperbolic embeddings are an alternative to Euclidean embeddings. Hyperbolic embedders tend to yield low-dimensional embeddings of scale-free graphs [NK17] [BFKL18]. However, the task of finding hyperbolic embeddings that outperform Euclidean embeddings poses some computational challenges [BHKM24] [Pen+22].

These problems partly arise, because shortest path between points in the hyperbolic space curve toward the origin and central vertices being generally embedded near the origin. Moreover, the mathematical complexity of hyperbolic space makes hyperbolic embeddings not very accessible.

Similar problems arise in the related topic of graph generation, where the efficient generation of hyperbolic random graphs (HRGs) is challenging. Here, these challenges were overcome by considering the alternative graph model of geometric inhomogeneous random graphs (GIRGs) [BKL19] [Blä+22]. In GIRGs the centrality of a vertex is explicitly given by a weight parameter w_v. It was shown that the generation of GIRGs is much easier and that HRGs and GIRGs are roughly equivalent.

The weighted geometry of GIRGs can also be used for embeddings: Similar to the Euclidean case, a d-dimensional *weighted embedding* of a graph G is an assignment of each vertex v to a position $p_v \in \mathbb{R}^d$ in Euclidean space and a positive weight $w_v \in \mathbb{R}_+$. We require

$$\frac{\|p_u - p_v\|}{(w_u w_v)^{1/d}} \leq 1 \iff \{u, v\} \in E(G),$$

for as many $u, v \in V(G)$ as possible.

As in the GIRG model, the weight w_v can be interpreted as the centrality of the vertex v. A natural way of choosing weights, based on the GIRG model, is to set the weight of a vertex based on it's degree. Indeed, existing embedders, such as WEmbed [BHKM24], choose weights in exactly that way. WEmbed computes w_v and p_v in two separate steps:

1. Set the weight of every vertex v to $w_v = (\deg v)^{d/8}$.

2. Calculate suitable positions p_v using gradient descent.

WEmbed reliably generates suitable embeddings on GIRGs and many real-world graphs.

However, even without knowing the details of the second step of the algorithm, it can be seen that WEmbed has at least one major flaw: In a complete binary tree, all vertices except the root and the leaves have a constant degree of 3. Thus, WEmbed assigns them the same weight. However, a weighted embedding with constant weights is equivalent to a Euclidean embedding and a complete binary tree can not be embedded into Euclidean space. This is due to the number of leafs of a tree growing exponentially, while the Euclidean space only grows polynomial. We state this argument in more depth in Section 3.1. This shows that WEmbed is not suitable for finding low-dimensional weighted embeddings of complete trees, which form a fundamental class of graphs. This is particularly interesting, as trees can be embedded trivially into hyperbolic space.

1.1. Contribution

In this thesis, we will fix this problem, by considering weight assignments based on more nuanced centrality measures as degree centrality. We will approach the problem from a theoretical perspective, i.e., we will not consider the technical details of computationally finding suitable positions p_v of a weighted embedding. Instead, we only consider the question of whether there exist such positions, given some assignment of weights.

Our contribution primarily consists of two parts.

First, we will introduce necessary and sufficient conditions for the weights of low-dimensional weighted embedding of high quality. In particular, we examine such conditions for 1-dimensional weighted embeddings of trees and 2-dimensional embeddings of grids. As a consequence, we show that there exist low-dimensional weighted embeddings of arbitrary trees, even though no such embeddings exist with weights based on degree centrality. These results are interesting on it's one, but are also useful for finding weaknesses of arbitrary weighted embedding algorithms

Second, we introduce concrete modifications to the weight assignment in the first step of WEmbed. Concretely, we consider assigning the weight of a vertex v based on it's *k-hop centrality* [NFWS14], i.e., the number of vertices with graph distance at most k to v, for some carefully chosen k. This alternative weight assignment outperforms degree based weights on trees and is on par with it on grids. We critically assess the quality of embeddings with k-hop based weights on more complex classes of graphs, namely arbitrary trees, induced subgraphs of grids and unit disk graphs.

1.2. Outline

The remainder of this work is structured as follows: In Chapter 2, we define the mathematical notions and concepts required. In Chapter 3, we analyze the existence of high-quality 1-dimensional weighted embeddings of trees with given weights. In Chapter 4, we introduce sufficient conditions of weights that yield weighted embeddings of high quality. In Chapter 5, we introduce and evaluate multiple alternative ways of assigning weights to vertices of complete trees and grids. We show that the k-hop centrality is suitable for that purpose. In Chapter 6, we assess the quality of this weight assignment on arbitrary trees, induced subgraphs of grids and unit disk graphs. Finally, Chapter 7 is a summary of our work and an outlook on possible future work.

2. Preliminary

In this chapter, we will discuss some fundamental prerequisites for working with weighted embeddings.

2.1. General Mathematical Notions

Sets and Numbers Let $\mathbb{R}$ denote the set of all *real numbers*, $\mathbb{R}_+$ the set of all positive real numbers and $\mathbb{R}_{\geq 0} \coloneqq \mathbb{R}_+ \cup \{0\}$. Furthermore, $\mathbb{Z}$ denotes the set of all *integers*, $\mathbb{N}_+$ the set of all positive integers and $\mathbb{N}_0 = \mathbb{N}_+ \cup \{0\}$. For a finite set A, $|A|$ denotes the cardinality of A. Let e, π be the usual mathematical constants and φ the golden ratio.

Geometry For each $d \in \mathbb{N}_+$, we consider the *d-dimensional (euclidean) space* $\mathbb{R}^d$ equipped with the standard norm $\|\cdot\|$. We represent the elements of $\mathbb{R}^d$ as row vectors $x = (x_1, x_2, \ldots, x_d)$. We define $\|x\|$ by

$$\|x\| \coloneqq \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}.$$

For $x, y \in \mathbb{R}^d$, we call $\|x - y\|$ the *(euclidean) distance* between x and y. For all $x, y, z \in \mathbb{R}^d$, the *triangle inequality*

$$\|x + y\| \leq \|x\| + \|y\|$$

and it's consequence

$$\|x - z\| \leq \|x - y\| + \|y - z\|$$

hold. For $d = 1$, $\|x\|$ is equal to the *absolute value* $|x|$ of x.

For $x \in \mathbb{R}^d$ and $r \in \mathbb{R}_{\geq 0}$, we define the *(euclidean) (d-dimensional) ball B* with radius r, centered at x by $B = \{y \in \mathbb{R}^d \mid \|x - y\| \leq r\}$. If $d = 2$, we call B a *disk*. We denote the *volume* of a set A by $\mathrm{Vol}(A)$ and consider the lemma about volumes of balls:

Lemma 2.1: *For any $d \in \mathbb{N}_+$ there exists a constant C_d such that for all $x \in \mathbb{R}^d$ and $r \in \mathbb{R}_{\geq 0}$, the volume of the d-dimensional ball B of radius r, centered at x has volume*

$$\mathrm{Vol}(B) = C_d r^d.$$

Furthermore, $C_1 = 2, C_2 = \pi$ and $C_3 = \frac{4}{3}\pi$.

Words Let A be as set of symbols. If $x_0, x_1, x_2, \ldots, x_{\ell-1} \in A$ are symbols, we call the finite sequence $v = x_0 x_1 x_2 \cdots x_{\ell-1}$ a *word* over the *alphabet* A and call $|v| \coloneqq \ell$ the *length* of the word v. The word of length 0 is called the *empty word* and is denoted by ε. For two words $u = x_0 x_1 \ldots x_{\ell-1}$ and $v = y_0 y_1 \ldots y_{r-1}$, we define the *concatenation uv* of u and v as $uv \coloneqq x_0 x_1 \ldots x_{\ell-1} y_0 y_1 \ldots y_{r-1}$. For any $k \in \mathbb{N}_0$, we define $A^{\leq k}$ to be the set of all words v over the alphabet A with $|v| \leq k$.

Big $\mathcal{O}$ Notation Let $f, g : \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$ such that the following limit exists:

$$q \coloneqq \lim_{k\to\infty} \frac{f(k)}{g(k)}$$

If $q = 0$, we write $f \in o(g)$. If $q \in (0, \infty)$, we write $f \in \Theta(g)$. If $q = \infty$, we write $f \in \omega(g)$. We define $\mathcal{O}(g) \coloneqq o(g) \cup \Theta(g)$ and $\Omega(g) \coloneqq \omega(g) \cup \Theta(g)$. We use this definitions not only for functions f, g, but also sequences of numbers and variables that depend on each other. In some situations, we will write $f = o(g)$ instead of $f \in o(g)$ for convenience.

Probability Theory For any events A, B and random variables X, Y, we use the notions $\mathbb{P}(A)$, $\neg A$, $\mathbb{E}[X]$, $\mathrm{Var}(X)$, $\mathrm{Cov}(X, Y)$, $\mathbb{1}_{\{A\}}$, $\mathbb{P}(A|B)$ and $\mathbb{E}[X|Y]$ as usual. We note that $\mathbb{E}\left[\mathbb{1}_{\{A\}}\right] = \mathbb{P}(A)$ and $\mathrm{Var}\left(\mathbb{1}_{\{A\}}\right) = \mathbb{P}(A) - \mathbb{P}(A)^2$. Furthermore, for any $\lambda > 0$, $\mathrm{Var}(\lambda X) = \lambda^2 \mathrm{Var}(X)$. If A_n is an event for each $n \in \mathbb{N}_+$, then we say that A_n occurs *with high probability* (w.h.p.), if $\mathbb{P}(\neg A_n) \in o(1)$.

2.2. Graph Theory

All graphs in this work are *finite, simple, undirected* and *unweighted*. For a graph G, we denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. The number of vertices of a graph is always denoted by n and the number of edges by m.

An edge is an unordered pair $\{u, v\}$ of vertices $u, v \in V(G)$. If $\{u, v\} \in E(G)$, we say that u and v are *adjacent* or that u is a *neighbor* of v. The number of neighbors of a fixed vertex v is called the *degree* of v and denoted by $\deg v$ or $\deg_G v$. A *path* of length ℓ from v to v' is a sequence of vertices $(v = u_0, u_1, \ldots, u_\ell)$, where u_i is adjacent to u_{i+1} for all $i \in \{0, \ldots, \ell - 1\}$. The path of smallest length from u to v is called the *distance* between u and v and is denoted by $\mathrm{dist}_G(u, v)$. If no path from u to v exists, we set $\mathrm{dist}_G(u, v) \coloneqq \infty$. If vertices $u, v \in V(G)$ exist with $\mathrm{dist}_G(u, v) = \infty$, we call G *disconnected* and otherwise *connected*. Connected graphs $H_1, H_2, \ldots, H_k$ are called the *connected components* of G, if $V(G) = \bigcup_{i=1}^k V(H_i)$ and $E(G) = \bigcup_{i=1}^k E(H_i)$ and k is minimal. The *diameter* of G is defined as $\operatorname{diam} G \coloneqq \max_{u,v\in V(G)} \mathrm{dist}_G(u, v)$. The *radius* of G is defined as $r(G) \coloneqq \min_{v\in V(G)} \max_{v\in V(G)} \mathrm{dist}_G(u, v)$. A set I of vertices is called an *independent set* of G, if for all $u, v \in I$, $\{u, v\} \notin E(G)$.

A graph H is called a *subgraph* of a graph G, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph H is called an *induced subgraph* of G, if H is a subgraph of G and $E(H) = \{\{u, v\} \in E(G) \mid u, v \in V(H)\}$.

Two graphs H, G are called *isomorphic*, if there exists a mapping $f : V(G) \to V(H)$ such that for all $u, v \in V(G)$,

$$\{u, v\} \in E(G) \iff \{f(u), f(v)\} \in E(H).$$

If this is the case, we write $G \cong H$.

Paths and Cycles For $k \in \mathbb{N}_+$, we define the *path graph* P_k as the graph with $V(P_k) \coloneqq \{0, 1, \ldots, k - 1\}$ and $E(P_k) \coloneqq \{\{0, 1\}, \{1, 2\}, \ldots, \{k - 2, k - 1\}\}$. The graph C_k defined by $V(C_k) \coloneqq V(P_k)$ and $E(C_k) \coloneqq E(P_k) \cup \{\{k - 1, 0\}\}$ is called a *cycle graph*.s

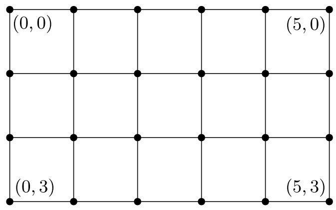


Figure 2.1.: Illustration of the grid $\Gamma_{6,4}$.

Grids For $a, b \in \mathbb{N}_+$, we call the graph $\Gamma_{a,b}$ defined by $V(\Gamma_{a,b}) = \{0, 1, \ldots, a-1\} \times \{0, 1, \ldots, b-1\}$ and

$$E(\Gamma_{a,b}) = \{\{(x, y), (i, j)\} \mid (x, y), (i, j) \in V(\Gamma_{a,b}), |i - x| + |j - y| = 1\}$$

a *grid*. See Figure 2.1 for an illustration. An induced subgraph G of $\Gamma_{a,b}$ is called *grid graph*.

Trees For $b \in \mathbb{N}_+$ and $h \in \mathbb{N}_0$, the *complete b-ary tree of height h* T_h^b is defined by $V(T_h^b) := \{0, 1, \ldots, b-1\}^{\leq h}$ and

$$E(T_h^b) := \{\{v, vx\} \mid v \in \{0, 1, \ldots, b-1\}^{\leq h-1}, x \in \{0, 1, \ldots, b-1\}\}.$$

We note that for any $v \in V(T_h^b)$, $|v| = \mathrm{dist}_G(\varepsilon, v)$. We call $|v|$ the *layer* of v. For each $v \in V(T_h^b)$, there exists a unique path p of length $|v|$ from ε to v. A vertex u is called an *ancestor* of v, if u is contained in p. An ancestor u of v with $\mathrm{dist}_G(u, v) = 1$ is called the *parent* of v. If u is an ancestor of v, then we call v a *descendant* of u. If u is a parent of v, then we call v a *child* of u.

All vertices of T_h^b have degree at most $b + 1$. A vertex v with degree 1 (i.e. $|v| = h$) is called a *leaf* of T_h^b. The set of all leaves of T_h^b is an independent set of size b^h. The vertex ε has degree b and is called the *root* of T_h^b. For a vertex v and $h' \in \mathbb{N}_0$, the *subtree* of height h' rooted in vertex v is defined as the subgraph T of T_h^b induced by $V(T) = \{u \in V(T_h^b) \mid u \text{ is a descendant of } v, \mathrm{dist}_G(u, v) \leq h'\}$.

If $b = 2$, we call T_h^b a *complete binary tree*. If $h = 1$, we call $S_b := T_1^b$ the *b-star*. A connected induced subgraph of T_h^b for any h and b is called a *tree*.

Note that some authors use the term ‘complete tree’ differently, referring to a tree in which all levels *except* possibly *the last* are fully populated and the last level is filled from left to right.

2.3. Weighted Embeddings

A d-dimensional weighted embedding of a graph G is an assignment of each vertex $v \in V(G)$ to a position $p_v \in \mathbb{R}^d$ and a weight $w_v \in \mathbb{R}_+$. More formally, for $d \in \mathbb{N}_0$, we call $\psi = (p_v, w_v)_{v \in V(G)}$ a *d-dimensional weighted embedding* of G if $p_v \in \mathbb{R}^d$ and $w_v \in \mathbb{R}_+$ for all $v \in V(G)$. For any $v \in V(G)$, we write $\psi_v := (p_v, w_v)$. We say that the embedding ψ is *perfect* if, for all $u, v \in V(G)$,

$$\frac{\|p_u - p_v\|}{(w_u w_v)^{1/d}} \leq 1 \iff \{u, v\} \in E(G).$$

For ease of notation, we define

$$\operatorname{dist}: (\mathbb{R}^d \times \mathbb{R}_+) \times (\mathbb{R}^d \times \mathbb{R}_+), ((p_1, w_1), (p_2, w_2)) \mapsto \frac{\|p_1 - p_2\|}{(w_1 w_2)^d}.$$

With that notion, a weighted embedding ψ is perfect, if and only if, for all $u, v \in V(G)$,

$$\operatorname{dist}(\psi_u, \psi_v) \leq 1 \iff \{u, v\} \in E(G). \tag{2.1}$$

We note that intuitively speaking, $\operatorname{dist}(\psi_u, \psi_v)$ can be interpreted as a weighted distance between ψ_u and ψ_v: However, dist is *not* a metric in the mathematical sense, as it does not satisfy the triangle-inequality (and is not positive definite): For instance consider $x_1 = (p_1, w_1) = (-1, 1)$, $x_2 = (p_2, w_2) = (0, 2)$ and $x_3 = (p_3, w_3) = (1, 1)$, then

$$\operatorname{dist}(x_1, x_2) + \operatorname{dist}(x_2, x_3) = \frac{|(-1) - 0|}{1 \cdot 2} + \frac{|0 - 1|}{2 \cdot 1} = 1 < 2 = \frac{|(-1) - 1|}{1 \cdot 1} = \operatorname{dist}(x_1, x_3).$$

We also note that dist must not be confused with the graph distance dist_G.

Like already mentioned in the introduction, we consider it desirable for an embedding ψ to be perfect. However, sometimes it is only possible to construct an embedding that is 'almost perfect', meaning that Condition 2.1 is only true for most, but not all, $u, v \in V(G)$. To formalize this idea, we define

$$E(\psi) \coloneqq \{\{u, v\} \mid u, v \in V(G), \operatorname{dist}(\psi_u, \psi_v) \leq 1\}$$

and call ψ an *embedding with error* $(s_{\text{fn}}, s_{\text{fp}})$ of G, if $s_{\text{fn}} = |E(G) \setminus E(\psi)|$ and $s_{\text{fp}} = |E(\psi) \setminus E(G)|$. Thus, s_{fn} counts the number of false negative edges and s_{fp} the number of false positive edges. If we don't want to distinguish between those two types of error, we just say that ψ has a *(total) error* of $s_{\text{fn}} + s_{\text{fp}}$. Note that ψ is perfect, if and only if, ψ is an embedding with total error 0.

In most parts of this thesis, we are not just interested in an arbitrary perfect embedding or arbitrary embedding with low total error, but an embedding that uses some given weights: For an assignment of vertices to weights $(w'_v)_{v \in V(G)}$, we say that $\psi = (p_v, w_v)_{v \in V(G)}$ is an *embedding with weights* $(w'_v)_{v \in V(G)}$, if $w_v = w'_v$ for all $v \in V(G)$. In that case, we also say that $(w'_v)_{v \in V(G)}$ *yields* the embedding ψ. If $w_v = 1$ for all $v \in V(G)$ then we say that $\psi = (p_v, w_v)_{v \in V(G)}$ is an embedding with *unit weights*.

For multiple results in this thesis, the minimal and maximal weight in an embedding will be relevant. Thus, we define the abbreviations

$$w_{\max}(\psi) = \max_{v \in V(G)} w_v \qquad \text{and} \qquad w_{\min}(\psi) = \min_{v \in V(G)} w_v$$

for an embedding $\psi = (p_v, w_v)_{v \in V(G)}$.

One way to make the geometry of dist intuitively more approachable is to consider the *weighted ball*

$$B_r(x_1) \coloneqq \{x_2 = (p_2, w_2) \in \mathbb{R}^d \times \mathbb{R}_+ \mid \operatorname{dist}(x_1, x_2) \leq r\}$$

for all $x_1 = (p_1, w_1) \in \mathbb{R}^d \times \mathbb{R}_+$ and $r \in \mathbb{R}_{\geq 0}$. This becomes especially illustrative if $d = 1$, as we can replace $\|\cdot\|$ with $|\cdot|$ and observe that

$$\begin{aligned}
\operatorname{dist}(x_1, x_2) \leq r &\iff \frac{|p_1 - p_2|}{w_1 w_2} \leq r \\
&\iff |p_1 - p_2| \leq r w_1 w_2 \\
&\iff p_1 - p_2 \leq r w_1 w_2 \wedge p_1 - p_2 \geq -r w_1 w_2 \\
&\iff p_1 - r w_1 w_2 \leq p_2 \wedge p_2 \leq p_1 + r w_1 w_2.
\end{aligned}$$

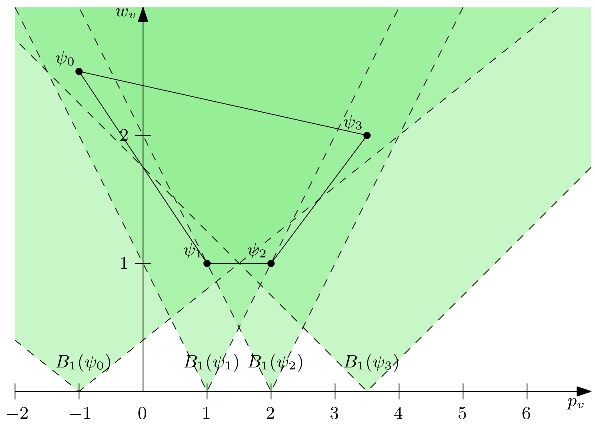


Figure 2.2.: Illustration of the embedding $\psi = (p_v, w_v)_{v \in V(C_4)}$ define by $p_0 = -1$, $w_0 = 2.5$, $p_1 = 1$, $w_1 = 1$, $p_1 = 2$, $w_0 = 1$ and $p_3 = 3.5$, $w_3 = 2$ and the balls $B_1(\psi_v)$ for all $v \in \{0, 1, 2, 3\}$. We observe that the balls $B_1(\psi_v)$ are cone-shaped. The lower w_v, the steeper is the slope of the boundary of $B_1(\psi_v)$. We note that ψ is perfect, as we can verify that for all different vertices $u, v \in \{0, 1, 2, 3\}$, the equivalence $\psi_u \in B_1(\psi_v) \iff \{u, v\} \in E(C_4)$ holds.

Thus:

Lemma 2.2: *For all $r \in \mathbb{R}_{\geq 0}$ and $x_1 = (p_1, w_1) \in \mathbb{R}^d \times \mathbb{R}_+$,*

$$B_r(x_1) = \{x_2 = (p_2, w_2) \in \mathbb{R} \times \mathbb{R}_+ \mid p_1 - rw_1w_2 \leq p_2 \wedge p_2 \leq p_1 + rw_1w_2\}$$

holds.

Figure 2.2 illustrates some weighted balls $B_r(x_1)$ for $d = 1$ in the half plane $\mathbb{R} \times \mathbb{R}_+$. By that illustration or Lemma 2.2, we observe that $B_r(x_1)$ is a cone with apex $(p_1, 0)$ and a boundary with slope rw_1w_2.

We will now discuss some general claims about weighted embeddings. First, we note that if $\psi = (p_v, w_v)_{v \in V(G)}$ is a perfect (d-dimensional) weighted embedding of a graph G, then for every $\lambda > 0$, the embedding $\psi' = (p'_v, w'_v)_{v \in V(G)}$ with $w'_v = \lambda w_v$ and $p'_v = \lambda^{2d} p_v$ for all $v \in V(G)$ is perfect too. This can be confirmed by verifying

$$\operatorname{dist}(\psi'_u, \psi'_v) = \frac{\left\|\lambda^{2d} p_u - \lambda^{2d} p_v\right\|}{(\lambda w_u \lambda w_v)^d} = \frac{\lambda^{2d} \left\|p_u - p_v\right\|}{\lambda^{2d} (w_u w_v)^d} = \operatorname{dist}(\psi_u, \psi_v),$$

for all $u, v \in V(G)$. Thus, if $(w_v)_{v \in V(G)}$ yields a perfect embedding, so does $(\lambda w_v)_{v \in V(G)}$.

Similarly, we observe that if $x \in \mathbb{R}^d$ and $\psi = (p_v, w_v)_{v \in V(G)}$ is a perfect d-dimensional weighted embedding of G, then so is $\psi' = (p'_v, w_v)_{v \in V(G)}$ with $p'_v = p_v + x$ for all $v \in V(G)$. Thus, the positions in an embedding can be translated arbitrarily without loosing it's perfect property.

Second, let G be a disconnected graph, with connected components $H_0, H_1, \ldots, H_{k-1}$. If there exist embeddings of those connected components with some error, then there exists an embedding ψ of G such that the sum of the errors of the embeddings of $H_0, \ldots, H_{k-1}$ is equal to the error of ψ. For that reason, we will only focus connected graphs in this work.

A consequence of this remark is that if there exists an embedding with error $(s_{\mathrm{fn}}, s_{\mathrm{fp}})$ of any induced subgraph H of G, then there exists an embedding of G with error

$$\left(s_{\mathrm{fn}} + \sum_{v \in V(G) \setminus V(H)} \deg_G v, s_{\mathrm{fp}}\right).$$

(Note that each vertex in $V(G) \setminus V(H)$ can be considered as a subgraph of G with one vertex.)

Third, if H is an induced subgraph of a graph G and there exists an embedding of G with error $(s_{\mathrm{fn}}, s_{\mathrm{fp}})$, then there exists an embedding of H with error $(s'_{\mathrm{fn}}, s'_{\mathrm{fp}})$ such that $s'_{\mathrm{fn}} \leq s_{\mathrm{fn}}$ and $s'_{\mathrm{fp}} \leq s_{\mathrm{fp}}$.

Last, for all $d, d' \in \mathbb{N}_+$ with $d' > d$, if $(w_v)_{v \in V(G)}$ yields a d-dimensional weighted embedding ψ of G with error $(s_{\mathrm{fn}}, s_{\mathrm{fp}})$, then $\left(w_v^{\frac{d'}{d}}\right)_{v \in V(G)}$ yields a d'-dimensional weighted embedding ψ' with error $(s_{\mathrm{fn}}, s_{\mathrm{fp}})$.

3. Embeddings of Trees

Trees are a fundamental class of graphs. However, the embedder WEmbed is not able to find embeddings (with low error) of them. In this chapter, we will find sufficient conditions and necessary conditions on the weights $(w_v)_{v \in V(T)}$ of any perfect embedding of a complete b-ary tree. In particular, we will show that perfect 1-dimensional weighted embeddings of arbitrary trees exist.

3.1. Necessary Condition

We claimed multiple times already that it is impossible to find a perfect d-dimensional weighted embedding with unit weights of a complete binary tree T_h^2 for all $d \in \mathbb{N}_+$ and sufficiently large height. This claim can be easily verified in 2 dimensions. Consider the binary tree T_h^2. Assume there exists a perfect 2-dimensional weighted embedding of this graph. For each leaf v of T_h^b, consider a disk of radius $\frac{1}{2}$ around the position p_v. By the definition of perfect embeddings, those disks do not intersect and the centers of the disks have distance at most h to the position p_ε of the root ε. As there are 2^h leafs, the union of the disks has area $2^h \pi \left(\frac{1}{2}\right)^2 \in \Theta(2^h)$, but the disks all fit into a bigger disk centered at p_ε with radius $h + \frac{1}{2}$ which has area $\pi(h+\frac{1}{2})^2 \in \Theta(h^2)$, a contradiction for sufficiently large h.

This argument can be generalized to a criterion about the quotient of the maximum and minimum weight in an arbitrary perfect d-dimensional weighted embedding of an arbitrary graph:

Theorem 3.1: *Let G be a graph, $r \in \mathbb{N}_0$ the radius of G, $I \subseteq V(G)$ an independent set of G and $\psi = (p_v, w_v)_{v \in V(G)}$ a perfect d-dimensional weighted embedding of G. Then $\frac{w_{max}(\psi)}{w_{min}(\psi)} > \sqrt{\frac{|I|}{(2r+1)^d}}$*

Proof. For two adjacent vertices $u, v \in V(G)$,

$$1 \geq \operatorname{dist}(\psi_u, \psi_v) = \frac{\|p_u - p_v\|}{(w_u w_v)^{1/d}} \geq \frac{\|p_u - p_v\|}{w_{\max}(\psi)^{2/d}},$$

holds, yielding $\|p_u - p_v\| \leq w_{\max}(\psi)^{2/d}$. Since r is the radius of G, there exists a vertex $c \in V(G)$ such that $\operatorname{dist}_G(c, v) \leq r$ for all $v \in V(G)$. It follows that

$$\|p_c - p_v\| \leq r \cdot w_{\max}(\psi)^{2/d} \tag{3.1}$$

for all $v \in V(G)$, as there exists a path of length at most r from c to v.

Now, let $u, v \in I$ be different vertices. u and v are not adjacent and thus

$$1 < \operatorname{dist}(\psi_u, \psi_v) = \frac{\|p_u - p_v\|}{(w_u w_v)^{1/d}} \leq \frac{\|p_u - p_v\|}{w_{\min}(\psi)^{2/d}}.$$

It follows that

$$\|p_u - p_v\| > w_{\min}(\psi)^{2/d}. \tag{3.2}$$

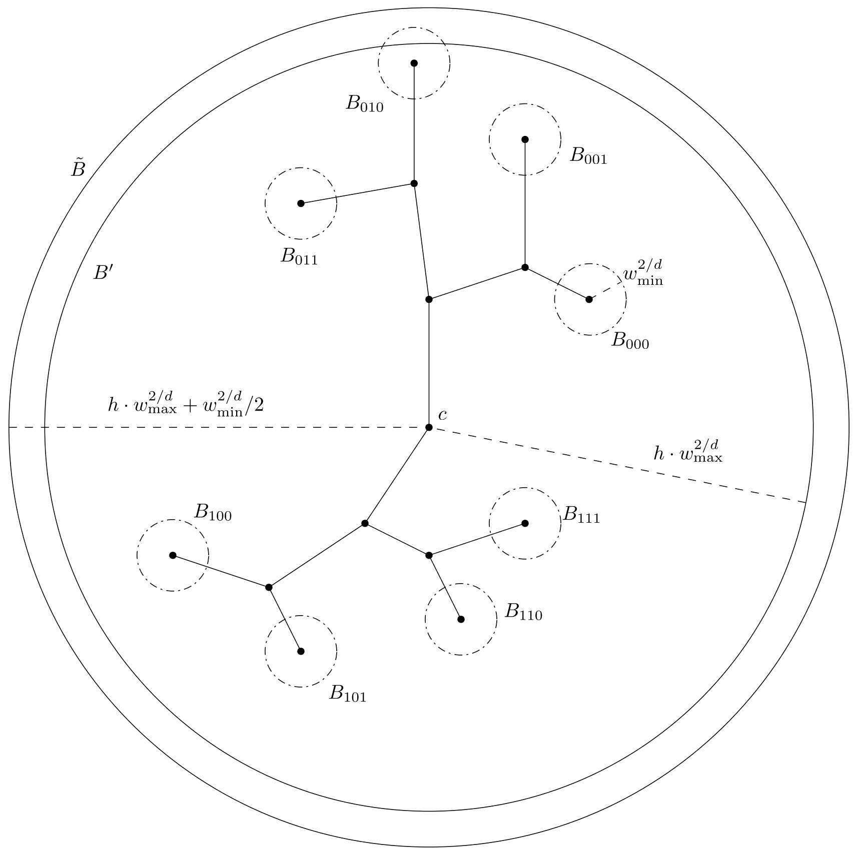


Figure 3.1.: Illustration of the construction of balls B_ℓ, B' and $\tilde{B}$ for $d = 2$ in the proof of Theorem 3.1 using $G = T_3^2$ with $r = 3$ and I being the set of leaves.

For every vertex $v \in I$ we construct the d-dimensional euclidean ball B_v centered at p_v with radius $w_{\min}(\psi)^{2/d}/2$. Note that $B_u \cap B_v = \emptyset$ for different $u, v \in I$, follows from Inequality 3.2. Furthermore, let B' be the d-dimensional euclidean ball centered at p_c with radius $r \cdot w_{\max}(\psi)^{2/d}$, and let $\tilde{B}$ be the d-dimensional euclidean ball centered at p_c with radius $r \cdot w_{\max}(\psi)^{2/d} + \frac{1}{2} w_{\min}(\psi)^{2/d}$. Figure 3.1 provides an illustration of this construction.

Using Inequality 3.1, we observe that $p_v \in B'$ for all $v \in I$ and thus $B_v \subseteq \tilde{B}$. Thus, $\dot{\bigcup}_{v \in V} B_v \subseteq \tilde{B}$. This subset relation is even strict, since a (d-dimensional) ball cannot be partitioned into multiple smaller (d-dimensional) balls. We compare the volumes of the sets to obtain

$$C_d(r w_{\max}(\psi)^{2/d} + w_{\min}(\psi)^{2/d}/2)^d = \mathrm{Vol}(\tilde{B}) > \sum_{v \in I} \mathrm{Vol}(B_v) = |I| \cdot C_d(w_{\min}(\psi)^{2/d}/2)^d. \quad (3.3)$$

Here, C_d is the constant introduced in Lemma 2.1 (e.g. $C_1 = 2$, $C_2 = \pi$ and $C_3 = \frac{4}{3}\pi$). Applying $r w_{\max}(\psi)^{2/d} + w_{\min}(\psi)^{2/d}/2 \le (r + \frac{1}{2}) w_{\max}(\psi)^{2/d}$ to Inequality 3.3 and simplifying yields

$$C_d(r + \frac{1}{2})^d w_{\max}(\psi)^2 > C_d \frac{|I|}{2^d} w_{\min}(\psi)^2.$$

Solving for $\frac{w_{\max}(\psi)}{w_{\min}(\psi)}$ results in

$$\frac{w_{\max}(\psi)}{w_{\min}(\psi)} > \sqrt{\frac{|I|}{(2r+1)^d}}.$$

■

Reiterman et al. used a somewhat similar idea to show a lower bound of the sphericity of a graph dependent on it's radius and size of an independent set [RRS89, Theorem 5.4]. To return to the topic of trees, we can apply this theorem directly on complete b-ary trees:

Corollary 3.2: *If ψ is a d-dimensional weighted embedding of the complete b-ary tree T_h^b, then* $\frac{w_{max}(\psi)}{w_{min}(\psi)} \in \Omega(b^{h/2}h^{-d/2})$. *More precisely,* $\frac{w_{max}(\psi)}{w_{min}(\psi)} \in \Omega\left(\sqrt{\frac{n}{(\log n)^d}}\right)$ *holds.*

Proof. Note that the radius of T_h^b is h. Let I be the set of leaves of T_h^b. I is an independent set with $|I| = b^h$. Thus by Theorem 3.1,

$$\frac{w_{\max}(\psi)}{w_{\min}(\psi)} > \sqrt{\frac{b^h}{(2h+1)^d}} \in \Omega\left(\sqrt{\frac{b^h}{h^d}}\right) = \Omega\left(b^{h/2}h^{-d/2}\right)$$

holds. Expressing this in terms of n, we observe that $h \in \Theta(\log n)$, $|I| = b^h \in \Theta(n)$ and thus

$$\frac{w_{\max}(\psi)}{w_{\min}(\psi)} \in \Omega\left(\sqrt{\frac{n}{(2\log n+1)^d}}\right) = \Omega\left(\sqrt{\frac{n}{(\log n)^d}}\right)$$

■

We note that for any constant $a < b$, it holds that $b^{h/2}h^{-d/2} \in \Omega\left(a^{h/2}\right)$. This shows that for any perfect embedding of T_h^b, if one exists at all, the fraction $\frac{w_{\max}(\psi)}{w_{\min}(\psi)}$ must grow at least exponential with h if we fix b to be constant. In particular, there exists no embedding of T_h^b, where the weights are constant or $\frac{w_{\max}(\psi)}{w_{\min}(\psi)}$ only grows polynomial with h. Note however that this requirement for exponentiality only applies for growing h and not necessarily for the number of vertices n: As far as Corollary 3.2 is concerned, $\frac{w_{\max}(\psi)}{w_{\min}(\psi)}$ might grow in $\Theta(\sqrt{n})$, as n increases.

3.2. Perfect Embeddings of Binary Trees

In the previous section we have found necessary conditions a perfect weighted embedding of T_h^b must satisfy. However, note that we have not proven the existence of such an embedding yet. We will make up for that now: We begin this section by finding perfect embeddings of binary trees and will follow up in the next section with perfect embeddings for arbitrary trees.

Corollary 3.2 shows that the maximal weight divided by the minimal weight of a perfect embedding of T_h^2 must grow exponentially with h. A natural assignment of weights that satisfies the criterion is

$$w_v := \alpha^{-|v|},$$

for each vertex v, where $\alpha > \sqrt{2}$ is a constant and $|v|$ denotes the layer of v (i.e. the distance to the root ε). Note that Corollary 3.2 is satisfied, as

$$\frac{\max_{v\in V(T_h^2)} w_v}{\min_{v\in V(T_h^2)} w_v} = \frac{w_\varepsilon}{w_{\text{any leaf}}} = \frac{\alpha^{-0}}{\alpha^{-h}} = \alpha^h \geq \left(\sqrt{2}\right)^h = 2^{h/2}.$$

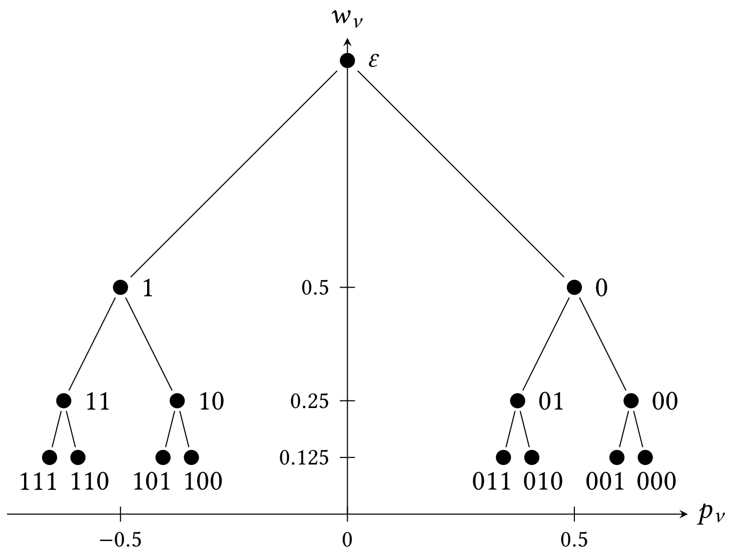


Figure 3.2.: Illustration of the embedding $\tilde{\psi}^{(\alpha)}$ of the binary tree T_3^2 of height 3 with $\alpha = 2$.

As we will show below, this construction of weights yields a perfect 1-dimensional weighted embedding $\tilde{\psi}^{(\alpha)} = (p_v, w_v)_{v \in V(T_h^2)}$ of T_h^2 (for some α). The positions in that embedding are set recursively as as follows: We set the position of the root ε to $p_\varepsilon = 0$. For any non-leaf vertex v we set the positions of its childs $v0$ and $v1$ to

$$p_{v1} = p_v - w_v w_{v1} = p_v - \alpha^{-2|v|-1} \qquad \text{and} \qquad p_{v0} = p_v + w_v w_{v0} = p_v + \alpha^{-2|v|-1}.$$

Figure 3.2 illustrates the embedding $\tilde{\psi}^{(\alpha)}$ constructed.

One benefit of this construction is that

$$\operatorname{dist}(\tilde{\psi}_v^{(\alpha)}, \tilde{\psi}_{v1}^{(\alpha)}) = \operatorname{dist}(\tilde{\psi}_v^{(\alpha)}, \tilde{\psi}_{v0}^{(\alpha)}) = \frac{|p_v - (p_v + w_v w_{v0})|}{w_v w_{v0}} = 1$$

for all non-leaves $v \in V(T_h^2)$. It would remain to show that $\operatorname{dist}(\tilde{\psi}_u^{(\alpha)}, \tilde{\psi}_v^{(\alpha)}) > 1$ for all non-adjacent vertices u, v. Instead of doing this for $\tilde{\psi}^{(\alpha)}$ explicitly, we will show a more general claim:

Theorem 3.3: *Let $\psi = (p_v, w_v)_{v \in V(T_h^2)}$ be any 1-dimensional weighted embedding of T_h^2 with $\frac{w_u}{w_v} > \varphi$ for all adjacent u, v with $|u| < |v|$. Additionally, let $p_{u_1} = p_v - w_{u_1} w_v$ and $p_{u_1} = p_v + w_{u_1} w_v$, for all vertex v with children u_1 and u_2. Then, ψ is perfect.*

Here, we recall that $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$ is defined as the positive solution of the equation

$$x^2 - x - 1 = 0.$$

Thus,

$$\varphi^2 - 1 = \varphi. \tag{3.4}$$

Proof of Theorem 3.3. We show this by induction over h. For $h = 0$ and $h = 1$, the claim obviously holds. Let $h \geq 2$. We assume the claim holds for $h - 1$ and show that it then holds for h. For that we will verify the defining property of perfect embeddings

$$\{u, v\} \in E(T_h^2) \iff \operatorname{dist}(\psi_u, \psi_v) \leq 1, \tag{3.5}$$

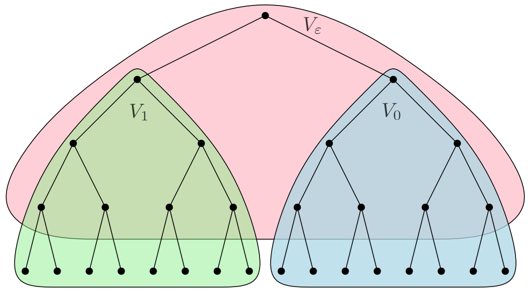


Figure 3.3.: Illustration of the vertex sets V_ε, V_0 and V_1 from the proof of Theorem 3.3.

for all pairs of vertices $u, v \in V(T_h^2)$. Consider the following 3 subsets of $V(T_h^2)$: $V_\varepsilon = \{v \in V(T_h^2) \mid |v| \neq h\}$, $V_0 = \{v \in V(T_h^2) \mid v(0) = 0\}$ and $V_1 = \{v \in V(T_h^2) \mid v(0) = 1\}$. Figure 3.3 illustrates these sets. W.l.o.g. let $p_1 < p_\varepsilon < p_0$.

Note that the subgraphs of T_h^2 induced by V_ε, V_0 and V_1 respectively are isomorphic to T_{h-1}^2 and thus Property 3.5 holds by the induction hypothesis for all u, v with either both $u, v \in V_\varepsilon$ or both $u, v \in V_0$ or both $u, v \in V_1$. Thus, we only have to verify Property 3.5 in the case where $u = \varepsilon$ is the root and v is a leaf, as well in the case where $v \in V_0$ and $u \in V_1$ are both leafs.

We start with the first case. So let $u = \varepsilon$ be the root and v any leaf. W.l.o.g. $v \in V_0$. As $|u| = 0$ and $|v| = h \geq 2$, ε and v are not adjacent. Thus, we have to verify that $\operatorname{dist}(\psi_\varepsilon, \psi_v) > w_\varepsilon w_v$. Let $(\varepsilon = v^0, v^1, v^2, \dots, v^h = v)$ be the shortest path from ε to v. We observe that

$$w_{v^i} = w_{v^1} \cdot \underbrace{\frac{w_{v^2}}{w_{v^1}}}_{<\frac{1}{\varphi}} \cdot \underbrace{\frac{w_{v^3}}{w_{v^2}}}_{<\frac{1}{\varphi}} \cdot \dots \cdot \underbrace{\frac{w_{v^i}}{w_{v^{i-1}}}}_{<\frac{1}{\varphi}} < w_{v^1} \left(\frac{1}{\varphi}\right)^{i-1} = w_{v^1}\varphi^{-i+1} = w_{v^0}\frac{w_{v^1}}{w_{v^0}}\varphi^{-i+1} < w_{v^0}\varphi^{-i}, \tag{3.6}$$

for all i, where the inequality is strict if $i \neq 0$. Now, applying a telescope sum and the geometric sum formula yields,

$$\begin{aligned}
p_v - p_\varepsilon = p_{v^h} - p_{v^0} &= \sum_{i=1}^{h} p_{v^i} - p_{v^{i-1}} \geq p_{v^1} - p_{v^0} - \sum_{i=2}^{h} |p_{v^i} - p_{v^{i-1}}| \\
&= w_{v^1} w_{v^0} - \sum_{i=2}^{h} w_{v^i} w_{v^{i-1}} \overset{(3.6)}{>} w_{v^1} w_{v^0} - \sum_{i=2}^{h} w_{v^1}\varphi^{-i+1} w_{v^0}\varphi^{-i+1} \\
&= w_{v^1} w_{v^0} - w_{v^0} w_{v^1}\varphi^2 \sum_{i=2}^{h} \varphi^{-2i} = w_{v^1} w_{v^0} - w_{v^0}^2 \varphi^2 \frac{\varphi^{-4} - \varphi^{-2h-2}}{1 - \varphi^{-2}} \\
&= w_{v^1} w_{v^0} \left(1 - \frac{\varphi^{-2} - \varphi^{-2h}}{1 - \varphi^{-2}}\right)
\end{aligned} \tag{3.7}$$

We will now show that for all $t \geq 2$ the inequality

$$1 - \frac{\varphi^{-2} - \varphi^{-2t}}{1 - \varphi^{-2}} - \varphi^{-t+1} \geq 0 \tag{3.8}$$

holds. For $t = 2$ this is true, because

$$1 - \frac{\varphi^{-2} - \varphi^{-2\cdot 2}}{1 - \varphi^{-2}} = 1 - \varphi^{-2}\frac{1 - \varphi^{-2}}{1 - \varphi^{-2}} = 1 - \varphi^{-2} = \varphi^{-2}\underbrace{(\varphi^2 - 1)}_{\overset{(3.4)}{=}\varphi} = \varphi^{-1} = \varphi^{-2+1}. \tag{3.9}$$

We now define $g : \mathbb{R}_+ \to \mathbb{R}, s \mapsto 1 - \frac{\varphi^{-2} - s^2}{1 - \varphi^{-2}} - \varphi s$ and note that Inequality 3.8 is equivalent to $g(\varphi^{-t}) \geq 0$. In addition to $g(\varphi^{-2}) \geq 0$ (by Equation 3.9), it follows from the chain rule that

$$\frac{d}{dt} g(\varphi^{-t}) = \underbrace{-\varphi^{-t}}_{<0} \cdot \underbrace{\left(\frac{2\varphi^{-t}}{1 - \varphi^{-2}} - \varphi\right)}_{\leq \frac{2\varphi^{-2}}{1-\varphi^{-2}} - \varphi \approx -0.381 < 0} > 0$$

for all $t \geq 2$ and thus $g(\varphi^{-t}) \geq 0$ or all $t \geq 2$. So we have finally shown that Inequality 3.8 holds. Inequalities 3.8 and 3.7 together imply that

$$p_v - p_\varepsilon > w_{v^1} w_{v^0} \varphi^{-h+1} \overset{(3.6)}{>} w_{v^h} w_{v^0} = w_v w_\varepsilon. \tag{3.10}$$

Thus, $\mathrm{dist}(\psi_\varepsilon, \psi_v) = \frac{|p_\varepsilon - p_v|}{w_u w_v} > 1$ which finishes this case.

Now, consider the second case: Let $v \in V_0$ and $u \in V_1$ be leafs of T_h^2. Recall that w.l.o.g. $p_1 < p_\varepsilon < p_o$ and thus $p_u < p_\varepsilon < p_v$. Hence,

$$\mathrm{dist}(\psi_u, \psi_v) = \frac{p_v - p_u}{w_u w_v} > \frac{p_v - p_\varepsilon}{w_u w_v} \overset{(3.10)}{>} \frac{w_v w_\varepsilon}{w_u w_v} = \frac{w_\varepsilon}{w_u} > 1.$$

We have also verified Property 3.5 for the second case, which concludes the prove. ∎

As a direct consequence, $\tilde{\psi}^{(\alpha)}$ from above is a perfect embedding of T_h^2 for all h, if $\alpha > \varphi$. However, we note that for $\alpha = \varphi$ and $h \geq 2$,

$$|p_\varepsilon - p_{01}| = \varphi^{-1} - \varphi^{-3} = \varphi^{-3}(\varphi^2 - 1) \overset{(3.4)}{=} \varphi^{-3}\varphi = \varphi^{-2} = w_{01} w_\varepsilon,$$

and thus, $\mathrm{dist}(\tilde{\psi}_\varepsilon^{(\varphi)}, \tilde{\psi}_{01}^{(\varphi)}) = 1 \leq 1$, despite $\{\varepsilon, 01\} \notin E(T_h^2)$. Hence, $\tilde{\psi}^{(\varphi)}$ is not perfect. This shows the importance of the strictness of the inequality $\frac{w_u}{w_v} > \varphi$ in Theorem 3.3.

Additionally, we consider the following corollary of Theorem 3.3:

Corollary 3.4: *For arbitrary given weights $(w_v \in \mathbb{R})_{v \in V(T_h^2)}$ with $\frac{w_u}{w_v} > \varphi$ for all adjacent u, v with $|u| < |v|$, there exist positions $(p_v \in \mathbb{R})_{v \in V(T_h^2)}$ such that $\psi = (p_v, w_v)_{v \in V(T_h^2)}$ is a perfect 1-dimensional weighted embedding of T_h^2.*

Proof. We set $p_\varepsilon = 0$ and for each vertex $v \in V(T_h^2)$ with $|v| < h$, we set recursively

$$p_{v0} = p_v + w_v w_{v0} \qquad \text{and} \qquad p_{v1} = p_v - w_v w_{v1}.$$

We then apply Theorem 3.3 on $\psi = (p_v, w_v)_{v \in V(T_h^2)}$. ∎

3.3. Perfect Embeddings of b-ary Trees

In the previous section, we have shown among other things that perfect 1-dimensional weighted embeddings of complete binary trees exist. For that purpose it was convenient that each vertex of a binary tree has two or none children. This allowed us for each vertex v to position one child of v to the left and the other child to the right of v. This idea can be generalized to find higher-dimensional perfect embeddings of general complete k-ary trees.

However, instead of generalizing the idea from the previous section, we will try a completely different approach. On the contrary to the previous approach, all children of a vertex v will be positioned to the right of v. Before we describe this embedding in detail however, we show a preliminary lemma about embeddings of disconnected graphs:

For a 1-dimensional weighted embedding $\psi = (p_v, w_v)_{v \in V(G)}$ of an arbitrary graph G we define $p_{\text{left}}(\psi)$, $p_{\text{right}}(\psi)$ and $\Delta(\psi)$ in a similar fashion as $w_{\max}(\psi)$ and $w_{\min}(\psi)$:

$$p_{\text{left}}(\psi) \coloneqq \min_{v \in V(G)} p_v, \quad p_{\text{right}}(\psi) \coloneqq \max_{v \in V(G)} p_v, \quad \Delta(\psi) \coloneqq p_{\text{right}}(\psi) - p_{\text{left}}(\psi)$$

Observe that $\Delta(\psi)$ can be alternatively characterized by

$$\Delta(\psi) = \max_{u,v \in V(G)} p_v - p_u.$$

With that notation, we observe:

Lemma 3.5: *Let $\delta > 0$ be arbitrary and G be a graph with k connected components $H_0, H_1, \dots H_{k-1}$, such that for each $i \in \{0, \dots, k-1\}$, a perfect 1-dimensional weighted embedding $\psi^{(i)} = (p_v^{(i)}, w_v^{(i)})_{v \in V(H_i)}$ of H_i exists. Then, there exists a perfect 1-dimensional weighted embedding $\psi = (p_v, w_v)_{v \in V(G)}$ of G such that $w_v = w_v^{(i)}$ for all $v \in V(H_i)$, $i \in \{0, \dots, k-1\}$ and*

$$\Delta(\psi) = \delta + (k-1) w_{max}(\psi)^2 + \sum_{i=0}^{k-1} \Delta(\psi^{(i)}). \tag{3.11}$$

Proof. W.l.o.g. assume that $p_{\text{left}}(\psi^{(i)}) = 0$ for all i. We define $\psi = (p_v, w_v)_{v \in V(G)}$ by $w_v \coloneqq w_v^{(i)}$ and $p_v \coloneqq p_v^{(i)} + s_i$ for all $v \in V(H_i)$, $i \in \{0, \dots, k-1\}$, where

$$s_i \coloneqq \frac{i}{k-1}\delta + \sum_{\ell=0}^{i-1} \left(\Delta(\psi^{(\ell)}) + w_{\max}(\psi)^2\right). \tag{3.12}$$

Figure 3.4 illustrates this construction.

We claim that ψ is perfect: Let $u \in V(H_i)$, $v \in V(H_j)$, $i \in \{0, \dots, k-1\}$ and $j \in \{0, \dots, k-1\}$ be arbitrary. If $i = j$, then

$$\operatorname{dist}(\psi_u, \psi_v) = \frac{\|p_u - p_v\|}{w_u w_v} = \frac{\|p_u^{(i)} + s_i - (p_v^{(i)} + s_i)\|}{w_u^{(i)} w_v^{(i)}} = \operatorname{dist}(\psi_u^{(i)}, \psi_v^{(i)})$$

and thus

$$\operatorname{dist}(\psi_u, \psi_v) > 0 \iff \operatorname{dist}(\psi_u^{(i)}, \psi_v^{(i)}) > 0 \iff \{u, v\} \in E(H_i).$$

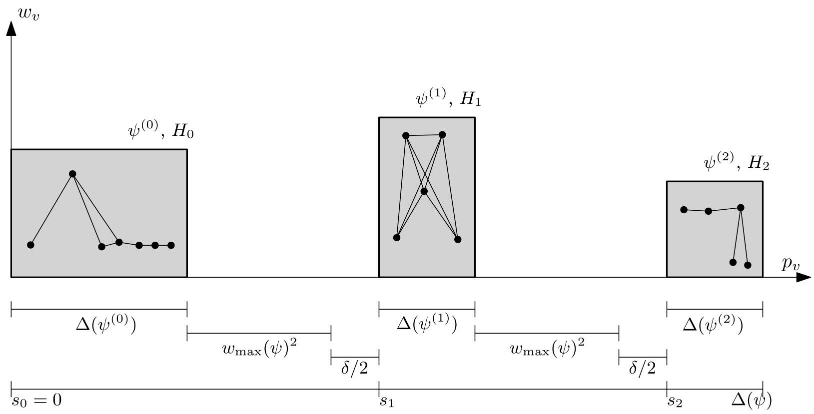


Figure 3.4.: Illustration of the construction of ψ and choice of s_i in the proof of Lemma 3.5, with $k = 3$. The connected components are positioned in such a way that the positions of two vertices u, v in different components have distance more than $w_{\max}(\psi)^2$. This guarantees that $\mathrm{dist}(\psi_u, \psi_v) > 1$ for all such u, v. The weights and relative positions of vertices in the same connected components remain unchanged.

Otherwise, if $i \neq j$ (w.l.o.g. $i > j, p_u \geq p_v$), then $\{u, v\} \notin E(G)$ and

$$\begin{aligned}
p_u - p_v = {} & \underbrace{p_u^{(i)}}_{\geq 0} + s_i - \underbrace{p_v^{(i)}}_{\leq \Delta(\psi^{(j)})} - s_j \\
\overset{(3.12)}{\geq} {} & -\Delta(\psi^{(j)}) + \frac{i-j}{k-1}\delta + \sum_{\ell=j}^{i-1} \left(\Delta(\psi^{(\ell)}) + w_{\max}(\psi)^2\right) \\
\geq {} & -\Delta(\psi^{(j)}) + \frac{i-j}{k-1}\delta + \left(\Delta(\psi^{(j)}) + w_{\max}(\psi)^2\right) \\
\geq {} & \frac{i-j}{k-1}\delta + w_{\max}(\psi)^2 > w_{\max}(\psi)^2.
\end{aligned}$$

Thus,

$$\mathrm{dist}(\psi_u, \psi_v) = \frac{\|p_u - p_v\|}{w_u w_v} > \frac{w_{\max}(\psi)^2}{w_{\max}(\psi)^2} = 1.$$

So, we have shown that ψ is perfect. We also confirm that Property 3.11 holds: Let $u = \arg\max_{u \in V(H_{k-1})} w_u$ and $v = \arg\min_{v \in V(H_0)} w_v$. Then,

$$\begin{aligned}
\Delta(\psi) = p_u - p_v &= p_u^{(k-1)} + s_{k-1} - p_v^{(0)} + s_0 \\
&= p_{\text{right}}(\psi^{(k-1)}) - p_{\text{left}}(\psi^{(0)}) + \frac{k-1}{k-1}\delta + \sum_{\ell=0}^{k-2} \left(\Delta(\psi^{(\ell)}) + w_{\max}(\psi)^2\right) \\
&= \Delta(\psi^{(k-1)}) - 0 + \delta + (k-1)w_{\max}(\psi)^2 + \sum_{i=0}^{k-2} \Delta(\psi^{(i)}) \\
&= \delta + (k-1)w_{\max}(\psi)^2 + \sum_{i=0}^{k-1} \Delta(\psi^{(i)}).
\end{aligned}$$

∎

With this in mind, we can tackle the construction of perfect 1-dimensional weighted embeddings of the complete b-ary tree T_h^b: As in the case with $b = 2$, we proof sufficient conditions for weights that yields a perfect 1-dimensional weighted embedding. For $b = 2$, the ratio for weights of connected vertices $\frac{w_u}{w_v}$ has to be larger than φ. For general b, this bound on the ratio is dependent on the branching factor b. We denote this bound by ξ_b and define it by:

Definition 3.6: *For all $b \in \mathbb{N}_+$ we define ξ_b as the positive solution x to the equation*

$$bx^{-1} + b = x. \tag{3.13}$$

With the quadratic equation, we can express ξ_b explicitly as

$$\xi_b = \frac{1}{2}\left(b + \sqrt{b^2 + 4b}\right),$$

which allows us to find the bounds,

$$b = \frac{b + \sqrt{b^2}}{2} < \xi_b < \frac{\overbrace{\sqrt{b^2 + 2b + 1}}^{=\sqrt{(b+1)^2}} + b + 1}{2} = b + 1. \tag{3.14}$$

As an additional change to the case $b = 2$, we require that all vertices in the same layer of T_h^b (i.e. with the same distance to the root) have the same given weight w_v. Thus, for each i there exists a number $\overline{w}_i$ such that for all vertices v in layer i (i.e. $|v| = i$) have weight $w_v = \overline{w}_i$. As a consequence, we can rephrase the condition $\frac{w_u}{w_v} > \xi_b$ for all vertices v with parent u, as $\frac{\overline{w}_i}{\overline{w}_{i+1}} > \xi_b$ for all $i \in \{0, \ldots, h-1\}$.

Theorem 3.7: *For all $h \in \mathbb{N}_0$, $b \in \mathbb{N}_+ \setminus \{1\}$ and $(\overline{w}_i)_{i\in\{0,\ldots,h\}}$, with $\frac{\overline{w}_i}{\overline{w}_{i+1}} > \xi_b$ for all $i \in \{0, \ldots, h-1\}$, there exists a perfect 1-dimensional weighted embedding $\psi = (p_v, w_v)_{v\in V(T_h^b)}$ of T_h^b such that $w_v = \overline{w}_{|v|}$ for all $v \in V(T_h^b)$ and*

$$\Delta(\psi) < \xi_b \overline{w}_0(\overline{w}_0 - \overline{w}_1), \tag{3.15}$$

if $h \geq 1$.

Proof. We will prove by induction over h that such an embedding ψ exists. The claim is obviously true for $h = 0$. For $h = 1$, we define $\psi = (p_v, w_v)_{v\in V(T_2^b)}$ by $p_\varepsilon = 0, w_\varepsilon = \overline{w}_0$ and $p_{u^\ell} = \left(\ell + \frac{\ell}{b-1}\right)\overline{w}_1^2, w_{u^\ell} = \overline{w}_1$ for the ℓ-th child u^ℓ of ε ($\ell \in \{0, 1, \ldots, b-1\}$). See Figure 3.5 for an illustration. Then,

$$\operatorname{dist}(\psi_\varepsilon, \psi_{u^\ell}) = \frac{(\ell + \frac{\ell}{b-1})\overline{w}_1^2}{\overline{w}_0\overline{w}_1} \leq \frac{(b - 1 + \frac{b-1}{b-1})\overline{w}_1^2}{\overline{w}_0\overline{w}_1} = b\frac{\overline{w}_1}{\overline{w}_0} \overset{(3.14)}{<} \xi_b\frac{\overline{w}_1}{\overline{w}_0} < \xi_b\xi_b^{-1} = 1$$

and

$$\operatorname{dist}(\psi_{u^\ell}, \psi_{u^k}) = \frac{\left(\ell - k + \frac{\ell}{b-1} - \frac{k}{b-1}\right)\bar{w}_1^2}{\overline{w}_1^2} \geq \frac{(1 + \frac{1}{b-1})\overline{w}_1^2}{w_1^2} > 1$$

for all $\ell, k \in \{0, 1 \ldots, b-1\}$, $\ell > k$. Thus, ψ is perfect.

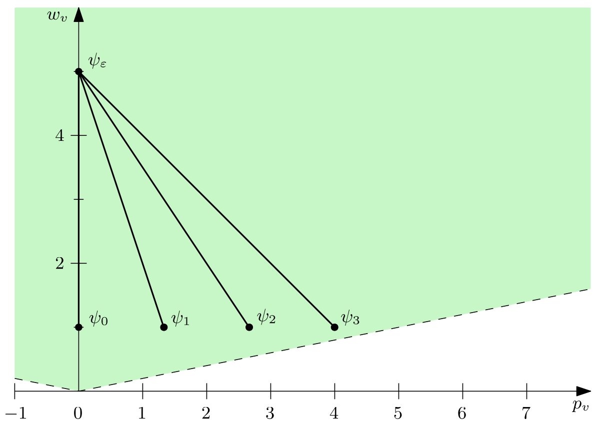


Figure 3.5.: Illustration of the induction base case in the proof of Theorem 3.7 for $b = 4, \overline{w}_0 = 5 > \xi_4$ and $\overline{w}_1 = 1$. The dashed, green area is the weighted ball $B_1(\psi_\varepsilon)$. We can see that all vertices are contained in this ball and we can verify that the distance between two child of ε is sufficiently large.

Additionally, Condition 3.15 is satisfied, as

$$\begin{aligned}\Delta(\psi) = p_{u^{b-1}} - p_\varepsilon = \left(b - 1 + \frac{b-1}{b-1}\right)\overline{w}_1^2 = b\overline{w}_1^2 \overset{(3.14)}{<} \xi_b\overline{w}_1^2 \\ < \xi_b\overline{w}_0\overline{w}_1 < \xi_b\overline{w}_0\overline{w}_1 \underbrace{\left(\frac{\overline{w}_0}{\overline{w}_1} - 1\right)}_{>\xi_b - 1 > b-1 \geq 1} = \xi_b\overline{w}_0(\overline{w}_0 - \overline{w}_1).\end{aligned}$$

Now let $h \geq 2$ be arbitrary and assume that the claim holds for all trees $T^b_{h'}$ of height $h' < h$. Consider the graph G' that is obtained by removing the root ε from T^b_h. Notet that G' has exactly b connected components $H_0, \ldots, H_{b-1}$. All connected components H_i are isomorphic to T^b_{h-1}. Thus, by the induction hypothesis, there exists a perfect 1-dimensional weighted embedding ψ'' of $T^b_{h-1} \cong H_i$ (for all i), that satisfies the claims. By Lemma 3.5, for all $\delta_1 > 0$, there exists a perfect 1-dimensional weighted embedding $\psi' = (p'_v, w'_v)_{v \in V(G')}$ of G' such that $w'_v = \overline{w}_{|v|}$ and

$$\begin{aligned}\Delta(\psi') &= \delta_1 + (b-1)\underbrace{w_{\max}(\psi')^2}_{=\overline{w}_1^2} + \sum_{i=0}^{b-1}\Delta(\psi'') \\ &= \delta_1 + (b-1)\overline{w}_1^2 + b\Delta(\psi''). \qquad (3.16)\end{aligned}$$

We will now show that the perfect embedding ψ' of G' can be extended to a perfect embedding $\psi = (p_v, w_v)_{v \in V(T^b_h)}$ of T^b_h with the required properties. In particular, we set $\psi_v := \psi'_v$ for all $v \in G'$ and $w_\varepsilon := \overline{w}_0$. It only remains to determine a suitable position p_ε of the root ε of T^b_h.

To show that ψ (with a suitable p_ε) is perfect, we only need to confirm that $\operatorname{dist}(\psi_\varepsilon, \psi_{r_i}) \leq 1$ for all $i \in \{0, \ldots, b-1\}$, where r_i is the root of H_i, and that $\operatorname{dist}(\psi_\varepsilon, \psi_v) > 1$ for all $v \in M_i$, where $M_i := V(H_i) \setminus \{r_i\}$. We observe that this is equivalent to $\psi_\varepsilon \in B_1(\psi_{r_i})$ and $\psi_\varepsilon \notin B_1(\psi_v)$

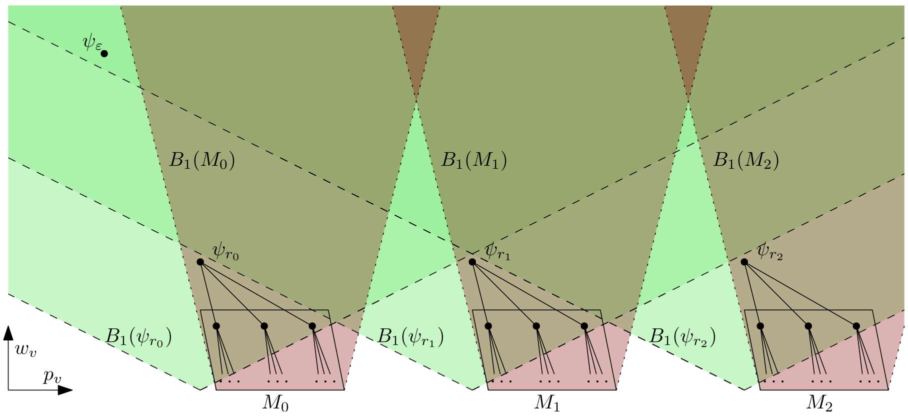


Figure 3.6.: This figure illustrates a part of the proof of Theorem 3.7. More precisely, it shows that it is possible for $k = 3$ to find a ψ_ε such that $\psi_\varepsilon \in B_1(\psi_{r_i})$ (green, dashed) and $\psi_\varepsilon \notin B_1(M_i) := \bigcup_{v \in M_i} B_1(\psi_v)$ (red, dotted) for all $i \in \{0, 1, 2\}$. This is the case, due to the boundary of the green, dashed cones $B_1(\psi_{r_i}$ having a less steep slope as the boundary of the red, dotted areas $B(M_i)$.

for all $i \in \{0, \dots, b-1\}, v \in M_i$, where we recall that $B_1(\psi_{r_i})$ denotes the set of all points $z \in \mathbb{R} \times \mathbb{R}_+$ with $\operatorname{dist}(\psi_{r_i}, z) \leq 1$ (compare Section 2.3). Figure 3.6 illustrates these conditions. By Lemma 2.2 it suffices to show that

$$p_\varepsilon \geq p_{r_i} - w_\varepsilon w_{r_i}, \tag{3.17}$$
$$p_\varepsilon \leq p_{r_i} + w_\varepsilon w_{r_i}, \tag{3.18}$$
$$p_\varepsilon < p_v - w_\varepsilon w_v \tag{3.19}$$

for all $i \in \{0, \dots, h-1\}, v \in M_i$. Note that instead of proving Inequality 3.19, it would be similarly sufficient to show that $p_\varepsilon > p_v + w_\varepsilon w_v$. However we will stick with Inequality 3.19. Now, applying $w_\varepsilon = \overline{w}_0, w_{r_i} = \overline{w}_1, w_v \leq \overline{w}_2$ and $p_{\text{left}}(\psi') \leq p_u \leq p_{\text{right}}(\psi')$ for all $i \in \{0, \dots, b-1\}, u \in V(G')$ and $v \in M_i$, shows that the Inequalities 3.17, 3.18 and 3.19 follow, if

$$p_\varepsilon \geq p_{\text{right}}(\psi') - \overline{w}_0\overline{w}_1, \tag{3.20}$$
$$p_\varepsilon \leq p_{\text{left}}(\psi') + \overline{w}_0\overline{w}_1, \tag{3.21}$$
$$p_\varepsilon < p_{\text{left}}(\psi') - \overline{w}_0\overline{w}_2 \tag{3.22}$$

We observe that Inequality 3.21 is a direct consequence of Inequality 3.22. Thus, it remains to find p_ε such that Inequalities 3.22 and 3.20 hold. We note that

$$\begin{aligned}
\exists p_\varepsilon : (3.22) \wedge (3.22) &\iff \exists p_\varepsilon : p_{\text{right}}(\psi') - \overline{w}_0\overline{w}_1 \leq p_\varepsilon < p_{\text{left}}(\psi') - \overline{w}_0\overline{w}_2 \\
&\iff p_{\text{right}}(\psi') - \overline{w}_0\overline{w}_1 < p_{\text{left}}(\psi') - \overline{w}_0\overline{w}_2 \\
&\iff \Delta(\psi') < \overline{w}_0\overline{w}_1 - \overline{w}_0\overline{w}_2 = \overline{w}_0(\overline{w}_1 - \overline{w}_2) \\
&\Longleftarrow \Delta(\psi') < \xi_b\overline{w}_1(\overline{w}_1 - \overline{w}_2)
\end{aligned}$$

The last statement holds by the induction hypothesis (Inequality 3.15) for trees of height $h-1$. Thus, we have proven that we can find a p_ε such that ψ is perfect. In particular, we can set $p_\varepsilon := p_{\text{left}}(\psi') - \overline{w}_0\overline{w}_2 - \delta_2$ for a sufficiently small $\delta_2 > 0$.

To complete the induction step, we need to prove that Inequality 3.15 also holds for ψ. We observe that $\Delta(\psi'') < \xi_b \overline{w}_1(\overline{w}_1 - \overline{w}_2)$ by the induction hypothesis (IH). Also note that $p_\varepsilon = p_{\text{left}}(\psi)$, as $p_\varepsilon < p_{\text{left}}(\psi') - \overline{w}_0\overline{w}_2 < p_{\text{left}}(\psi')$. Hence, $p_{\text{right}}(\psi) = p_{\text{right}}(\psi')$. We apply those observations and obtain

$$\begin{aligned}
\Delta(\psi) &= p_{\text{right}}(\psi) - p_{\text{left}}(\psi) = p_{\text{right}}(\psi') - p_\varepsilon = p_{\text{right}}(\psi') - (p_{\text{right}}(\psi') - \overline{w}_0\overline{w}_2 - \delta_2) \\
&= \delta_2 + \Delta(\psi') + \overline{w}_0\overline{w}_2 \overset{(3.16)}{=} \delta_1 + \delta_2 + (b-1)\overline{w}_1^2 + b\Delta(\psi'') + \overline{w}_0\overline{w}_2 \\
&\overset{(IH)}{<} \delta_1 + \delta_2 + (b-1)\overline{w}_1^2 + b\xi_b\overline{w}_1 \underbrace{(\overline{w}_1 - \overline{w}_2)}_{<\overline{w}_1 < \xi_b^{-1}\overline{w}_0} + \overline{w}_0\overline{w}_2 \\
&< \delta_1 + \delta_2 + (b-1)\xi_b^{-1}\overline{w}_1\overline{w}_0 + b\xi_b\overline{w}_1\xi_b^{-1}\overline{w}_0 + \xi_b^{-1}\overline{w}_0\overline{w}_1 \\
&= \delta_1 + \delta_2 + \overline{w}_0\overline{w}_1 \underbrace{(b\xi_b^{-1} + b)}_{=\xi_b,\text{ by (3.13)}} = \delta_1 + \delta_2 + \overline{w}_0\overline{w}_1\xi_b \underbrace{\left(\frac{\overline{w}_0}{\overline{w}_1} - 1\right)}_{>\xi_b - 1 > b-1 \geq 1} \\
&= \xi_b\overline{w}_0\left(\overline{w}_0 - \overline{w}_1\right),
\end{aligned}$$

for some $\delta_1, \delta_2 > 0$. As we can choose $\delta_1 + \delta_2$ arbitrarily small, this implies

$$\Delta(\psi) < \xi_b\overline{w}_0(\overline{w}_0 - \overline{w}_1).$$

∎

This theorem is a useful criterion: If given weights satisfy the condition, then there exists a perfect 1-dimensional embedding that uses those weights. Note, however, that the criterion is not necessary: It is easy to verify that there exists an assignment of weights to the vertices of T_h^b that yields a perfect embedding, but does not assign all vertices in the same layer the same weight. Additionally, we note that $\varphi \approx 1.618 < 2.732 \approx \xi_2$, thus Corollary 3.4 can be applied in much more situations than Theorem 3.7, if $b = 2$. It seems probable that the requirements of Theorem 3.7 can also be relaxed for other values of b, however this is not done in this work.

As another note, consider the following corollary:

Corollary 3.8: *For any tree T there exists a perfect 1-dimensional weighted embedding of T.*

Proof. We select $b \in \mathbb{N}_+$ such that T has maximal degree $b + 1$ and $h := r(T)$ to the radius of T. Then, T is an isomorphic to an induced subgraph of T_h^b. It remains to show that T_h^b can be embedded in 1-dimension.

We set $w_v := (b+1)^{-|v|}$ for each vertex $v \in V(T_h^b)$. Then the claim follows directly by observing that $b + 1 > \xi_b$ and applying Theorem 3.7. ∎

4. Embeddings of Grids

In this chapter, we will discuss weighted embeddings of grids. Even though grids, like complete b-ary trees, have a homogeneous degree distribution, we will show embeddings of grids behave very differently to those of trees.

We start by considering 1-dimensional weighted embeddings of the grid $\Gamma_{a,b}$. It seems unlikely that there exists a perfect 1-dimensional weighted embedding $\Gamma_{a,b}$, if a and b are sufficiently large, since such an embedding must satisfy various seemingly contradicting properties. We even conjecture that no such embedding exists for the case $a = b = 3$. However, we were not able to formulate a rigorous proof of this. Surprisingly, a perfect 1-dimensional weighted embedding of $\Gamma_{a,2}$ exists for all $a \in \mathbb{N}_+$:

Lemma 4.1: *For all $a \in \mathbb{N}_+$, there exists a perfect 1-dimensional weighted embedding of the grid $\Gamma_{a,2}$.*

Proof Sketch. Consider the embedding $\psi = (p_v, w_v)_{v \in V(\Gamma_{a,2})}$ defined by

$$w_{(x,y)} \coloneqq 2^{-x}$$

and

$$p_{(x,y)} \coloneqq (-1)^y 2^{-2x-1},$$

for all $(x, y) \in V(\Gamma_{a,2})$. Verify that ψ is perfect. $\square$

For a complete proof of this lemma, see Section A.1. See Figure 4.1 for an illustration of the ψ constructed in the proof sketch. A consequence of Lemma 4.1 is that if $a \leq 2$ or $b \leq 2$, then there exists a perfect 1-dimensional embedding of $\Gamma_{a,b}$. However, as mentioned above, it seems like 1-dimensional embeddings are not suitable for arbitrary grids. Hence, we consider 2-dimensional embeddings instead:

We observe that for all $a, b \in \mathbb{N}_+$, the trivial 2-dimensional weighted embedding $\psi = ((x, y), 1)_{(x,y) \in V(\Gamma_{a,b})}$ of $\Gamma_{a,b}$ is perfect. This is, because the positions of two adjacent vertices differ by exactly 1, while the difference between the positions of two non-adjacent vertices is at least $\sqrt{2}$. We note that the distance between the positions of any two vertices is never in the open interval $(1, \sqrt{2})$. This observation allows us to make a more general claim, which is stated in the next lemma. First, however, for an ease in notation, we define

$$w_{\max} \coloneqq \max_{v \in V(G)} w_v \qquad \text{and} \qquad w_{\min} \coloneqq \min_{v \in V(G)} w_v$$

for all graphs G and weight assignments $w = (w_v)_{v \in V(G)}$.

Lemma 4.2: *For all $a, b \in \mathbb{N}_+$, an induced subgraph H of $\Gamma_{a,b}$, and given weights $w = (w_v)_{v \in V(H)}$, with $\frac{w_{max}}{w_{min}} < \sqrt{2}$, there exists a perfect 2-dimensional embedding $\psi = (p_v, w_v)_{v \in V(H)}$ of H.*

Proof. We set $p_{(x,y)} \coloneqq (x \cdot w_{\min}, y \cdot w_{\min})$ for all $(x, y) \in V(H)$ and claim that $\psi = (p_v, w_v)_{v \in V(H)}$ is perfect. For all adjacent vertices $u, v \in V(H)$, it holds that $\|p_u - p_v\| = w_{\min}$ and thus

$$\operatorname{dist}(\psi_u, \psi_v) = \frac{w_{\min}}{(w_u w_v)^{1/2}} \leq \frac{w_{\min}}{(w_{\min} w_{\min})^{1/2}} = 1.$$

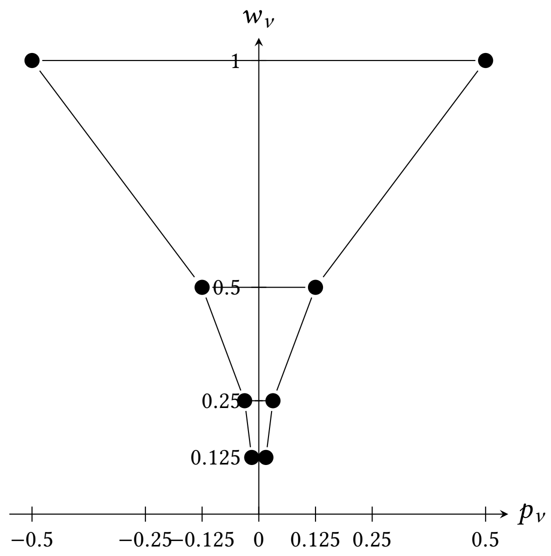


Figure 4.1.: Plot of the perfect 1-dimensional weighted embedding $\psi = (p_v, w_v)_{v \in V(\Gamma_{4,2})}$ of $\Gamma_{4,2}$ with $p_{(x,y)} = (-1)^y 2^{-2x-1}$ and $w_{(x,y)} = 2^{-x}$ for all $(x, y) \in V(\Gamma_{4,2})$,

For all non-adjacent vertices $(i, j) \in V(H)$ and $(x, y) \in V(H)$, we observe that $|i-x|+|j-y| \geq 2$. It follows

$$\begin{aligned}\|p_{(i,j)} - p_{(x,y)}\| &= \sqrt{(i-x)^2 w_{\min}^2 + (j-y)^2 w_{\min}^2} \\ &= w_{\min}\sqrt{(i-x)^2 + (j-y)^2} \geq w_{\min}\sqrt{2}.\end{aligned}$$

and thus

$$\operatorname{dist}(\psi_{(i,j)}, \psi_{(x,y)}) \geq \frac{w_{\min}\sqrt{2}}{(w_{\max} w_{\max})^{1/2}} = \sqrt{2} \cdot \frac{w_{\min}}{w_{\max}} > \sqrt{2} \cdot \frac{1}{\sqrt{2}} = 1.$$

■

This is a useful criterion for weight assignments that yield perfect embeddings of the grid $\Gamma_{a,b}$. However, we note that the requirement $\frac{w_{\max}}{w_{\min}} < \sqrt{2}$ is very strict: For instance, if we consider the weights $w = (w_v)_{v \in \Gamma_{a,a}}$ defined by $w_v = \deg v$ for all $v \in V(\Gamma_{a,a})$, then $\frac{w_{\max}}{w_{\min}} = 2 > \sqrt{2}$. Thus, we cannot apply Lemma 4.2 directly.

However, instead of only focusing on perfect embeddings, we can more generally search for embeddings with low total error. What exactly we consider as *low* total error, highly depends on the use case. Yet, we note that if an embedding of $\Gamma_{a,a}$ has error $(o(a^2), o(a^2)) = (o(n), o(n))$, then

$$\lim_{a\to\infty} \frac{|E(\psi) \setminus E(G) \cup E(G) \setminus E(\psi)|}{|E(\Gamma_{a,a})|} = \lim_{a\to\infty} \frac{o(n) + o(n)}{\mathcal{O}(n)} = 0.$$

We will show a very general claim about embeddings of grids with sublinear total error. However, we first introduce the following definition:

Definition 4.3 ((c_x, c_y)-Monotonicity)**:** *Let $(c_x, c_y) \in \mathbb{R}^2$ be a point and $w = (w_v)_{v \in V(\Gamma_{a,b})}$ be a mapping with $w_v \in \mathbb{R}$ for all $v \in V(\Gamma_{a,b})$. We say that w is (c_x, c_y)-*monotone*, if for all $(x_1, y_1), (x_2, y_2) \in V(\Gamma_{a,b})$ with $|x_1 - c_x| \leq |x_2 - c_x|$ and $|y_1 - c_y| \leq |y_2 - c_y|$, the inequality $w_u \geq w_v$ holds.*

Intuitively speaking, a weight assignment is called (c_x, c_y)-monotone, if the weight of vertices decreases, as they get further away from the center point (c_x, c_y). We note that this monotonicity seems natural in the case where the weight assignment is a centrality measure.

We claim:

Theorem 4.4: *For arbitrary $a \in \mathbb{N}_+$, $(c_x, c_y) \in \mathbb{R}^2$ and a (c_x, c_y)-monotone mapping $w = (w_v)_{v \in V(\Gamma_{a,a})}$, there exist $p_v \in \mathbb{R}^2$ for all $v \in V(\Gamma_{a,a})$, such that $\psi = (p_v, w_v)_{v \in V(\Gamma_{a,a})}$ is a 2-dimensional weighted embedding of $\Gamma_{a,a}$ with error $(s_{fn}, 0)$, where*

$$s_{fn} \leq 4\left\lceil 2\log_2\left(\frac{w_{max}}{w_{min}}\right)\right\rceil a \in \mathcal{O}(\log\left(\frac{w_{max}}{w_{min}}\right)\sqrt{n}).$$

Before we prove this theorem, we consider the following two corollaries:

Corollary 4.5: *For arbitrary $a \in \mathbb{N}_+$, $(c_x, c_y) \in \mathbb{R}^2$ and a (c_x, c_y)-monotone mapping $w = (w_v)_{v \in V(\Gamma_{a,a})}$ with $\frac{w_{max}}{w_{min}} \in \Theta(1)$, there exist $p_v \in \mathbb{R}^2$ for all $v \in V(\Gamma_{a,a})$, such that $\psi = (p_v, w_v)_{v \in V(\Gamma_{a,a})}$ is a 2-dimensional weighted embedding of $\Gamma_{a,a}$ with error $\left(\mathcal{O}\left(\sqrt{n}\right), 0\right)$.*

Corollary 4.6: *For arbitrary $a \in \mathbb{N}_+$, $(c_x, c_y) \in \mathbb{R}^2$ and a (c_x, c_y)-monotone mapping $w = (w_v)_{v \in V(\Gamma_{a,a})}$ with $\frac{w_{max}}{w_{min}} \in 2^{o(\sqrt{n})}$, there exist $p_v \in \mathbb{R}^2$ for all $v \in V(\Gamma_{a,a})$, such that $\psi = (p_v, w_v)_{v \in V(\Gamma_{a,a})}$ is a 2-dimensional weighted embedding of $\Gamma_{a,a}$ with error $(o\,(n)\,, 0)$.*

Especially the latter is surprising, as it shows that even for some weights that differ super-polynomial, there exists a corresponding embedding of the grid with sublinear total error, as long as the condition of monotonicity is satisfied. We remark that another criterion for the existence of embeddings with sublinear error with given weights, that does *not* rely on any monotonicity, will be discussed as a consequence of a more general theorem in Chapter 6 (Corollary 6.5).

The fundamental idea of the proof of Theorem 4.4 relies on the idea of partitioning the vertices of the grid $\Gamma_{a,a}$ into multiple subgraphs such that each of those subgraphs satisfies the requirements of Lemma 4.2. Thus, we can find perfect embeddings of those subgraphs and, thus, a perfect embedding ψ of the (disjoint) union H of all the subgraphs. If we show, that $E(\Gamma_{a,a}) \setminus E(H)$ only contains few edges, then ψ is an embedding of $\Gamma_{a,a}$ with low error. We note that $E(\Gamma_{a,a}) \setminus E(H)$ is the set of all edges $\{u, v\}$ of $E(\Gamma_{a,a})$ such that u and v are not contained in the same subgraph. In this last step of the proof, the following definition and lemma will be useful:

Definition 4.7: *We call a set of vertices $V' \subseteq V(\Gamma_{a,b})$ a w-*upset*, if $w_u > w_v$ for all $u \in V'$ and $v \in V(\Gamma_{a,b}) \setminus V'$.*

Figure 4.2 shows an example of a w-upset and a set that is not a w-upset.

Lemma 4.8: *For a point $(c_x, c_y) \in \mathbb{R}^2$, a (c_x, c_y)-monotone mapping $w = (w_v)_{v \in V(\Gamma_{a,b})}$ and a w-upset $V' \subseteq V(\Gamma_{a,b})$, it holds that*

$$\underbrace{|\{\{u, v\} \in E(\Gamma_{a,b}) \mid u \in V', v \in V(\Gamma_{a,b}) \setminus V'\}|}_{=:\tilde{E}} \leq 2a + 2b.$$

Proof. Every edge $\{(x_1, y_1), (x_2, y_2)\} \in E(\Gamma_{a,b})$ is either a *horizontal edge* (i.e. $y_1 = y_2$ and $|x_1 - x_2| = 1$) or a *vertical edge* (i.e. $x_1 = x_2$ and $|y_1 - y_2| = 1$). We show that there are at most $2a$ vertical edges and at most $2b$ horizontal edges in $\tilde{E}$. We will only prove that there are at most $2a$ vertical edges in $\tilde{E}$. The other case follows analogously.

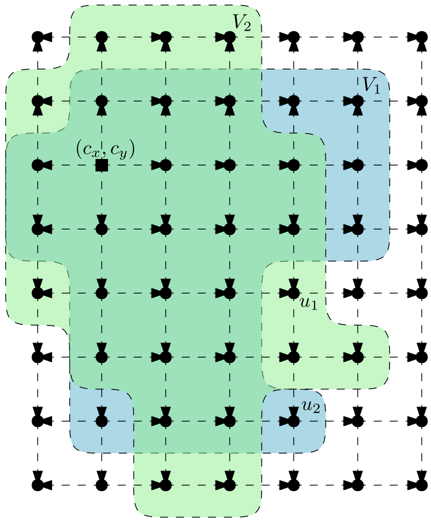


Figure 4.2.: Example of a (c_x, c_y)-monotone mapping $w = (w_v)_{v \in \Gamma_{a,b}}$, a w-upset and a set that is not a w-upset. The direction of the edges shows in which direction w_v gets smaller: An edge $\{u, v\} \in E(\Gamma_{a,b})$ is drawn as directed from u to v, if $w_u > w_v$. The mapping w for which this is shown, is (c_x, c_y)-monotone, where $(c_x, c_y) = (1, 2)$. Additionally, we note that the set of vertices V_2 (highlighted in green and dashed) is a w-upset. The set V_1 (highlighted in blue) is not a w-upset, as there are vertices $u_1 \notin V_1$ and $u_2 \in V_1$ such that $w_{u_1} > w_{u_2}$.

For the sake of contradiction, we assume that there exist (at least) 3 vertical edges

$$\{(x_1, y), (x_2, y)\}, \{(x_3, y), (x_4, y)\}, \{(x_5, y), (x_6, y)\} \in \tilde{E}$$

in the same column with $x_1 < x_2 \leq x_3 < x_4 \leq x_5 < x_6$. Note that either $x_4 \leq c_x$ or $c_x \leq x_3$ ($x_3 < c_x < x_4$ is not possible, because $|x_3 - x_4| = 1$). W.l.o.g. we assume $c_x \leq x_3$. Now, $(x_i, y) \in V(\Gamma_{a,b}) \setminus V'$ for some $i \in \{3, 4\}$ and $(x_j, y) \in V'$ for some $j \in \{5, 6\}$, by the definition of $\tilde{E}$. Since V' is a w-upset, it follows that $w_{(x_j,y)} > w_{(x_i,y)}$. However, we observe that

$$|x_i - c_x| = x_i - c_x \leq x_j - c_x = |x_j - c_x|$$

and, since w is c-monotone, it follows that $w_{(x_i,y)} \geq w_{(x_j,y)}$. This is a contradiction. Hence there exist at most 2 vertical edges in the same column. There are exactly b columns in $\Gamma_{a,b}$, which implies the claim. ■

We can now prove Theorem 4.4:

Proof of Theorem 4.4. Let $r := \left\lceil 2\log_2\left(\frac{w_{\max}}{w_{\min}}\right)\right\rceil$. For every $k \in \{0, 1, \ldots, r-1\}$ we define

$$V_k := \{v \in V(\Gamma_{a,a}) \mid w_{\min} \cdot (\sqrt{2})^k \leq w_v < w_{\min} \cdot (\sqrt{2})^{k+1}\}.$$

Now, $V_0, V_1, \ldots, V_{r-1}$ is a partition of all vertices, since each vertex $v \in V(\Gamma_{a,a})$ is contained in one of the sets, as

$$\begin{aligned} w_{\min}(\sqrt{2})^0 &= w_{\max}(w) \leq w_v \leq w_{\max} \leq w_{\min}\frac{w_{\max}}{w_{\min}} = w_{\min}2^{\log_2\left(\frac{w_{\max}}{w_{\min}}\right)} \\ &\leq w_{\min}(\sqrt{2})^{(r-1)+1} \end{aligned}$$

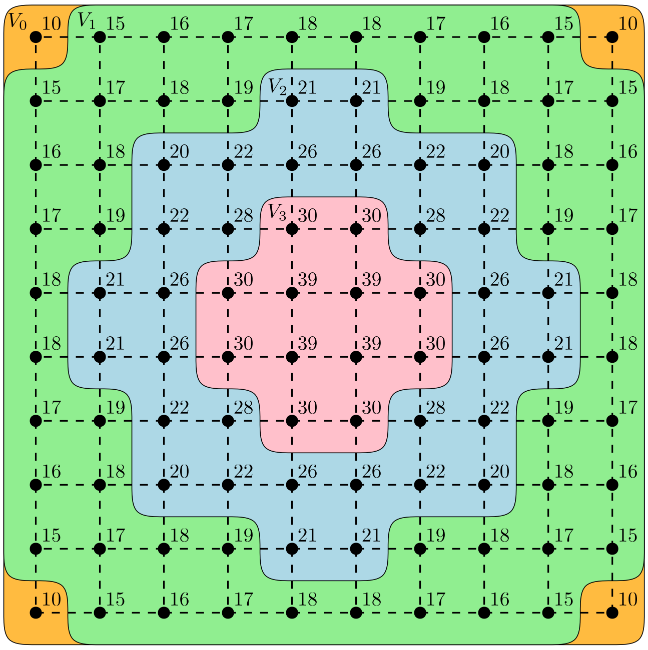


Figure 4.3.: Illustration of the proof of Theorem 4.4. Each vertex v in the grid $\Gamma_{10,10}$ has a given weight w_v, which is displayed next to it. We note that this weight assignment is c-monotone, where c is a point in the center of the grid. The partition of vertices into sets V_0 (orange), V_1 (green), V_2 (blue) and V_3 (pink) is according to the proof. To verify that this sets are illustrated correctly, we note that $10(\sqrt{2})^0 = 10$, $10(\sqrt{2})^1 \approx 14.1$, $10(\sqrt{2})^2 = 20$, $10(\sqrt{2})^3 \approx 28.3$ and $10(\sqrt{2})^4 = 40$.

Figure 4.3 illustrates this partition.

By Lemma 4.2, for each $k \in \{0, 1, \ldots, r-1\}$ there exists a perfect 2-dimensional weighted embedding $\psi^{(k)} = (p_v^{(k)}, w_v)_{v \in V(G_k)}$ of the subgraph G_k of $\Gamma_{a,a}$ induced by V_k. Thus, there exists a perfect 2-dimensional weighted embedding $\psi = (p_v, w_v)_{v \in V(H)}$ of the graph H defined by $V(H) := V(\Gamma_{a,a})$ and

$$E(H) := E(G_0) \cup \cdots \cup E(G_{r-1}).$$

Now, it holds that $E(\psi) = E(H) \subseteq E(\Gamma_{a,a})$ and thus $E(\psi) \setminus E(\Gamma_{a,a}) = \emptyset$. Additionally,

$$\begin{aligned} E(\Gamma_{a,a}) \setminus E(\psi) &= \{\{u, v\} \in E(\Gamma_{a,a}) \mid u \in V_k, v \in V_{k'}, k \neq k'\} \\ &= \bigcup_{k=0}^{r-1} \underbrace{\{\{u, v\} \in E(\Gamma_{a,a}) \mid u \in V_0 \cup \cdots \cup V_k, v \in V_{k+1} \cup \cdots \cup V_{r-1}\}}_{=:\tilde{E}_k}. \end{aligned} \tag{4.1}$$

Observe that $V_0 \cup \cdots \cup V_k$ is, by construction, a w-upset. By Lemma 4.8 follows, that $|\tilde{E}_k| \leq 4a$. With Equation 4.1 follows that

$$|E(\Gamma_{a,a}) \setminus E(\psi)| \leq \sum_{k=0}^{r-1} |\tilde{E}_k| \leq r \cdot 4a \in \mathcal{O}(\log\left(\frac{w_{\max}}{w_{\min}}\right)\sqrt{n}).$$

∎

5. Specific Weight Assignments

In the previous two chapters, we discussed criteria for weights that yield embeddings with small error for complete trees and grids, respectively. The main goal of this work is to find a general procedure that assigns each vertex v of an arbitrary graph G a weight, such that a good embedding (i.e. an embedding with low error), using those weights, exists for as many G as possible.

Given a procedure, it seems impossible (or at least very hard), to assess whether it generally yields good embeddings for all graphs. Instead, we will focus on a few basic graph classes and show the suitability of the procedure on them. We then hope that the procedure also yields good embeddings for other graphs.

The two classes of graphs we will focus on in this chapter are complete trees and grids. We require that a suitable procedure yields perfect 1-dimensional embeddings of complete trees T_h^b and 2-dimensional embeddings with error at most $\mathcal{O}(\sqrt{n})$ for all grids $\Gamma_{a,a}$. The second requirement may seem a bit arbitrary, however it is motivated by the fact that using $(\deg v)^\beta$ (where β is a constant) as weights in a grid results in embeddings with a total error of $\mathcal{O}(\sqrt{n})$ (see further below). The degree centrality functions as our baseline procedure as it was introduced in the original paper [BHKM24], thus we require an alternative procedure to yield an embedding with asymptotically at most the same error.

We state the goal from above more formally: A *weight-setter* (or *(weight-)procedure*) is a function f that maps each pair (v, G) to a positive real number $f(v, G) \in \mathbb{R}_+$, where G is any graph and $v \in V(G)$. We desire a weight-setter f that satisfies the requirements

- for all $h, b \in \mathbb{N}_+$ there exist position $(p_v \in \mathbb{R})_{v \in T_h^b}$ such that $\psi = (p_v, f(v, T_h^b))_{v \in V(T_h^b)}$ is perfect and
- for all $a \in \mathbb{N}_+$ there exist position $\left(p_v \in \mathbb{R}^2\right)_{v \in \Gamma_{a,a}}$ such that $\psi = (p_v, f(v, \Gamma_{a,a}))_{v \in V(\Gamma_{a,a})}$ has total error $\mathcal{O}(\sqrt{n})$.

Even though not formally stated, we wish for f to not just be a case distinction (distinguishing between grids and complete graphs) as this would probably not yield any desirable embeddings of graphs outside those two classes of graphs.

Before we start discussing some concrete weight-setters, we note that a suitable weight-setter f must satisfy Corollary 3.1 and thus $q_f := \frac{\max_{v \in V(T_h^2)} f(v, T_h^2)}{\min_{v \in V(T_h^2)} f(v, T_h^2)} \in \Omega\big((\sqrt{2})^h\big)$ on binary trees. However, some weight-setter will not satisfy this, but only $q_f \in \Omega(y^h)$ for another basis $y > 1$. Even though that disqualifies f, we can instead consider the modified weight-setter f' with $f'(v, G) = f(v, G)^\beta$ for some constant $\beta > 1$. In that case $q_{f'} \in \Omega\big((y^\beta)^h\big)$, which satisfies the condition of Corollary 3.1 for sufficiently large β. However, note that if q_f is sub-exponential, then so is $q_{f'}$ and this trick fails.

We will now discuss some possible weight-setters and whether they satisfy the 2 requirements stated above. Promising candidates as weight-setters are measures that measure the centrality of a vertex in a graph. Intuitively speaking, this is because the weight of a vertex in an embedding with low error often correlates with the centrality of that vertex (e.g. compare to trees or GIRGs). We note that the centrality measures mentioned are already well-studied [New16].

Degree Centrality A possible such centrality measure is the *degree centrality* $f(v, G) = \deg(v)^\beta = \deg_G(v)^\beta$. This measure was used as a weight-setter in [BHKM24], where it was experimentally shown to yield embeddings of GIRGs with low error. Furthermore, $\deg_G(v)^\beta$ satisfies all requirements of Corollary 4.5 and thus yields embeddings with error $\mathcal{O}(\sqrt{n})$ for grids $\Gamma_{a,a}$ for all $a \in \mathbb{N}_+$. However, like also discussed in [BHKM24], this weight-setter does not yield embeddings with low error on complete binary trees. We can proof formally that no perfect weighted embeddings of T_h^2 exists (for sufficiently high h), by observing that $\frac{\max_{v \in V(T_h^2)} \deg v}{\min_{v \in V(T_h^2)} \deg v} = \frac{3}{1} \in \Theta(1)$ for growing h and applying Corollary 3.2. Thus, $f(v, G) = \deg_G(v)$ does not satisfy our goals.

Closeness Centrality The *closeness centrality* [Bav50] is defined as

$$C_{\mathrm{cl}}(v, G) := \sum_{\substack{u \in V(G) \\ \mathrm{dist}_G(u,v) < \infty}} \mathrm{dist}_G(u, v).$$

Closeness centrality has a similar problem as degree centrality: Consider $G = T_h^2$ and any vertex $v \in V(G)$. Note that at least half the leafs have distance at least h from v. Thus,

$$C_{\mathrm{cl}}(v, G) = \sum_{u \in V(G)} \mathrm{dist}_G(u, v) \geq \sum_{\substack{u \in V(G) \\ \mathrm{dist}_G(u,v) \geq h}} \mathrm{dist}_G(u, v) \geq \sum_{\substack{u \in V(G) \\ \mathrm{dist}_G(u,v) \geq h}} h = \frac{2^h}{2} h$$

On the other hand,

$$C_{\mathrm{cl}}(v, G) = \sum_{u \in V(G)} \mathrm{dist}_G(u, v) \leq \sum_{u \in V(G)} \mathrm{diam}\, T_h^2 = n \,\mathrm{diam}\, T_h^2 = (2^{h+1} - 1)(2h) \leq 4 \cdot 2^h h.$$

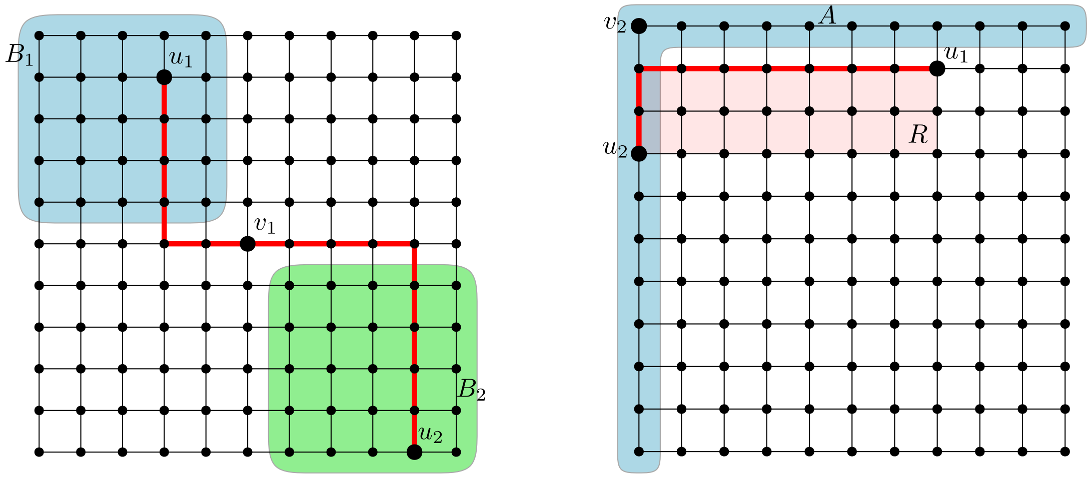
This implies $\frac{\max_{v \in V(T_h^2)} C_{\mathrm{cl}}(v, T_h^2)}{\min_{v \in V(T_h^2)} C_{\mathrm{cl}}(v, T_h^2)} \leq 8 \in \Theta(1)$, which yields the same problem as for degree centrality.

Inbetweenness Centrality When considering *inbetweenness centrality* [Fre77], a vertex is called central, if it lies on many shortest paths between two other vertices. There are multiple variants of this measure. We define it as follows:

$$C_{\mathrm{in}}(v, G) := |\{(u_1, u_2) \in V(G)^2 \mid u_1 \neq v \neq u_2, p \text{ is a shortest path between } u_1 \text{ and } u_2, v \text{ lies on } p\}|.$$

Note that $0 \leq C_{\mathrm{ib}}(v, G) \leq (n-1)(n-2) < n^2$, where $n := |V(G)|$. We will first analyze $C_{\mathrm{in}}(v, G)$ on the grid $\Gamma_{a,a}$ for some odd $a \in \mathbb{N}_+$. Let $v_1 = (v_1^x, v_1^y)$ be the vertex in the center of the $\Gamma_{a,a}$. Observe that for any $u_1 = (u_1^x, u_1^y), u_2 = (u_2^x, u_2^y) \in V(\Gamma_{a,a})$ with $u_1^x < v_1^x < u_2^x$ and $u_1^y < v_1^y < u_2^y$, there exists a shortest path from u_1 to u_2 that contains v_1. For an illustration see Figure 5.1a. Thus,

$$C_{\mathrm{ib}}(v_1, \Gamma_{a,a}) \geq \left(\frac{a-1}{2}\right)^2 \cdot \left(\frac{a-1}{2}\right)^2 \in \Theta(a^4) = \Theta(n^2).$$



(a) The set B_1 (highlighted in blue) contains all vertices that are to the upper-left of the center v_1. B_2 (highlighted in green) contains all vertices that are to the lower-right of v_1. For any $u_1 \in B_1$ and $u_2 \in B_2$, there exists a shortest path from u_1 to u_2 that passes through the center v_1. Such a shortest path between two vertices u_1 and u_2 is illustrated in fat red in the figure.

(b) If any of two vertices u_1, u_2 is not contained in the highlighted set A, then no path from u_1 to u_2 pass through the corner v_2: In this example, u_2 is contained in A, but u_1 is not. A shortest path between u_1 and u_2 is highlighted in bold red. Any shortest path between u_1 and u_2 is fully contained by the rectangle R highlighted in light red. Note, that v_2 is never contained in that rectangle.

Figure 5.1.: Illustrates the difference between the inbetweenness centrality of the central vertex and a corner vertex in the grid $\Gamma_{11,11}$.

On the other hand, consider the vertex v_2 in the upper left corner of $\Gamma_{a,a}$. Let A be the set of all vertices that have no vertex above them or no vertex to the left of them. For any two vertices u_1, u_2 with $u_1 \notin A$ or $u_2 \notin A$, there exists no shortest path from u_1 to u_2 that contains v_2 (see Figure 5.1b). Thus,

$$C_{\text{ib}}(v_2, \Gamma_{a,a}) \leq |A|^2 = a^2 = n.$$

Hence,

$$\frac{\max_{v \in V(\Gamma_{a,a})} C_{\text{ib}}(v, \Gamma_{a,a})}{\min_{v \in V(\Gamma_{a,a})} C_{\text{ib}}(v, \Gamma_{a,a})} \geq \frac{\Theta(n^2)}{n} = \Theta(n).$$

In turn, Corollary 4.5 cannot be used to show the existence of an embedding with total error $\mathcal{O}(\sqrt{n})$ with weights $C_{\text{ib}}(\cdot, G)$. The lowest with Theorem 4.4 achievable upper bound for the total error is $\mathcal{O}(\sqrt{n} \log n)$, assuming we could show that $C_{\text{ib}}(\cdot, G)$ is v_1-monotone. This does not mean, that no embedding with smaller error exists, just that we currently have no way of showing the opposite. Since we are interested in a weight-setter that provably satisfies our goals, we dismiss C_{ib} for now.

***k*-Hop Centrality** For a graph G, $k \in \mathbb{N}_0$ and $v \in V(G)$ we define the *k-hop neighborhood* of v as

$$\text{Hop}_k(v) := \text{Hop}_k(v, G) := \{u \in V(G) \mid \text{dist}_G(u, v) \leq k\}.$$

and call $\mathrm{hop}_k(v) := \mathrm{hop}_k(v, G) := |\mathrm{Hop}_k(v, G)|$ the *k-hop centrality* [NFWS14] of v. For any $k = k(G)$, $\mathrm{hop}_k(\cdot)$ is a weight-setter. We will show that a variant of $\mathrm{hop}_k(\cdot)$ satisfies all the requirements we stated above, if we choose k carefully: In the next section, we will analyze the k-hop centrality on trees and find a variation that yields perfect embeddings on them. After that, we will show that this variation, also yields satisfying results on grids.

5.1. k-Hop Centrality on Complete Trees

5.1.1. Choice of k

We start by analyzing $\mathrm{hop}_k(v)$ as a weight-setter on the complete b-ary tree $G = T_h^b$ for different choices of $k = k(G)$. We will see that most choices of k immediately lead to undesirable results, namely the nonexistence of a perfect embedding with the given weights. We consider $b \in \mathbb{N}_+ \setminus \{1\}$ as fixed. Throughout this subsection, all bounds in big $\mathcal{O}$ notation are given for increasing h while keeping b constant.

For instance, consider any constant $k \in \Theta(1)$. Since the maximal degree of T_h^b is $b+1 \in \Theta(1)$ (as h increases, recall that b is considered fixed), for all $v \in V(T_h^2)$,

$$1 \leq \mathrm{hop}_k(v) \leq (b+1)^k \in \Theta(1)^{\Theta(1)} = \Theta(1).$$

Hence, by Theorem 3.2, no perfect weighted embedding of T_h^b with weights $\mathrm{hop}_k(\cdot)$ exists (for sufficiently high h).

Thus, we must consider non constant $k(G)$. However, we will observe in the following Lemma that only few choices of k make $\mathrm{hop}_k(\cdot)$ a suitable weight-setter. For the sake of generality, we claim this not just for $\mathrm{hop}_k(\cdot)$, but for $\mathrm{hop}_k(\cdot)^\beta$ for all constant $\beta \geq 0$. We will discuss later, why $(\cdot)^\beta$ is a useful modification to the k-hop centrality.

Lemma 5.1: *For all constant $b \in \mathbb{N}_+ \setminus \{1\}$, $\beta \geq 0$ and $d \in \mathbb{N}_+$, there exist constants c_1, c_2 such that for all $k = k(T_h^b) \notin [h - c_1, h + c_2]$ and sufficiently high h, there exists no perfect d-dimensional weighted embedding of T_h^b with weights $\mathrm{hop}_k(\cdot)^\beta$.*

Proof. First, consider any k with $k(T_h^b) > h$ and $k - h \in \omega(1)$ (as h increases). Then, let V' be the set of all vertices of T_h^b with distance at most $k - h$ to the root ε. Now, for all $v \in V'$ and $u \in V(T_h^b)$,

$$\mathrm{dist}_G(u, v) \leq \underbrace{\mathrm{dist}_G(u, \varepsilon)}_{\leq k-h} + \underbrace{\mathrm{dist}_G(\varepsilon, v)}_{\leq h} \leq k - h + h = k.$$

Thus, $\mathrm{hop}_k(v) = n$ for all $v \in V'$. If we assume the existence of a perfect embedding with weights $w_v = \mathrm{hop}_k(v, T_h^b)^\beta$ of T_h^b for all $h \in \mathbb{N}_+$, then there also exists a perfect embedding with the same weights w_v of the subtree of T_h^b that is induced by V'. However, as all vertices in V' have the same weight $w_v = n^\beta$, this contradicts Theorem 3.2. Hence, if $k(T_h^b) > h$ either $k - h \in \mathcal{O}(1) \leq c_2$ for a constant c_2 or $\mathrm{hop}_k(v, T_h^b)^\beta$ yields no perfect embeddings of T_h^b for sufficiently large h. So, $k > h + c_2$ implies the latter.

Second, consider any k with $k(T_h^b) < h$ and $h' := h - k \in \omega(1)$. Similarly as above, let V'' be the set of all vertices with distance at most h' to the root ε. Furthermore, let $n_k := |V(T_k^b)|$ be the number of vertices in a complete b-ary tree of height k. First note, that $\mathrm{hop}_k(v) \geq n_k$, since $\mathrm{Hop}_k(v)$ contains at least all vertices in the subtree of height k rooted in v. For any

$v \in V''$ and $u \in \mathrm{Hop}_k(v)$, there exists a unique vertex v' such that v' is an ancestor of v and of u and lies on the shortest path from v to u (This is a general property of trees). We note, that a fixed $v \in V''$ has at most h' ancestors and any fixed ancestor v' of v has at most n_k descendants u with $\mathrm{dist}_G(u, v') \leq k$. Thus, $\mathrm{hop}_k(v) \leq h'n_k$. It follows

$$\frac{\max_{v \in V''} \mathrm{hop}_k(v)^\beta}{\min_{v \in V''} \mathrm{hop}_k(v)^\beta} \leq \left(\frac{h'n_k}{n_k}\right)^\beta = (h')^\beta$$

As $(h')^\beta \notin \Omega(b^{h'/2})$, Theorem 3.2 implies the non-existence of a perfect embedding of the subgraph of T_h^b induced by V'' with weights $\mathrm{hop}_k(\cdot, T_h^b)^\beta$ (for sufficiently high h'). Thus, T_h^b has no such embedding too (for sufficiently high h). Hence, there exists a constant c_1 such that $k(T_h^b) < h - c_1$ implies the non-existence of a perfect weighted embedding of T_h^b with weights $\mathrm{hop}_k(\cdot)^\beta$ (for sufficiently high h). ■

This lemma severely restricts the choice of $k(T_h^b)$, as only $k(T_h^b) = h + c$ is feasible where $c \in [-c_1, c_2] \cap \mathbb{Z}$. We could explicitly calculate c_1 and c_2 and check the suitability of the finite remaining options. Instead, however, we will directly take a closer look at the arguably most natural choice of $k(T_h^b)$, namely $k(T_h^b) = h$.

5.1.2. Calculation of the k-Hop Centrality

So far, we have shown that there are only few choices of $k(T_h^b)$ that make $\mathrm{hop}_k(\cdot)^\beta$ a suitable weight-setter for our purposes. One of those choices is $k(T_h^b) = h$, on which we will focus from now on. We will try to show that in this case $\mathrm{hop}_k(\cdot)$ yields perfect weighted embeddings on complete b-ary trees. For that reason, we will explicitly calculate the h-hop centrality for all vertices in T_h^b.

We fix h and b and consider the complete b-ary tree $G = T_h^b$. Let v_h be any fixed leaf of T_h^b and $(v_h, v_{h-1}, \ldots, v_2, v_1, v_0 = \varepsilon)$ be the shortest path from v_h to the root v_ε. Note that $\mathrm{dist}_G(v_i, \varepsilon) = i$ for all $i \in \{0, \ldots, h\}$. Furthermore, for any vertex u with $\mathrm{dist}_G(u, \varepsilon) = i$, by reasons of symmetry, $\mathrm{hop}_h(u) = \mathrm{hop}_h(v_i)$. Hence, it suffices to evaluate $\mathrm{hop}_h(v_i)$ for all i. For ease in notation, let $n_{h'} = |V(T_h^b)|$ denote the number of vertices in a complete b-ary tree of height h'.

Lemma 5.2: *For all $b \geq 2$ and $0 \leq i \leq h$,*

$$\mathrm{hop}_h(v_i) = \begin{cases} \frac{b^{h-i/2}(b+1) - b^{h-i} - 1}{b-1} & \text{if } i \text{ is even} \\ \frac{2b^{h-(i-1)/2} - b^{h-i} - 1}{b-1} & \text{if } i \text{ is odd.} \end{cases}$$

Proof. Let $i \in \{0, \ldots, h\}$ be arbitrary, but fixed. We will now partition $\mathrm{Hop}_h(v_i)$ into 3 parts, as can be seen in Figure 5.2. First, note that all vertices in the subtree rooted in v_i are in the h-hop neighborhood of v_i. This subtree has exactly n_{h-i} vertices. Furthermore, for each $0 \leq j < i$, the vertex v_j is also contained in $\mathrm{Hop}_h(v_i)$. Additionally, for each $0 \leq j < i$, let $u \neq v_{j+1}$ be any child of v_j (there are $(b-1)$ such childs) and let u' be any descendant of u. Then

$$\begin{aligned} \mathrm{dist}_G(u', v_i) \leq h \iff& \mathrm{dist}_G(u', u) + \underbrace{\mathrm{dist}_G(u, v_j)}_{=1} + \underbrace{\mathrm{dist}_G(v_j, v_i)}_{=i-j} \leq h \\ \iff& \mathrm{dist}_G(u', u) \leq h - i + j - 1. \end{aligned}$$

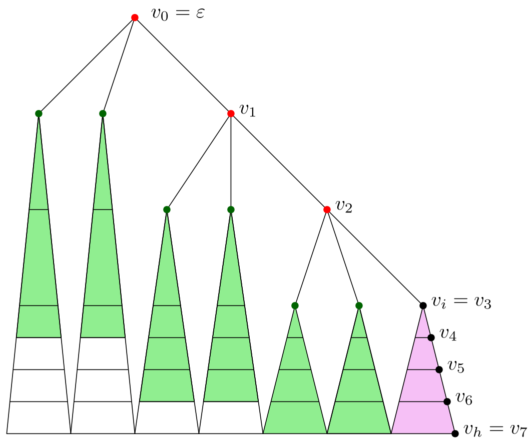


Figure 5.2.: This shows a complete 3-ary tree T_7^3 of height 7. For visibility, not all vertices have been drawn individually. Each triangle in the figure represents a subtree of T_7^3. We set $i = 3$ and consider the set $\mathrm{Hop}_h(v_i)$. All vertices in the subtree below v_i (highlighted in violet) are in $\mathrm{Hop}_h(v_i)$. All vertices v_j with $j < i$ (highlighted in red) are in $\mathrm{Hop}_h(v_i)$. The remaining vertices in $\mathrm{Hop}_h(v_i)$ are descendants of the vertices v_i and are highlighted in green.

Since $\mathrm{dist}_G(u, \varepsilon) = j + 1$ (and thus has $n_{h-(j+1)} = n_{h-j-1}$ descendants, including u itself), it has exactly $n_{\min\{h-i+j-1,h-j-1)\}}$ descendants with $\mathrm{dist}_G(u', u) \leq h - i + j - 1$. In total we get

$$\begin{aligned}\mathrm{hop}_h(v_i) &= n_{h-i} + \sum_{j=0}^{i-1} 1 + (b-1)n_{\min\{h-i+j-1,h-j-1\}} \\ &= n_{h-i} + \sum_{j=0}^{\lceil i/2\rceil -1} 1 + (b-1)n_{h-i+j-1} + \sum_{j=\lceil i/2\rceil}^{i-1} 1 + (b-1)n_{h-j-1},\end{aligned} \tag{5.1}$$

where we used that $h - i + j - 1 \leq h - j - 1 \iff j \leq i/2$. We recall that

$$n_{h'} = \sum_{j=0}^{h'} b^j = \frac{1 - b^{h'+1}}{1-b}$$

and thus

$$1 + (b-1)n_{h'} = 1 - (1-b)\frac{1-b^{h'+1}}{1-b} = 1 - (1 - b^{h'+1}) = b^{h'+1}. \tag{5.2}$$

We now find simpler expressions for the three terms in Equation 5.1 separately:

$$\begin{aligned}n_{h-i} &= \frac{1-b^{h-i+1}}{1-b} = \frac{b^{h-i+1}-1}{b-1} \\ \sum_{j=0}^{\lceil i/2\rceil -1} 1 + (b-1)n_{h-i+j-1} &\overset{(5.2)}{=} \sum_{j=0}^{\lceil i/2\rceil -1} b^{h-i+j} = b^{h-i} \sum_{j=0}^{\lceil i/2\rceil -1} b^j = b^{h-i}\frac{b^{\lceil i/2\rceil}-1}{b-1} \\ \sum_{j=\lceil i/2\rceil}^{i-1} 1 + (b-1)n_{h-j-1} &\overset{(5.2)}{=} \sum_{j=\lceil i/2\rceil}^{i-1} b^{h-j} = b^h \sum_{j=\lceil i/2\rceil}^{i-1} (b^{-1})^j = b^h \frac{b^{-\lceil i/2\rceil} - b^{-i}}{1-b^{-1}} \\ &= b^{h+1}\frac{b^{-\lceil i/2\rceil} - b^{-i}}{b-1}\end{aligned}$$

Putting them together, we have

$$\begin{aligned}
\mathrm{hop}_h(v_i) &= \frac{b^{h-i+1} - 1 + b^{h-i}(b^{\lceil i/2 \rceil} - 1) + b^{h+1}(b^{-\lceil i/2 \rceil} - b^{-i})}{b-1} \\
&= \frac{b^{h-i+1} - 1 + b^{h-i+\lceil i/2 \rceil} - b^{h-i} + b^{h+1-\lceil i/2 \rceil} - b^{h-i+1}}{b-1} \\
&= \frac{b^{h-i+\lceil i/2 \rceil} + b^{h+1-\lceil i/2 \rceil} - b^{h-i} - 1}{b-1} \\
&= \begin{cases} \frac{b^{h-i/2}(b+1) - b^{h-i} - 1}{b-1} & \text{if } i \text{ is even} \\ \frac{2b^{h-(i-1)/2} - b^{h-i} - 1}{b-1} & \text{if } i \text{ is odd.} \end{cases}
\end{aligned}$$

∎

With this explicit formula, we can calculate (assuming h is even),

$$\begin{aligned}
\frac{\max_{v \in V(T_h^b)} \mathrm{hop}_h(v)}{\min_{v \in V(T_h^b)} \mathrm{hop}_h(v)} &= \frac{\mathrm{hop}_h(v_0)}{\mathrm{hop}_h(v_h)} = \frac{\frac{b^{h-0/2}(b+1) - b^{h-0} - 1}{b-1}}{\frac{b^{h-h/2}(b+1) - b^{h-h} - 1}{b-1}} = \frac{b^h(b+1) - b^h - 1}{b^{h/2}(b+1) - 2} \\
&= \frac{b^{h+1} - 1}{b^{h/2}(b+1) - 2} \in \Theta\left(\frac{b^{h+1}}{b^{h/2+1}}\right) = \Theta(b^{h/2}). \qquad (5.3)
\end{aligned}$$

We get the same result, if we assume h is odd. This is great news, as it satisfies the necessary asymptotic condition of Theorem 3.2. This shows, that $\mathrm{hop}_h(\cdot)$ respects the need of exponential decay of weights in complete b-ary trees.

However, as Theorem 3.2 is only a necessary condition, this does not prove the existence of a perfect embedding of T_h^b with those weights. For that, we must show that $\mathrm{hop}_h(v)^\beta$ satisfies the sufficient conditions of Corollary 3.4 (for $b = 2$) and Theorem 3.7 (for $b \geq 2$). The latter requires us, to show that $\frac{\mathrm{hop}_h(v_i)}{\mathrm{hop}_h(v_{i+1})} > \xi_b$ for all i, where $\xi_b < b + 1$ is a real number that only depends on b (compare to Definition 3.6).

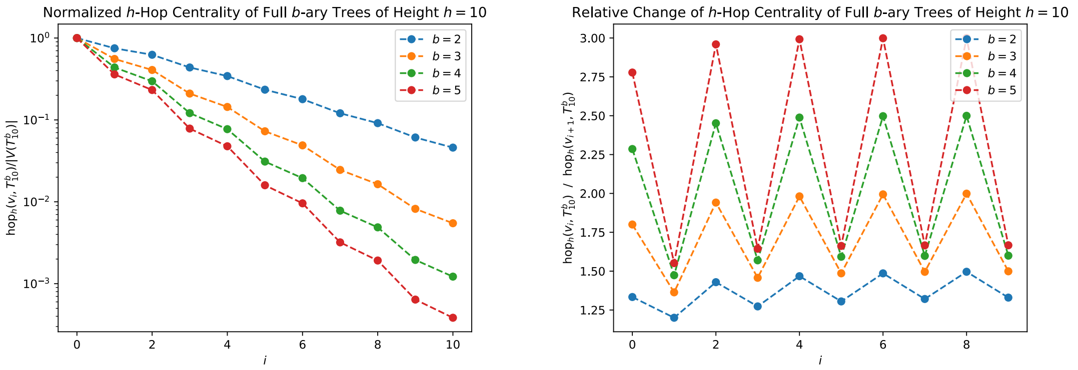
If we now (wrongfully) assume, that $x := \frac{\mathrm{hop}_h(v_i)}{\mathrm{hop}_h(v_{i+1})}$ is identical for all i, we could follow from Equation 5.3 that

$$x^h \geq x^{h-1} = \prod_{i=0}^{h-1} \frac{\mathrm{hop}_h(v_i)}{\mathrm{hop}_h(v_{i+1})} = \frac{\mathrm{hop}_h(v_0)}{\mathrm{hop}_h(v_h)} \in \Theta(b^{h/2})$$

and thus,

$$x \in \sqrt[h]{\Theta(b^{h/2})} = \Theta\left(b^{\frac{h}{2h}}\right) = \Theta(\sqrt{b}),$$

as b increases. For sufficiently high $\beta \geq 2$, this would allow us to apply Theorem 3.7 on $\mathrm{hop}_h(v)^\beta$. The β in that case would allow us to increase the exponential growth factor of $\mathrm{hop}_h(v_i)$ from $\Theta(\sqrt{b})$ to ξ_b. However, the assumption that the fraction $\frac{\mathrm{hop}_h(v_i)}{\mathrm{hop}_h(v_{i+1})}$ is equal for all i, is wrong. In reality, the fraction changes drastically, depending on whether i is even or odd. Figure 5.3 illustrates this behavior.



(a) The diagram illustrates for multiple values of b and fixed $h = 10$, how the value $\mathrm{hop}_h(v_i, T_h^b)$ changes for growing i. For reasons of readability, the y-axis is logarithmic and the h-hop centrality was normalized by the total number of vertices in T_h^b. Note, that if $\mathrm{hop}_h(v_i)$ was purely exponential, the logarithmic plot would be a straight line with a constant slope. Here, however, the slope of a line segment alternates between two values. Figure 5.3b shows the slope in more detail.

(b) This diagram illustrates $q_i := \frac{\mathrm{hop}_h(v_i, T_h^b)}{\mathrm{hop}_h(v_{i+1}, T_h^b)}$ for different values of b and fixed $h = 10$, as i increases. We notice that for fixed b, q_i takes vastly different values, depending on whether i is even or odd. We also observe that for a fixed odd i, q_i takes similar values for different b.

Figure 5.3.: Illustration of the h-hop centrality on the tree T_{10}^b for different values of b.

The problem of this already becomes apparent, when focusing on the fraction for $i = 1$. Apply Lemma 5.2:

$$\frac{\mathrm{hop}_h(v_1)}{\mathrm{hop}_h(v_2)} = \frac{2b^{h-(1-1)/2} - b^{h-1} - 1}{b^{h-2/2}(b+1) - b^{h-2} - 1} = \frac{2b^h - b^{h-1} - 1}{b^{h-1}(b+1) - b^{h-2} - 1}$$
$$= \frac{b^h(2 - b^{-1} - b^{-h})}{b^h(1 + b^{-1} - b^{-2} - b^{-h})} \xrightarrow{(h\to\infty)} \frac{2 - b^{-1}}{1 + b^{-1} - b^{-2}} < \frac{2 - \overbrace{b^{-1}}^{>0}}{1 + \underbrace{b^{-1}}_{>b^{-2}} - b^{-2}} < 2. \tag{5.4}$$

Thus, for any constant $\beta > 0$ and sufficiently high h, we consider $b = \lceil 2^\beta \rceil$ and note that

$$\frac{\mathrm{hop}_h(v_1)^\beta}{\mathrm{hop}_h(v_2)^\beta} < 2^\beta \leq \lceil 2^\beta \rceil = b < \xi_b.$$

Thus, we can not apply Theorem 3.7 to prove the existence of a perfect weighted embedding of T_h^b with weights $\mathrm{hop}_h(\cdot)^\beta$. However, as we will show in the next subsection, a small modification of $\mathrm{hop}_h(\cdot)^\beta$ fixes this problem.

5.1.3. Smooth k-Hop Centrality

The problem of $\mathrm{hop}_h(\cdot)$ as a weight-setter is that the fraction $\frac{\mathrm{hop}_h(v_i)}{\mathrm{hop}_h(v_{i+1})}$ is much smaller for odd i, than for even i, as can be seen in Figure 5.3b. For that reason, $\mathrm{hop}_h(\cdot)^\beta$ does not satisfy the requirements of Theorem 3.7 for odd i and we can not apply this theorem.

This problem can be solved: Note that $\text{hop}_{h+1}(v_0) = \text{hop}_h(v_0)$ and for all $i > 0$, $\text{hop}_{h+1}(v_i) = \text{hop}_h(v_{i-1})$. This implies, that the fraction

$$\frac{\text{hop}_{h+1}(v_i)}{\text{hop}_{h+1}(v_{i+1})} = \frac{\text{hop}_h(v_{i-1})}{\text{hop}_h(v_i)}$$

for $i > 0$ has the exact opposite problem: For even i this fraction is significantly smaller than for odd i.

The key idea of the *smooth k-hop centrality* is to consider the geometric mean

$$\sqrt{\text{hop}_h(v) \cdot \text{hop}_{h+1}(v)}$$

between both centrality measures and hoping that the problematic effects cancel each other out. Since, we will take this centrality measure to the power of some constant, the square root has no relevance and we simply define the smooth k-hop centrality as

$$\text{s-hop}(v) := \text{s-hop}(v, T_h^b) := \text{hop}_h(v, T_h^b) \cdot \text{hop}_{h+1}(v, T_h^b).$$

We will show in this subsection, that $\text{s-hop}(\cdot)$ yields perfect weighted embeddings of T_h^b for all h and b. We first note that

$$\frac{\text{s-hop}(v_i)}{\text{s-hop}(v_{i+1})} = \frac{\text{hop}_h(v_i)\,\text{hop}_h(v_{i-1})}{\text{hop}_h(v_{i+1})\,\text{hop}_h(v_i)} = \frac{\text{hop}_h(v_{i-1})}{\text{hop}_h(v_{i+1})}$$

for $i > 1$ and

$$\frac{\text{s-hop}(v_0)}{\text{s-hop}(v_1)} = \frac{\text{hop}_h(v_0)\,\text{hop}_h(v_0)}{\text{hop}_h(v_1)\,\text{hop}_h(v_0)} = \frac{\text{hop}_h(v_0)}{\text{hop}_h(v_1)}$$

for $i = 0$. This observation allows us to find a lower bound of $\frac{\text{s-hop}(v_i)}{\text{s-hop}(v_{i+1})}$:

Lemma 5.3: *For all $h > i \geq 0$ and $b \geq 2$,*

$$\frac{\text{s-hop}(v_i)}{\text{s-hop}(v_{i+1})} \geq \begin{cases} \frac{23}{20} & \text{if } b = 2 \\ \frac{1}{2}b & \text{if } b \geq 3 \end{cases}$$

holds.

The proof of this lemma is somewhat tedious and offers little additional insight, so we defer it to the appendix (see Section A.2). This is partly because we need to distinguish between several cases for the possible values of i and b. Moreover, we are concerned with exact values rather than asymptotic results or limits. If one is only interested in correctness for sufficiently large h, i and b, a similar approach as in Equation 5.4 can be used.

Equipped with this lemma, we can finally prove that $\text{s-hop}(\cdot)$ is suitable for all complete b-ary trees:

Theorem 5.4: *For any fixed $\beta \geq 4$, there exists a perfect 1-dimensional weighted embedding $\psi = (p_v, w_v)_{v \in V(T_h^b)}$ of T_h^b with $w_v = (\text{s-hop}(v))^\beta$ for all $b \in \mathbb{N}_+, h \in \mathbb{N}_0$.*

Proof. We first assume $b \geq 3$. We will apply Theorem 3.7, where we set $\overline{w}_i := w_{v_i} = (\text{s-hop}(v)_i)^\beta$. We observe that all vertices in the same layer of T_h^b have identical k-hop centralities. Thus, $w_v = \overline{w}_{|v|}$ for all $v \in V(T_h^b)$. Let $0 \leq i < h$ be arbitrary. Now, by Lemma 5.3,

$$\frac{\overline{w}_i}{\overline{w}_{i+1}} = \left(\frac{\text{s-hop}(v_i)}{\text{s-hop}(v_{i+1})}\right)^\beta \geq \left(\frac{1}{2}b\right)^\beta \geq \left(\frac{1}{2}b\right)^4 = \frac{1}{16}\underbrace{b^3}_{\geq 3^3} b$$

$$= \frac{27b}{16} = b + \underbrace{\frac{11}{16}b}_{\geq \frac{33}{16}} \geq b + \frac{33}{16} > b + 2 > \xi_b.$$

Thus, all assumptions of Theorem 3.7 are satisfied, which yields the desired result.

Now, we focus on the case $b = 2$. Again, by Lemma 5.3,

$$\left(\frac{\text{s-hop}(v_i)}{\text{s-hop}(v_{i+1})}\right)^\beta \geq \left(\frac{23}{30}\right)^4 = 1.749\ldots > \varphi$$

for all $0 \leq i < h$. The claim follows by Theorem 3.4.

Lastly, the claim holds obviously for $b = 1$, since T_h^1 is a path, which has a trivial perfect embedding for arbitrary given weights. ∎

Remark that The lower bound of 4 for β is not optimized. It can be decreased by considering less generous estimates in Lemma 5.3, as well as Theorem 5.4. However, a lower bound of 1 is not achievable.

Additionally, remark that if weights $(w_v)_{v\in V(T_h^b)}$ yield a perfect 1-dimensional embedding of T_h^b then $(w_v^d)_{v\in V(T_h^b)}$ yields a perfect d-dimensional embedding of T_h^b (compare Section 2.3). Thus, $\text{s-hop}(\cdot)^\beta$ yields perfect d-dimensional embeddings of T_h^b, if $\beta \geq 4/d$.

In the next section and chapter, we will try to assess the suitability of $\text{hop}_h(\cdot)$ and $\text{s-hop}(\cdot)^\beta$ on other classes of graphs. Before we can do so, however, we have one remaining problem: The definition of $\text{s-hop}(v, T_h^b)$ depends on the k-hop centrality $\text{hop}_k(v, T_h^b)$ for $k \in \{h, h+1\}$. If we try to apply this definition to graphs G that are no complete trees, we notice that the height h is not necessary defined for G. Thus, we have to find a general property $p(G)$ of G, that happens to equal h, if $G = T_h^b$. The only three somewhat natural such properties $p(G)$ we found are:

- $p(G) = r(G)$, the radius of G,
- $p(G) = \left\lfloor \frac{\operatorname{diam} G}{2} \right\rfloor$, half the diameter of G, rounded down and
- $p(G) = \left\lceil \frac{\operatorname{diam} G}{2} \right\rceil$, half the diameter of G, rounded up.

As all three properties behave very similar on the graphs we consider in this work (except RGGs, see Section 6.3), we were not able to find qualitative differences of them. For no particular reason, we commit to the latter of the three options for the rest of this work. Most results, however, can be easily transferred to the other two options, except when noted otherwise.

Definition 5.5: *We define, for any G and $v \in V(G)$, the* smooth k-hop centrality *of v (in G) as*

$$\text{s-hop}(v) := \text{s-hop}(v, G) := \text{hop}_k(v)\,\text{hop}_{k+1}(v),$$

where $k := \left\lceil \frac{\operatorname{diam} G}{2} \right\rceil$.

5.2. k-Hop Centrality on Grids

In the last section, we defined s-hop$(v)^\beta$ and showed that it yields perfect embeddings of complete b-ary trees, if $\beta \geq 4$. To achieve the goal of the start of this chapter, it remains to show that s-hop$(v)^\beta$ yields 2-dimensional weighted embeddings of $\Gamma_{a,a}$ with total error $\mathcal{O}(\sqrt{n})$. To make this section more readable, we will not show this result for the *smooth* k-hop centrality s-hop$(v)^\beta$, but instead for the 'standard' k-hop centrality $\mathrm{hop}_k(\cdot)^\beta$, where $k := \left\lceil \frac{\operatorname{diam} G}{2} \right\rceil$. However, we note that the same result can be shown for s-hop$(v)^\beta$ in a very similar way.

We start by noting that $\operatorname{diam} \Gamma_{a,a} = 2(a-1)$ and thus $k = a-1$. We will use Corollary 4.5 to show the desired result. It has two requirements: First, we have to show that $\frac{\max_{v \in V(\Gamma_{a,a})} \mathrm{hop}_{a-1}(v)}{\min_{v \in V(\Gamma_{a,a})} \mathrm{hop}_{a-1}(v)}$ is constant. Second, we have to show that there exists a point $(c_x, c_y) \in \mathbb{Z}^2$, such that $\left(\mathrm{hop}_{a-1}(v)\right)_{v \in V\Gamma_{a,a}}$ is (c_x, c_y)-monotone (as defined in Definition 4.3).

We start with the second requirement.

Lemma 5.6: *For all $a \in \mathbb{N}_+$, let $(c_x, c_y) := \left(\frac{a-1}{2}, \frac{a-1}{2}\right)$. Then, $\left(\mathrm{hop}_{a-1}(v)\right)_{v \in V(\Gamma_{a,a})}$ is (c_x, c_y)-monotone.*

Proof. Let $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$ be arbitrary vertices of $\Gamma_{a,a}$ with $|x_1 - c_x| \leq |x_2 - c_x|$ and $|y_1 - c_y| \leq |y_2 - c_y|$. We need to show that $\mathrm{hop}_{a-1}(v_1) \geq \mathrm{hop}_{a-1}(v_2)$. W.l.o.g. assume that $x_1, y_1, x_2, y_2 \leq c_x = c_y$. Thus, $x_2 \leq x_1 \leq c_x$ and $y_2 \leq y_1 \leq c_y$. Let $u = (x_1, y_2)$. We will show that $\mathrm{hop}_{a-1}(u) \leq \mathrm{hop}_{a-1}(v_1)$. Figure 5.4 illustrates the remainder of this proof.

We consider the set $A := \mathrm{Hop}_{a-1}(u) \setminus \mathrm{Hop}_{a-1}(v_1)$. Furthermore, we consider the point $p = (x_1, \frac{y_1+y_2}{2})$ directly between v_1 and u. Let g be the horizontal line that goes through p. Let B be the set of all vertices above g.

We note that $A \subseteq B$, since all vertices below g are closer to v_1 than to u. Now, let A' (respectively B') be the set obtained by reflecting A (respectively B) across g. We note that still $A' \subseteq B' \subseteq V(\Gamma_{a,a})$. By symmetry, $A' \subseteq \mathrm{Hop}_{a-1}(v_1) \setminus \mathrm{Hop}_{a-1}(u)$. We conclude,

$$\begin{aligned}\mathrm{hop}_{a-1}(v_1) - \mathrm{hop}_{a-1}(u) &= |\mathrm{Hop}_{a-1}(v_1) \setminus \mathrm{Hop}_{a-1}(u)| - |\mathrm{Hop}_{a-1}(u) \setminus \mathrm{Hop}_{a-1}(v_1)| \\ &\geq |A'| - |A| = 0.\end{aligned}$$

Thus, $\mathrm{hop}_{a-1}(v_1) \geq \mathrm{hop}_{a-1}(u)$. Analogously follows $\mathrm{hop}_{a-1}(u) \geq \mathrm{hop}_{a-1}(v_2)$, which finishes the proof. ∎

We can now verify the first requirement:

Lemma 5.7: *For all $a \in \mathbb{N}_+$ and $k = a-1$,*

$$\frac{\max_{v \in V(\Gamma_{a,a})} \mathrm{hop}_k(v)}{\min_{v \in V(\Gamma_{a,a})} \mathrm{hop}_k(v)} \leq 2.$$

Proof. Let $v_{\mathrm{ct}} := \left(\left\lfloor \frac{a-1}{2} \right\rfloor, \left\lfloor \frac{a-1}{2} \right\rfloor\right)$ be a vertex in the center of $\Gamma_{a,a}$ and $v_{\mathrm{co}} = (0,0)$ be the vertex in the upper left corner of $\Gamma_{a,a}$. By Lemma 5.6 follows that v_{ct} maximizes $\mathrm{hop}_{a-1}(\cdot)$ and v_{co} minimizes $\mathrm{hop}_{a-1}(\cdot)$. Thus,

$$\max_{v \in V(\Gamma_{a,a})} \mathrm{hop}_k(v) = \mathrm{hop}_k(v_{\mathrm{ct}}) \leq n$$

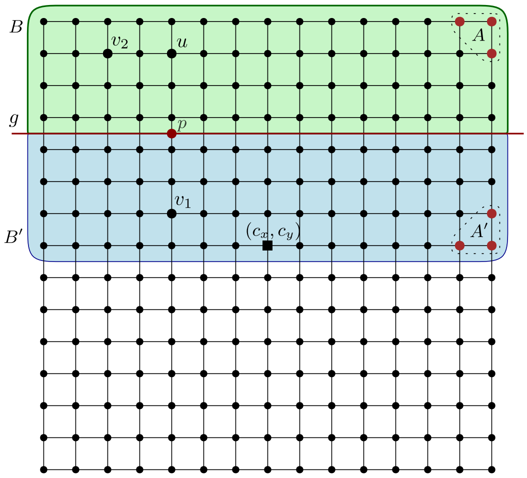


Figure 5.4.: Illustration of the proof of Lemma 5.6.

and

$$\min_{v \in V(\Gamma_{a,a})} \mathrm{hop}_k(v_{\mathrm{co}}) = \mathrm{hop}_k(v_{\mathrm{co}}) = \sum_{i=0}^{a-1} |\mathrm{Hop}_k(v_{\mathrm{co}}) \cap \{(i,j) \in V(\Gamma_{a,a}) \mid j \in \mathbb{N}_0\}|$$
$$\geq \sum_{i=0}^{a-1}(a-i) = \sum_{i=1}^{a} i = \frac{a(a+1)}{2} \geq \frac{a^2}{2} = \frac{n}{2}.$$

The claim follows. ∎

We conclude:

Corollary 5.8: *For all $\beta \geq 0$ and $a \in \mathbb{N}_+$, there exist positions $p_v \in \mathbb{R}^2$ for all $v \in V(\Gamma_{a,a})$ such that $\psi = (p_v, \mathrm{hop}_k(v)^\beta)_{v \in V(\Gamma_{a,a})}$ is a weighted embedding of $\Gamma_{a,a}$ with error $\big(\mathcal{O}(\sqrt{n}), 0\big)$, where $k := \left\lceil \frac{\operatorname{diam} G}{2} \right\rceil$.*

Proof. Note that $k = a - 1$, apply Lemma 5.7, Lemma 5.6 and Corollary 4.5 to show the claim for $\beta = 1$. Note, that $(\cdot)^\beta$ is monotone and maps constants to constants, thus the claim also holds for any other $\beta \geq 0$. ∎

Like noted in the beginning of this section, we can show that s-hop$(\cdot)^\beta$ is suitable for all grids $\Gamma_{a,a}$ in a very similar way. This has the consequence of s-hop$(\cdot)^\beta$ yielding perfect embeddings of complete b-ary trees T_h^b and embeddings of error $\big(\mathcal{O}(\sqrt{n}), 0\big)$ on grids $\Gamma_{a,a}$, when $\beta \geq 4$. Thus, we have found a weight-setter that satisfies the requirements from the start of this chapter.

6. Challenges of k-Hop Centrality

In the previous chapter, we established the weight-setter s-hop$(\cdot)^\beta$ (see Definition 5.5). We showed that it yields good embeddings for complete trees T_h^b and grids $\Gamma_{a,a}$, which are the classes of graphs we set our main focus on. In this chapter, however, we will focus on some other, more generic classes of graphs and examine if s-hop$(\cdot)^\beta$ also yields good embeddings on those. We will see that this is not necessary the case. In particular, we will focus on arbitrary trees (Section 6.1), subgraphs of grids (Section 6.2) and unit disk graphs (Section 6.3).

6.1. Arbitrary Trees

Corollary 3.8 shows that for any arbitrary tree T there exists a perfect 1-dimensional weighted embedding. However, when we fix the weights to s-hop$(\cdot)^\beta$, the previous chapter only showed the existence of perfect embeddings of *complete* trees T_h^b. Thus, the question remains, whether such embeddings exist for all trees. As we will show in this section, the answer to that question is no. We will introduce a counterexample.

For any $h \in \mathbb{N}_0$ and $b \in \mathbb{N}_+ \setminus \{1\}$, we define the tree $\widetilde{T}_h^b$ as the subgraph of T_h^b induced by the vertex set

$$V\left(\widetilde{T}_h^b\right) := \{\varepsilon\} \cup \{0^i x \mid i \in \{0, \dots, h-1\}, x \in \{0, 1, 2, \dots, b-1\}\}.$$

Alternatively, a graph that is isomorphic to $\widetilde{T}_h^b$ can be obtained as follow: Consider the path graph P_h of length h and name one of the two endpoints v' For each vertex $v \in V(P_h) \setminus \{v'\}$: Add $b-1$ new vertices that are only adjacent to v. Figure 6.1 illustrates the tree $\widetilde{T}_6^4$.

We will show that $w_v :=$ (s-hop$(v))^{\beta/d}$ does not yield embeddings of $\widetilde{T}_h^b$ with small error for sufficiently large b and h. The key idea of the proof is based on the observation that $\widetilde{T}_h^b$ contains multiple induced subgraphs that are isomorphic to the star graph S_{b-1} with $b-1$ edges. We will show in the next lemma, that the weights of any embedding of S_{b-1} must satisfy some necessary condition. However, we will observe that the weights given by s-hop$(\cdot)^\beta$ do not satisfy these conditions for sufficiently large b and h.

Lemma 6.1: *For $d \in \mathbb{N}_+$ and $b \in \mathbb{N}_+$, let $\psi = (p_v, w_v)_{v \in V(S_b)}$ be an arbitrary perfect d-dimensional weighted embedding of S_b. Then,*

$$\frac{w_{max}(\psi)}{w_{min}(\psi)} > \sqrt{\frac{b}{3^d}}$$

Proof. Let c be the distinct vertex of S_b with $\deg c = b$. Then $\mathrm{dist}_{S_b}(c, v) \leq 1$ for all $v \in V(S_b)$. Thus S_b has radius 1. Additionally, $V(S_b) \setminus \{c\}$ is an independent set of size b of S_b. The claim follows directly by Lemma 3.1. ∎

Lemma 6.2: *For all $i \in \{0, \dots, h-1\}$ and $x \in \{1, \dots, b-1\}$,*

$$1 \leq \frac{\text{s-hop}(0^i)}{\text{s-hop}(0^i x)} \leq 1 + \frac{8b}{h}$$

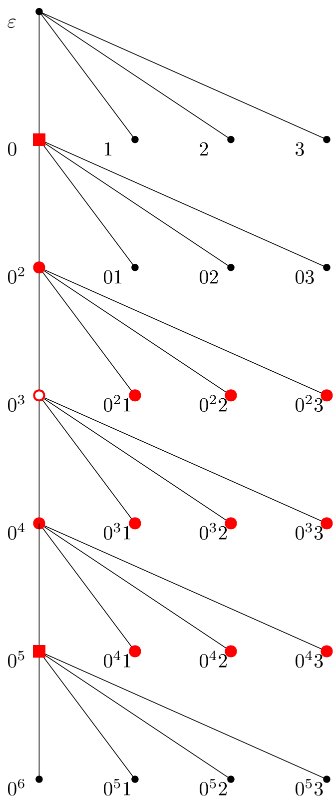


Figure 6.1.: Illustration of the tree $\widetilde{T}_6^4$. The 2-hop neighborhood of the vertex 0^3 is highlighted in red. The vertices 0 and 0^5 (marked as squares) are the only two vertices in the 2-hop neighborhood of 0^3 that have neighbors that are not in the 2-hop neighborhood of 0^3.

holds.

Proof. We first observe that for any $k' \geq 2$,

$$\mathrm{Hop}_{k'}(0^i x) = \{0^i x\} \cup \mathrm{Hop}_{k'-1}(0^i) = \mathrm{Hop}_{k'-1}(0^i).$$

Since, there exist at most 2 vertices $u_1, u_2 \in \mathrm{Hop}_{k'}(0^i)$ that have a neighbor which is not an element of $\mathrm{Hop}_{k'}(0^i)$ (see Figure 6.1), we obtain

$$\begin{aligned}\left|\mathrm{Hop}_{k'+1}(0^i)\right| &= \left|\mathrm{Hop}_{k'}(0^i)\right| + \sum_{v \in \{u_1,u_2\}} \underbrace{\left|\mathrm{Hop}_1(v) \setminus \mathrm{Hop}_{k'}(0^i)\right|}_{\leq b} \\ &\leq \left|\mathrm{Hop}_{k'}(0^i)\right| + 2b. \end{aligned} \tag{6.1}$$

We set $k = \left\lceil \frac{\operatorname{diam} \widetilde{T}_h^b}{2} \right\rceil = \left\lceil \frac{h+1}{2} \right\rceil$ and apply Inequality 6.1 twice to obtain,

$$\begin{aligned}\frac{\text{s-hop}(0^i)}{\text{s-hop}(0^i x)} &= \frac{\mathrm{hop}_k(0^i)\,\mathrm{hop}_{k+1}(0^i)}{\mathrm{hop}_k(0^i x)\,\mathrm{hop}_{k+1}(0^i x)} = \frac{\mathrm{hop}_k(0^i)\,\mathrm{hop}_{k+1}(0^i)}{\mathrm{hop}_{k-1}(0^i)\,\mathrm{hop}_k(0^i)} = \frac{\mathrm{hop}_{k+1}(0^i)}{\mathrm{hop}_{k-1}(0^i)} \\ &\leq \frac{\mathrm{hop}_{k-1}(0^i) + 2b + 2b}{\mathrm{hop}_{k-1}(0^i)} = 1 + \frac{4b}{\mathrm{hop}_{k-1}(0^i)} \\ &\leq 1 + \frac{4b}{k} = 1 + \frac{4b}{\left\lceil \frac{h+1}{2} \right\rceil} \leq 1 + \frac{4b}{\frac{h+1}{2}} \leq 1 + \frac{8b}{h}.\end{aligned}$$

Also, obviously,

$$\frac{\text{s-hop}(0^i)}{\text{s-hop}(0^i x)} = \frac{\text{hop}_{k+1}(0^i)}{\text{hop}_{k-1}(0^i)} \geq 1.$$

∎

Equipped with those two lemmas, we show:

Theorem 6.3: *For all $\beta > 0$ and $d \in \mathbb{N}_+$, choose $b = 3^d + 2$ and a sufficiently large h. Let $\psi = (p_v, w_v)_{v \in V(\widetilde{T}_h^b)}$ be an arbitrary d-dimensional weighted embedding of $\widetilde{T}_h^b$ with error (s_{fn}, s_{fp}), where $w_v = (\text{s-hop}(v))^{\beta/d}$ for all $v \in V\left(\widetilde{T}_h^b\right)$. Then, $s_{fn} + s_{fp} \geq h \in \Theta(n/b)$. In particular, ψ is not perfect.*

Proof. For an arbitrary $i \in \{0, \ldots, h-1\}$, let H_i be the subgraph of $\widetilde{T}_h^b$ induced by the vertices $V_i := \{0^i, 0^i 1, 0^i 2, \ldots, 0^i(b-1)\}$. We observe, that H_i is isomorphic to the star graph S_{b-1}. Let ψ^i be the weighted embedding of S_{b-1}, that is obtained canonically from ψ, by the isomorphism. For the sake of contradiction, we assume ψ^i to be a perfect embedding of S_{b-1}. Then, by Lemma 6.2,

$$\frac{w_{\max}(\psi^i)}{w_{\min}(\psi^i)} = \frac{w_{0^i}}{w_{0^i x}} = \left(\frac{\text{s-hop}(0^i)}{\text{s-hop}(0^i x)}\right)^{\beta/d} \leq \left(1 + \frac{8b}{h}\right)^{\beta/d} \tag{6.2}$$

for some $x \in \{1, \ldots, b-1\}$. On the other hand, by Lemma 6.1,

$$\frac{w_{\max}(\psi^i)}{w_{\min}(\psi^i)} > \sqrt{\frac{b-1}{3^d}} = \sqrt{\frac{3^d+1}{3^d}} = \sqrt{1 + \frac{1}{3^d}}. \tag{6.3}$$

We now choose h sufficiently large, such that

$$\sqrt{1 + \frac{1}{3^d}} > \left(1 + \frac{8b}{h}\right)^{\beta/d}. \tag{6.4}$$

This is possible, since $\sqrt{1 + \frac{1}{3^d}} > 1$ is independent of h and $\lim_{h\to\infty} \left(1 + \frac{8b}{h}\right)^{\beta/d} = 1$. Now, Inequalities 6.2, 6.3 and 6.4 lead to a direct contradiction. Thus, ψ^i is not perfect and there exist vertices $u, v \in V_i$ such that $\{u, v\} \in (E(G) \setminus E(\psi)) \cup (E(\psi) \setminus E(G))$ for all $i \in \{0, \ldots, h-1\}$. The claim follows by

$$s_{\text{fn}} + s_{\text{fp}} = |E(G) \setminus E(\psi)| + |E(\psi) \setminus E(G)| = |(E(G) \setminus E(\psi)) \cup (E(\psi) \setminus E(G))| \geq h.$$

∎

So, there exist trees that not only have no perfect embedding with weights $\text{s-hop}(\cdot)^\beta$, but they don't even have embeddings with sublinear error. Furthermore, we notice that alternatively the weights $\deg(\cdot)$ yields perfect 1-dimensional embeddings of the graph $\widetilde{T}_h^b$ for all h and b. This give as a first example for a graph, where setting the weights to the degree instead of using $\text{s-hop}(\cdot)^\beta$ gives (significantly) better results. This shows a big problem of using the k-hop centrality for non-constant k as weights: Intuitively speaking, the k-hop centrality only considers the global structure of the graph and loses sight over the local structure. On the other hand, $\deg v$ only considers the local structure (the direct neighborhood of v) and not the global structure of the graph. Thus, $w_v := \deg v$ is not suitable for trees with exponential growth of vertices. We claim that a weight-setter that works on all trees (if it exists) must depend not only on one of those elements, but both: the local and global structure. This thesis does not delve deeper in this idea.

6.2. Subgraphs of the Grid

In Section 5.2, we discussed why the k-hop centrality yields embeddings with total error $\mathcal{O}(\sqrt{n})$ for all grids $\Gamma_{a,a}$. One might be interested whether there also exist low-error embeddings of *grid graphs* (i.e. induced subgraphs of $\Gamma_{a,a}$). As this problem gets trivial if we allow disconnected graphs, we will focus only on connected grid graphs. We will try to answer this question, however we will examine the existence of sublinear total error ($o(n)$) instead of $\mathcal{O}(\sqrt{n})$.

We introduce some notation: Let $\mathcal{S}$ be the set of all connected graphs G, for which $a \in \mathbb{N}_+$ exists, such that G is an induced subgraph of $\Gamma_{a,a}$. Furthermore, for $G \in \mathcal{S}$, let $a(G)$ be the smallest a such that G is an induced subgraph of $\Gamma_{a,a}$.

We are not able to show the existence of embeddings with sublinear total error for all graphs in $\mathcal{S}$, but instead will try to show the existence of such embeddings for all graphs in subsets $\mathcal{F}$ of $\mathcal{S}$. We will first consider the following lemma, which is a general statement about embeddings of grid graphs with given weights, not just $\text{hop}_k(\cdot)$.

Theorem 6.4: *Let $\mathcal{F} \subseteq \mathcal{S}$ be an infinite family of graphs and $(w_v)_{v \in V(G)}$ given weights for each $G \in \mathcal{F}$ such that for all adjacent $u, v \in V(G)$, $\frac{w_u}{w_v} \leq 1 + o(1)$ for increasing $n = |V(G)|$. Furthermore, assume $|V(G)| \in \Theta\left(a(G)^2\right)$ for all $G \in \mathcal{F}$. Then, there exist positions $(p_v)_{v \in V(G)}$, such that $\psi = (p_v, w_v)_{v \in V(G)}$ is a 2-dimensional weighted embedding of G with error $(o(n), 0)$.*

Proof. Fix any $G \in \mathcal{F}$. By assumption, there exists $g \in o(1)$ such that $\frac{w_u}{w_v} \leq 1 + g$. Note, that $g^{-1} := \frac{1}{g} \in \omega(1)$. For a fixed $\gamma > 0$, let $u, u' \in V(G)$ be vertices with $k' := \text{dist}_G(u, u') \leq \gamma g^{-1}$. Then, there exists a path $(u, v_1, \ldots, v_{k'-1}, u')$ of length k' from u to u'. Thus,

$$\begin{aligned}
\frac{w_{u'}}{w_u} &= \underbrace{\frac{w_{u'}}{w_{v_{k'-1}}}}_{\leq 1+g} \cdot \underbrace{\frac{w_{v_{k'-1}}}{w_{v_{k'-2}}}}_{\leq 1+g} \cdots \underbrace{\frac{w_{v_2}}{w_{v_1}}}_{\leq 1+g} \cdot \underbrace{\frac{w_{v_1}}{w_u}}_{\leq 1+g} \\
&\leq (1+g)^{k'} \leq (1+g)^{\gamma g^{-1}} = \left(\left(1 + \frac{1}{g^{-1}}\right)^{g^{-1}}\right)^{\gamma} \\
&< \mathrm{e}^{\gamma},
\end{aligned}$$

where $\mathrm{e} := \sup_{x>0} \left(1 + \frac{1}{x}\right)^x \approx 2.718$ is Euler's number. If we set $\gamma := \ln(\sqrt{2})$, then

$$\frac{w_{u'}}{w_u} < \sqrt{2} \tag{6.5}$$

follows.

We now partition the vertices of G into subgraphs with diameter at most γg^{-1}. There will exist perfect embeddings of the individual subgraphs and we will show that only few edges between two different subgraphs exist. A suitable partition is the partition into squares with side length $s := \sqrt{\gamma g^{-1}}$:

Let $H_{i,j}$ be the subgraph of G induced by the vertices

$$V_{i,j} := \{(x, y) \in V(G) \mid x \in [is, (i+1)s), y \in [js, (j+1)s)\}$$

for all $i, j \in \{0, 1, \ldots, \left\lceil \frac{a(G)}{s} \right\rceil - 1\}$. For fixed i, j, let H' be an arbitrary connected component of $H_{i,j}$. Since

$$\text{diam}\, H' \leq |V(H')| \leq |V(H_{i,j})| \leq s^2 = \gamma g^{-1}(k),$$

we have $\frac{w_{u'}}{w_u} < \sqrt{2}$ for all vertices $u, u' \in V(H')$ by Inequality 6.5. Thus, by Lemma 4.2, there exists a perfect embedding of H' with the given weights. The existence of a perfect embedding of $H_{i,j}$ with the given weights follows directly (as $H_{i,j}$ is a union of all connected components H' of $H_{i,j}$). Let G' be the union of all graphs $H_{i,j}$. As all connected components of G' can be embedded perfectly with the given weights, there exist $p_v \in \mathbb{R}^2$ for all $v \in V(G)$, such that $\psi = (p_v, w_v)_{v \in V(G')}$ is a perfect embedding of G'.

As G' is a subgraph of G, we have

$$|E(\psi) \setminus E(G)| \leq |E(\psi) \setminus E(G')| = 0.$$

Furthermore,

$$\begin{aligned}
|E(G) \setminus E(\psi)| &\leq |\{\{u, v\} \in E(G) \mid u \in V(H_{i_1,j_1}), v \in V(H_{i_2,j_2}), (i_1, j_1) \neq (i_2, j_2)\}| \\
&= \frac{1}{2} \sum_{i_1,j_1} |\{\{u, v\} \in E(G) \mid u \in V(H_{i_1,j_1}), v \notin V(H_{i_1,j_1})\}| \\
&\leq \frac{1}{2} \sum_{i_1,j_1} 4s = \frac{1}{2} \left\lceil \frac{a(G)}{s} \right\rceil^2 4s \in \Theta\left(\frac{a(G)^2}{s}\right) = \Theta\left(\frac{n}{s}\right) \\
&= o(n).
\end{aligned}$$

■

Remark that this theorem requires $\frac{w_u}{w_v} \leq 1 + o(1)$ for all adjacent vertices $u, v \in V(G)$. However, it is actually sufficient if this inequality holds for at most $o(n)$ pairs of adjacent vertices, as noted in Section 2.3.

As another side note, if we set $\mathcal{F} = \{\Gamma_{a,a} \mid a \in \mathbb{N}_+\}$, Theorem 6.4 yields the following general criterion for the existence of embeddings of complete grids for given weights:

Corollary 6.5: *For all $a \in \mathbb{N}_+$ let $(w_v)_{v \in V(\Gamma_{a,a})}$ be weights with $\frac{w_u}{w_v} > 1 + o(1)$ for all adjacent vertices $u, v \in V(\Gamma_{a,a})$. Then there exist positions $(p_v)_{v \in V(\Gamma_{a,a})}$ such that $\psi = (p_v, w_v)_{v \in V(\Gamma_{a,a})}$ is a 2-dimensional weighted embedding of $\Gamma_{a,a}$ with error $(o(n), 0)$.*

Note, that this corollary is different from Corollary 4.6, as it does not require some form of monotonicity of the weights.

We will now apply Theorem 6.4 particularly for the case that the given weights are the k-hop centrality. Again, like in Section 5.2, we will only focus on the normal k-hop centrality $\mathrm{hop}_k(\cdot)^\beta$ (with $k = \lceil \frac{\operatorname{diam} G}{2} \rceil$) instead of the smooth k-hop centrality s-hop$(\cdot)^\beta$. However, all results in this section also apply to s-hop$(\cdot)^\beta$ and can be proven in a very similar fashion. We start by rephrasing the requirement $\frac{w_u}{w_v} \leq 1 + o(1)$ in Theorem 6.4, which is a statement about a pair of adjacent vertices, into a statement about single vertices:

Corollary 6.6: *Let $\beta \geq 0$ and $\mathcal{F} \subseteq \mathcal{S}$ be an infinite family of graphs with $|V(G)| \in \Theta\left(a(G)^2\right)$ for all $G \in \mathcal{F}$. Furthermore, assume*

$$|\{u \in V(G) \mid \mathrm{dist}_G(u, v) = k\}| \in o\left(|\{u \in V(G) \mid \mathrm{dist}_G(u, v) \leq k\}|\right) \tag{6.6}$$

for all $v \in V(G)$, where $k := \lceil \frac{\operatorname{diam} G}{2} \rceil$. Then, for all $G \in \mathcal{F}$, there exists a 2-dimensional weighted embedding $\psi = (p_v, \mathrm{hop}_k(v)^\beta)_{v \in V(G)}$ of G with error $(o(n), 0)$.

We note that Relation 6.6 is equivalent to

$$\mathrm{hop}_k(v) - \mathrm{hop}_{k-1}(v) \in o(\mathrm{hop}_k(v)). \tag{6.7}$$

Proof of Corollary 6.6. We will apply Theorem 6.4. First, let $v_1, v_2 \in V(G)$ be arbitrary adjacent vertices. Then, by the fact that $\mathrm{Hop}_{k-1}(v_1) \subseteq \mathrm{Hop}_k(v_2)$ and Relation 6.7, we obtain

$$\begin{aligned}\frac{\mathrm{hop}_k(v_1)}{\mathrm{hop}_k(v_2)} &\leq \frac{\mathrm{hop}_k(v_1)}{\mathrm{hop}_{k-1}(v_1)} = \frac{\mathrm{hop}_k(v_1) - \mathrm{hop}_{k-1}(v_1) + \mathrm{hop}_{k-1}(v_1)}{\mathrm{hop}_{k-1}(v_1)} \\ &\leq \frac{o(\mathrm{hop}_k(v_1)) + \mathrm{hop}_{k-1}(v_1)}{\mathrm{hop}_{k-1}(v_1)} \leq 1 + \frac{o(\mathrm{hop}_k(v_1))}{\mathrm{hop}_{k-1}(v_1)}.\end{aligned}$$

We note, that the maximal degree of G is 4 and thus, $\mathrm{hop}_k(v_1) \leq (4+1)\,\mathrm{hop}_{k-1}(v_1)$. It follows,

$$\frac{\mathrm{hop}_k(v_1)^\beta}{\mathrm{hop}_k(v_2)^\beta} \leq \left(1 + \frac{o(\mathrm{hop}_k(v_1))}{\mathrm{hop}_{k-1}(v_1)}\right)^\beta \leq \left(1 + \frac{o(5\,\mathrm{hop}_{k-1}(v_1))}{\mathrm{hop}_{k-1}(v_1)}\right)^\beta = (1 + o(1))^\beta = 1 + o(1).$$

Thus, we have shown all requirements of Theorem 6.4 and are done. ∎

The proof above heavily relies on the assumption that Relation 6.6 holds. For ease in notation, we will call this assumption $\mathcal{A}_2$. We denote the assumption that $|V(G)| \in \Theta\left(a(G)^2\right)$ with $\mathcal{A}_1$. Now, the usefulness of this Corollary heavily depends on the number of graphs (or more precisely, family of graphs) that satisfy assumption $\mathcal{A}_2$. For that reason, we will investigate $\mathcal{A}_2$ below.

For a fixed $a \in \mathbb{N}_+$, we consider the complete grid $\Gamma_{a,a}$. We verify easily, that $\mathrm{hop}_k(v) \in \Theta(k^2)$ for all $v \in \Gamma_{a,a}$ (and $k \in \{\lceil \frac{\operatorname{diam} G}{2} \rceil, \lceil \frac{\operatorname{diam} G}{2} \rceil + 1\}$), and that the number of vertices with distance exactly k to v is at most $4k$. This shows, that the family $\{\Gamma_{a,a} \mid a \in \mathbb{N}_+\}$ satisfies $\mathcal{A}_2$ (and obviously $\mathcal{A}_1$ too). Using Theorem 6.6 yields an alternative proof for Corollary 5.8.

With this result in mind, one might guess that any family of graphs for which $\mathcal{A}_1$ holds, does also satisfy $\mathcal{A}_2$. The existence of a family that doesn't satisfy $\mathcal{A}_2$ might even sound absurd. However, we will now show that not every family $\mathcal{F} \subseteq \mathcal{S}$ satisfying $\mathcal{A}_1$ does also satisfy $\mathcal{A}_2$:

For $s \in \mathbb{N}_+$, set $a = 2^{s+1} - 3$ and define H_s as an induced subgraph of $\Gamma_{a,a}$. The vertex set $V(H_s)$ consists of all vertices lying on paths that start at the center of $\Gamma_{a,a}$ and proceed as follows: first, take 2^{s-1} steps either left or right; then 2^{s-1} steps either up or down; followed by 2^{s-2} steps left or right; then 2^{s-2} steps up or down; continuing in this pattern until ending with 2 steps up or down. Figure 6.2 illustrates H_s. H_s is known as a *H-tree* [1] . All leafs of H_s have distance exactly

$$k := \frac{\operatorname{diam} H_s}{2} = \sum_{i=1}^{s-1} 2 \cdot 2^i = 2(2^s - 2) \tag{6.8}$$

to the center vertex v_c of $\Gamma_{a,a}$. We observe that there are exactly 2^{2s-2} leafs, since H_s is a balanced binary tree with $2s - 2$ forks. Thus,

$$\begin{aligned}&|\{u \in V(G) \mid \mathrm{dist}_G(u, v_c) = k\}| = 2^{2s-2} = 2^{-4}\left(2^{s+1}\right)^2 = 2^{-4}(a+3)^2 \quad \text{and} \\ &|\{u \in V(G) \mid \mathrm{dist}_G(u, v_c) \leq k\}| \leq a^2.\end{aligned} \tag{6.9}$$

We consider the family $\mathcal{F}_H = \{H_s \mid s \in \mathbb{N}_+\}$. The first equation implies assumption $\mathcal{A}_1$. Both equations together show that Relation 6.6 does *not* hold for the vertex v_c, which contradict $\mathcal{A}_2$. Which is the counterexample we searched for. In particular, we cannot apply Corollary 6.6 on $\mathcal{F}_H$. From this alone, however, does not follow that no embeddings for the graphs in $\mathcal{F}_H$ with sublinear error and weights $\mathrm{hop}_k(\cdot)$ exist. However, we will now modify H_s to a graph for which provably no such embeddings exist.

[1] The idea of considering H-trees as subgraphs of a grid is inspired by an answer on *math.stackexchange.com* of dtldarek [dtl17].

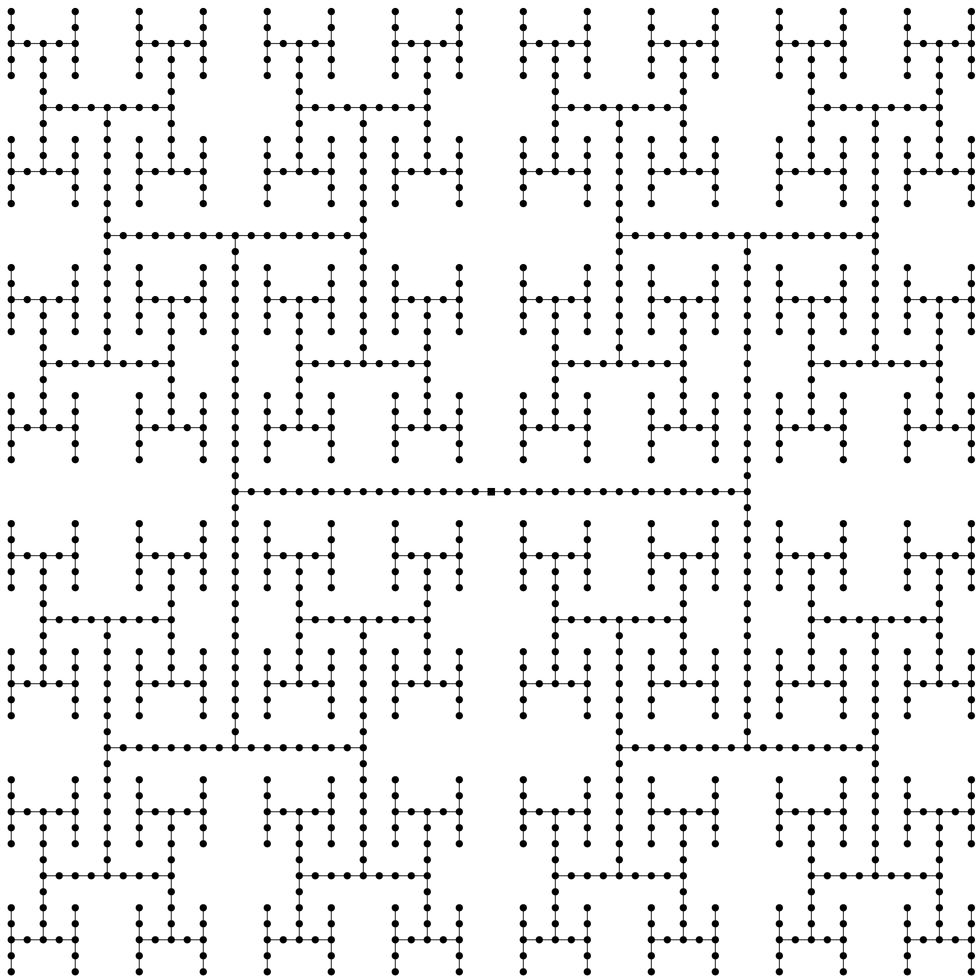


Figure 6.2.: The H-tree H_5 is illustrated in black. The fractal nature of the H-tree allows it leafs to cover a fixed proportion of the grid. This can visually be confirmed by observing that in each 4 times 4 square of the grid, there is exactly one leaf of H_5 contained. The blue crosses show a path P. The union of H_5 and P is the modified H-tree $\overline{H}_5$.

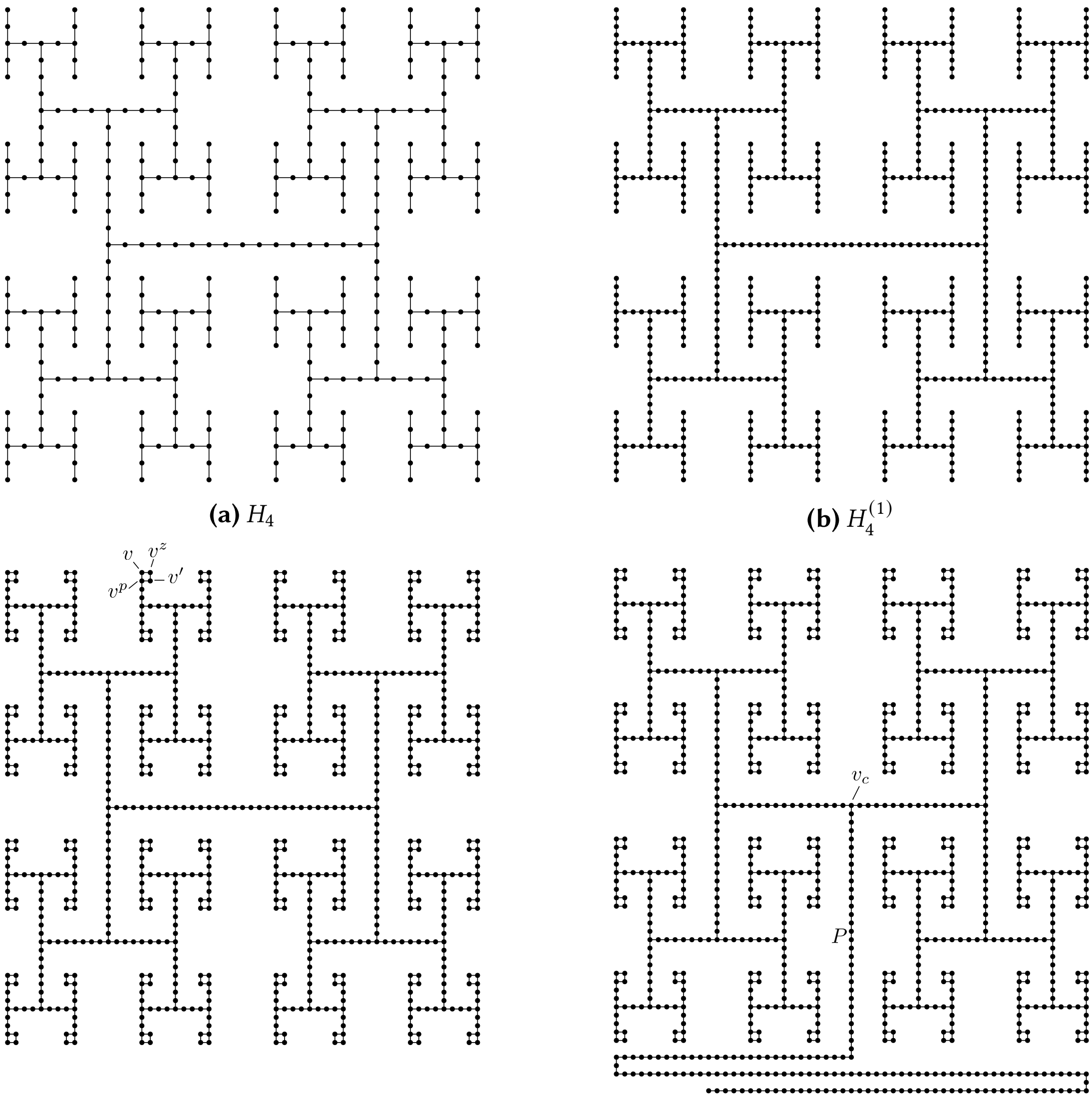
We describe this modification in 3 steps. Figure 6.3 illustrates all 3 steps of the construction. Firstly, let $H_s^{(1)}$ be the graph obtained by replacing each edge of H_s with a path P_2 of length 2. For precisely,

$$V(H_s^{(1)}) = V(H_s) \cup \{u_e \mid e \in E(H_s)\} \quad \text{and} \quad E(H_s^{(1)}) = \{\{v, u_e\} \mid e = \{v, v'\} \in E(H_s)\}$$

Secondly, let $H_s^{(2)}$ be the graph obtained in the following way: For each leaf v of $H_s^{(1)}$, let v^p be it's unique neighbor in $H_s^{(1)}$ and let v', v^z be new vertices such that $\{v, v^p, v', v^z\}$ induces a 4-cycle C_4 in $H_s^{(2)}$ (i.e. $\{v, v^p\}, \{v_p, v'\}, \{v', v_z\}, \{v_z, v\} \in E(H_s^{(2)})$ and $\{v, v'\}, \{v^p, v^z\} \notin E(H_s^{(2)})$).

Finally, let $H_s^{(3)}$ be the union of $H_s^{(2)}$ and a path P of length $6k - 1 = 12 \cdot 2^s - 25$ that starts at the center of $H_s^{(2)}$ and is otherwise disjoint from $H_s^{(2)}$ (here $k = \frac{\operatorname{diam} H_s}{2}$, which we analyzed in Equation 6.8).

We note that H_s is an induced subgraph of $\Gamma_{a,a}$ by definition. Thus, $H_s^{(1)}$ is an induced subgraphs of $\Gamma_{2a,2a}$. $H_s^{(2)}$ is still an induced subgraph of $\Gamma_{2a,2a}$, as the vertices v' and v^z in the construction can be embedded into $\Gamma_{2a,2a}$ as the vertex one to the left of v^p and v respectively



(a) H_4 (b) $H_4^{(1)}$ (c) $H_4^{(2)}$ (d) $H_4^{(3)}$

Figure 6.3.: Illustration of the construction of the graphs $H_s^{(1)}$, $H_s^{(2)}$ and $H_s^{(3)}$ for $s = 4$ and how they are induced subgraphs of a grid.

(compare Figure 6.3c). Now, $H_s^{(3)}$ is an induced subgraph of $\Gamma_{2a,2a+6}$ (which is an induced subgraph of $\Gamma_{2a+6,2a+6}$), as P can be constructed as follows: Start at the center of $H_s^{(2)}$, take $2^{s+1} - 2$ steps down, then $2^{s+1} - 4$ steps left, then 2 steps down, then $2^{s+2} - 8$ steps right, then 2 steps down, then $2^{s+2} - 19$ steps left. Now P is a path of length $12 \cdot 2^s - 25 = 6k - 1$ that does not contain any vertices of $H_s^{(2)}$ (except the central vertex). Thus, $H_s^{(3)}$ is a grid graph. Again, Figure 6.3d illustrates this construction. We now claim:

Theorem 6.7: *For all $\beta \geq 90$, $s \in \mathbb{N}_+$ and 2-dimensional weighted embedding $\psi = (p_v, \mathrm{hop}_{k'}(v)^\beta)_{v \in V(H_s^{(3)})}$ of the grid graph $H_s^{(3)}$ with error (s_{fn}, s_{fp}), it holds that $s_{fn} + s_{fp} \in \Omega(n)$, where $k' := \left\lceil \frac{\operatorname{diam} H_s^{(3)}}{2} \right\rceil$.*

The key idea behind the proof of this theorem is to realize that for each leaf v of H_s, the vertices v, v^z and v' (see construction of $H_s^{(2)}$) induce a subgraph of $H_s^{(3)}$ that is isomorphic to the 3-path P_3. Thus, before we prove Theorem 6.7, we make the following axillary claim about the 3-path P_3.

Lemma 6.8: *If $\psi = (p_v, w_v)_{v \in V(P_3)}$ is a perfect d-dimensional weighted embedding of the path P_3 with $E(P_3) = \{\{0,1\},\{1,2\}\}$ and $w_0 = w_2$, then $\frac{w_0}{w_1} < 2^d$.*

Proof. By the definition of perfect embeddings,

$$\begin{aligned}
\{0,1\} \in E(P_3) &\implies \|p_0 - p_1\| \le (w_0 w_1)^{1/d}, \\
\{1,2\} \in E(P_3) &\implies \|p_1 - p_2\| \le (w_1 w_2)^{1/d} = (w_0 w_1)^{1/d} \quad \text{and} \\
\{2,0\} \notin E(P_3) &\implies \|p_2 - p_0\| > (w_2 w_0)^{1/d} = w_0^{2/d}.
\end{aligned}$$

As p_0, p_1, p_2 are points in $\mathbb{R}^2$, the triangle inequality $\|p_2 - p_0\| \le \|p_0 - p_1\| + \|p_1 - p_2\|$ must hold, thus

$$w_0^{2/d} < 2(w_0 w_1)^{1/d},$$

which implies

$$w_0^2 < 2^d w_0 w_1 \implies \left(\frac{w_0}{w_1}\right)^2 < 2^d \frac{w_0}{w_1} \implies \frac{w_0}{w_1} < 2^d.$$

■

Proof of Theorem 6.7. We recall that the distance of each leaf of H_s to the center of H_s is exactly $k := 2(2^s - 2)$ (see Equation 6.8). Thus, the diameter of H_s is $\operatorname{diam} H_s = 2k$. We note, that the diameter is doubled in the construction of $H_s^{(1)}$, and then is increased by 2 in the construction of $H_s^{(2)}$. Thus $\operatorname{diam} H_s^{(2)} = 4k + 2$. We note, that the largest distance between any nodes in $H_s^{(3)}$ lies between the end vertex of the path P and v^z for any leaf v in $H_s^{(1)}$. Thus,

$$\operatorname{diam} H_s^{(3)} = \frac{\operatorname{diam} H_s^{(2)}}{2} + |E(P)| = \frac{4k+2}{2} + 6k - 1 = 8k.$$

So,

$$k' := \frac{\operatorname{diam} H_s^{(3)}}{2} = 4k.$$

Now let $v \in V(H_s^{(3)}$ be any vertex on the left side of the center v_c (i.e. the first edge of shortest path from the center to v goes left) with $\operatorname{dist}_{H_s^{(3)}}(v, v_c) = 2k$. Note, that v was a leaf in the original tree H_s and we constructed vertices v^p, v', v^z with $\{v, v^z\}, \{v', v^z\} \in E(H_s^{(3)})$ and $\{v, v'\} \notin E(H_s^{(3)})$. Now consider any vertex u on the right side of the center v_c with $\operatorname{dist}_{H_s^{(3)}}(u, v_c) = 2k$. Again let u^p, u' and u^z be as constructed. Now observe that

$$\operatorname{dist}_{H_s^{(3)}}(v, u) = \operatorname{dist}_{H_s^{(3)}}(v, u') = 2(2k+1) = 4k = k',$$

which implies

$$\operatorname{dist}_{H_s^{(3)}}(v^z, u) = \operatorname{dist}_{H_s^{(3)}}(v^z, u') = k' + 1 > k'.$$

Thus, $u, u' \in \operatorname{Hop}_{k'}(v, H_s^{(3)})$, but $u, u' \notin \operatorname{Hop}_{k'}(v^z, H_s^{(3)})$. As H_s has exactly 2^{2s-2} leafs, it has 2^{2s-3} leafs on the left side of v_c and 2^{2s-3} leafs on the right side of v_c. So, there are 2^{2s-3} possible choices of u and thus,

$$\operatorname{hop}_{k'}(v, H_s^{(3)}) - \operatorname{hop}_{k'}(v^z, H_s^{(3)}) \geq 2 \cdot 2^{2s-3} = 2^{2s-2} = 2^{-4}(a+3)^2$$

(recall that $a = 2^{s+1} - 3$). Now,

$$\begin{aligned}
\frac{\operatorname{hop}_{k'}(v, H_s^{(3)})}{\operatorname{hop}_{k'}(v^z, H_s^{(3)})} &= \frac{\operatorname{hop}_{k'}(v^z, H_s^{(3)}) + \operatorname{hop}_{k'}(v, H_s^{(3)}) - \operatorname{hop}_{k'}(v^p, H_s^{(3)})}{\operatorname{hop}_{k'}(v^z, H_s^{(3)})} \\
&\geq 1 + \frac{\operatorname{hop}_{k'}(v, H_s^{(3)}) - \operatorname{hop}_{k'}(v^z, H_s^{(3)})}{|V(H_s^{(3)})|} \geq 1 + \frac{2^{-4}(a+3)^2}{2a(2a+6)} \\
&\geq 1 + 2^{-6} = 1.015625.
\end{aligned}$$

For any $\beta \geq 90$, we note that $\{v, v^z, v'\}$ induce a copy of the path P_3 with $\operatorname{hop}_{k'}(v', H_s^{(3)})^\beta = \operatorname{hop}_{k'}(v, H_s^{(3)})^\beta$ and

$$\frac{\operatorname{hop}_{k'}(v, H_s^{(3)})^\beta}{\operatorname{hop}_{k'}(v^z, H_s^{(3)})^\beta} \geq 1.015625^\beta \geq 1.015625^{90} \approx 4.036 > 2^2.$$

Thus, by Lemma 6.8, there exist no perfect 2-dimensional weighted embedding of P_3 with weights $\operatorname{hop}_{k'}(\cdot, H_s^{(3)})^\beta$. Since there are exactly $2^{2s-3} \in \Theta(a^2) = \Theta(n)$ possible choices of v, resulting in $\Theta(n)$ different induced copies of P_3 (which have no perfect embedding), the total error $s_{\mathrm{fn}} + s_{\mathrm{fp}}$ of ψ is at least $\Theta(n)$ (if $\beta \geq 90$). ∎

This theorem shows that the weight-setter $\operatorname{hop}_k(\cdot, G)^\beta$ (with $k := \lceil \frac{\operatorname{diam} G}{2} \rceil$) is not suitable for all grid graphs G, if $\beta \geq 90$. We claim (without proof) that the same is true for the weight-setter $\operatorname{s-hop}(\cdot, G)^\beta$. This is especially interesting, if we compare this result with Theorem 5.4, which only shows the suitability of $\operatorname{s-hop}(\cdot, G)^\beta$ as weight-setter for all complete trees T_h^b, if $\beta \geq 4$. Thus, if we need a weight-setter that is suitable for both classes of graphs, we need to select β carefully between 4 and 90 (if a suitable β exists at all).

Additionally, we note that we did not optimize the lower bound of β in Theorem 6.7. We conjecture that it can be reduced significantly by adjusting the construction of $H_s^{(3)}$. The question, whether β can be reduced down to 4 and we come in direct conflict with Theorem 5.4, remains unanswered.

Even though we have found a graph for which $\operatorname{hop}_k(\cdot)^\beta$ is not necessary suitable, we claim that this is *no* sign of superiority of the baseline weight-setter $\deg(v)^\beta$ over the k-hop centrality: Consider the graph $G_a^{\searrow}$, which is defined as the subgraph of $\Gamma_{a,a}$ induced by $V(G_a^{\searrow}) := \{(x, y) \in V(\Gamma_{a,a}) \mid |x - y| \leq 1\}$. For an illustration see Figure 6.4. We make the following claim:

Theorem 6.9: *For $\beta \geq 2$, let $\psi = (p_v, (\deg v)^\beta)_{v \in V(G_a^{\searrow})}$ be an arbitrary 2-dimensional weighted embedding of $G_a^{\searrow}$ with error (s_{fn}, s_{fp}). Then, $s_{fn} + s_{fp} \geq a - 3 \in \Omega(a) = \Omega(n)$.*

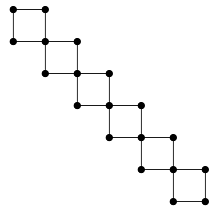


Figure 6.4.: Illustration of the graph $G_7^{\searrow}$.

Proof. For any $i \in \{1, \ldots, a-3\}$ consider the vertices $v_i := (i,i)$, $v_{i+1} := (i+1, i+1)$ and $u := (i+1, i)$. Note that $\deg v_i = \deg v_{i+1} = 4$ and $\deg u = 2$. Thus, $\frac{(\deg v_i)^\beta}{(\deg u)^\beta} = 2^\beta \geq 2^2 = 4$. Additionally, note that $\{v_i, u, v_{i+1}\}$ induces a copy of P_3. Thus, by Lemma 6.8, $(\deg(\cdot, G_a^{\searrow}))^\beta$ yields no perfect 2-dimensional weighted embedding of P_3. As we can choose $a-3$ different values of i and thus have $a-3$ such induced copies of P_3, we have shown $s_{\text{fn}} + s_{\text{fp}} \geq a - 3$. ∎

So, we have also found a counterexample for $\deg(\cdot)^\beta$. The severity of this counterexample differs from Theorem 6.7 in at least 2 ways: First, the construction of $G_a^{\searrow}$ is much simpler then the one of $H_s^{(3)}$. As a consequence, it seems much easier to find variations of $G_a^{\searrow}$ that act as counterexamples too, then to find such variations of $H_s^{(3)}$. Second, Theorem 6.7 requires β to be at least 90, while Theorem 6.9 already gets relevant for $\beta = 2$.

In summary, if we also keep in mind the result of Corollary 6.6, it seems like using the k-hop centrality as weights still is a better (or at least equally good) option for embedding grid graphs, as using the degree centrality. However, we have to keep in mind that there exist grid graphs that cannot be embedded with sublinear error with weights $\text{hop}_k(\cdot)^\beta$.

6.3. Unit Disk Graphs

In this section, we will discuss embeddings of unit disk graphs, which are a generalization of grid graphs:

A graph G is called a *unit disk graph (UDG)*, if there exist positions $(p_v \in \mathbb{R}^2)_{v \in V(G)}$ such that

$$\|p_u - p_v\| \leq 1 \iff \{u, v\} \in E(G)$$

for all $u, v \in V(G)$. Alternatively, a graph G is an UDG iff. there exists a perfect 2-dimensional weighted embedding of G with unit weights.

Notably, all grid graphs (i.e. induced subgraphs of the grid $\Gamma_{a,a}$) are UDGs. Consequently, we have already established in the previous section that neither $\text{hop}_k(\cdot)^\beta$ nor $\deg(\cdot)^\beta$ yield embeddings with sublinear error for *all* UDGs when β is sufficiently large (see Theorems 6.7 and 6.9). This presents a methodological challenge in our comparison of $\text{hop}_k(\cdot)^\beta$ and $\deg(\cdot)^\beta$, not only for UDGs but also for any class of graphs that includes the counterexample from the previous section.

One way to address this issue is to shift the focus from determining whether suitable embeddings exist for *all* graphs in a class to asking whether they exist for *most* graphs in the class. This reformulation is equivalent to determining whether a randomly chosen graph from the class is likely to admit a suitable embedding. To explore this question, we introduce random geometric graphs:

For $n \in \mathbb{N}_+$ and $r = r(n)$, we consider the *random geometric graph (RGG)* $\mathcal{G} = \mathcal{G}(n, r)$, which has vertex set $\{X_1, \ldots, X_n\}$, where $X_i \in \mathcal{D} := [-\sqrt{n}/2, \sqrt{n}/2]^2$ are uniform and independent random variables [Pen03] [DMPP14]. Two vertices X_i and X_j are adjacent in $\mathcal{G}$ iff. $\left\|X_i - X_j\right\| \leq r$. $\mathcal{G}$ is a random graph and all properties of $\mathcal{G}$ are random variables.

One useful fact is that for any fixed $A \subseteq \mathcal{D}$ and $i \in \{1, \ldots, n\}$, $\mathbb{P}\left(X_i \in A\right) = \frac{|A|}{|\mathcal{D}|} = \frac{|A|}{n}$, where $|A|$ is the area of A. More generally, let $\mathcal{C}$ be an event that is independent of X_i, then $\mathbb{P}\left(X_i \in A | \mathcal{C}\right) = \frac{|A|}{n}$. With that fact, we can calculate the probability that the vertices X_i and X_j ($i \neq j$) are adjacent, given that X_i has distance at least r from the boundary of $\mathcal{D}$ (we denote this assumption with $\mathcal{B}_i$):

$$\mathbb{P}\left(\{X_i, X_j\} \in E(\mathcal{G}) \middle| \mathcal{B}_i\right) = \mathbb{P}\left(X_j \in B_r(X_i) \middle| \mathcal{B}_i\right) = \frac{|B_r(X_i)|}{n} = \frac{\pi r^2}{n}. \tag{6.10}$$

Recall that $B_\rho(x)$ denotes the Euclidean disk of radius $\rho \in \mathbb{R}_+$, centered at $x \in \mathbb{R}^2$ and that $B_\rho(x)$ has area $\pi\rho^2$. As a consequence of Equation 6.10, we can calculate the expected degree of a vertex X_i under the assumption $\mathcal{B}_i$,

$$\mathbb{E}\left[\deg X_i | \mathcal{B}_i\right] = \sum_{\substack{j=1 \\ j \neq i}}^{n} \mathbb{P}\left(\{X_i, X_j\} \in E(\mathcal{G}) \middle| \mathcal{B}_i\right) = (n-1)\frac{\pi r^2}{n}.$$

Similarly, we can show that under the assumption that X_i is in the upper left corner $(-\frac{\sqrt{n}}{2}, -\frac{\sqrt{n}}{2})$ of $\mathcal{D}$, the expected degree is given by

$$\mathbb{E}\left[\deg X_i \middle| X_i = \left(-\sqrt{n}/2, -\sqrt{n}/2\right)\right] = (n-1)\frac{\pi r^2}{4n}.$$

Thus, the degree of a vertex in the corner of $\mathcal{D}$ is expected to be smaller by a factor of 4 in comparison to the expected degree of a more central vertex. These effects on the boundary of $\mathcal{D}$ can make the analysis of properties of $\mathcal{G}$ tedious. Hence, we introduce a toroidal variant of RGGs:

For $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in \mathcal{D}$, we define the toroidal distance between z_1 and z_2 by

$$d_{\mathrm{T}}(z_1, z_2) := \min_{i,j \in \mathbb{Z}} \left\|(x_1 - x_2 + i\sqrt{n}, y_1 - y_2 + j\sqrt{n})\right\|.$$

Figure 6.5 illustrates, how this toroidal distance can be visualized. For $n \in \mathbb{N}_+$ and $r \in \mathbb{R}_+$, we define the *toroidal random geometric graph* $\mathcal{G}_{\mathrm{T}}$ as the graph with the same vertices $X_1, \ldots, X_n$ as $\mathcal{G}$, but with an edge $\{X_i, X_j\}$ iff. $d_{\mathrm{T}}(X_i, X_j) \leq r$ [BGPS25].

We note that $d_{\mathrm{T}}(X_i, X_j) \leq \left\|X_i - X_j\right\|$ and thus, $\mathcal{G}$ is a subgraph of $\mathcal{G}_{\mathrm{T}}$. The main advantage of $\mathcal{G}_{\mathrm{T}}$ over $\mathcal{G}$ is that for any $z_1 \in \mathcal{D}$ and $\rho \in \mathbb{R}_+$ with $\rho \leq \sqrt{2n}$, the set

$$\{z_2 \in \mathcal{D} \mid d_{\mathrm{T}}(z_1, z_2) \leq \rho\}$$

has area exactly $\pi\rho^2$. Thus,

$$\mathbb{P}\left(d_{\mathrm{T}}(X_i, X_j) \leq \rho\right) = \frac{\pi\rho^2}{n},$$

independent on whether X_i and X_j are near the boundary of $\mathcal{D}$ or not. Thus, $\mathbb{E}\left[\deg X_i\right] = (n-1)\frac{\pi r^2}{n}$.

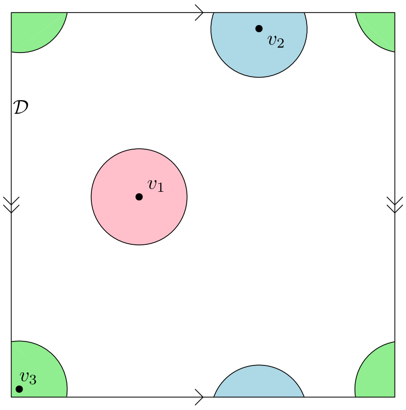


Figure 6.5.: The toroidal distance on $\mathcal{D}$ can be visualized as the top and bottom, aswell the left and right boundary of $\mathcal{D}$ being identified. The figure shows three toroidal disks centered at the vertices v_1, v_2 and v_3 with a fixed radius. We observe that all those disks have the same area.

We now consider embeddings of toroidal RGGs. For that reason, we need to consider a toroidal variant of weighted embeddings: We define a 2-dimensional weighted toroidal embedding $\psi = (p_v, w_v)_{v \in V(G)}$ analogously to standard weighted embeddings, with the difference that $p_v \in \mathcal{D}$ for all $v \in V(G)$. However, ψ is called perfect, if for all $u, v \in V(G)$,

$$\{u, v\} \in E(G) \iff \frac{d_{\mathrm{T}}(p_u, p_v)}{(w_u w_v)^{1/2}} \leq r,$$

for some $r \in \mathbb{R}_+$. We define $E(\psi)$ and the error of ψ analogously to standard weighted embeddings.

We are now interested, which weight assignments yield a toroidal weighted embedding of $\mathcal{G}_{\mathrm{T}}$ with sublinear error. If we assume that the weight of a vertex X_i is based on some centrality measure, as for instance $\deg X_i$ or $\operatorname{hop}_k(X_i)$, then the weight W_i assigned to X_i depends on $X_1, \ldots, X_n$. Thus, W_i is a random variable. Furthermore, for any reasonable weight assignment, the variables W_i and W_j should be distributed equally for all i, j. Hence, we assume that $W_i \sim W_1$ for all i.

We note that, if $W_i = 1$ is constant for all i, then the toroidal weighted embedding $\psi = (X_i, W_i)_{X_i \in \mathcal{G}_{\mathrm{T}}}$ is perfect by the definition of toroidal weighted embeddings and $\mathcal{G}_{\mathrm{T}}$. W_i being constant is equivalent to $\operatorname{Var}(W_i) = 0$. As we are interested in an embedding with sublinear error, one might hope that ψ has sublinear error not only when $\operatorname{Var}(W_i) = 0$, but when $\operatorname{Var}(W_i)$ is sufficiently small. This hope turns out to be true, if W_i satisfies some additional properties:

Theorem 6.10: *Let $W_1, \ldots, W_n$ be random variables, such that $\mathbb{E}[W_i] = 1$, $Var(W_i) \in o(1)$ and there exists a constant $c \geq 1$ such that $W_i < c$ deterministically. Additionally, let W_i and W_j be independent of $d_T(X_i, X_j)$ for all i, j. Then, w.h.p. $\psi = (X_i, W_i)_{X_i \in \mathcal{G}_T}$ is a toroidal weighted embedding of $\mathcal{G}_T$ with error $(o(nr^2), o(nr^2))$.*

Proof. We define $F_1 := E(\mathcal{G}) \setminus E(\psi)$ and $F_2 := E(\psi) \setminus E(\mathcal{G})$. We need to show that $|F_1| \in o(nr^2)$ and $|F_2| \in o(nr^2)$. We start with the first goal. We note that for any $i \neq j$ and $t \in (0,1)$:

$$\begin{aligned}\mathbb{P}\left(\{X_i, X_j\} \in F_1 \middle| W_i, W_j > 1-t\right) &= \mathbb{P}\left(r\sqrt{W_iW_j} < d_{\mathrm{T}}(X_i, X_j) \leq r \middle| W_i, W_j > 1-t\right)\\ &= \mathbb{P}\left(r\sqrt{(1-t)(1-t)} < d_{\mathrm{T}}(X_i, X_j) \leq r \middle| W_i, W_j > 1-t\right)\\ &= \frac{\pi r^2}{n} - \frac{\pi r^2(1-t)}{n} = \frac{\pi r^2 t}{n} \end{aligned} \tag{6.11}$$

Furthermore,

$$\begin{aligned}\mathbb{P}\left(\{X_i, X_j\} \in F_1 \middle| W_i \leq 1-t \vee W_j \leq 1-t\right) &= \mathbb{P}\left(r\sqrt{W_iW_j} < d_{\mathrm{T}}(X_i, X_j) \leq r \middle| W_i \leq 1-t \vee W_j \leq 1-t\right)\\ &\leq \mathbb{P}\left(d_{\mathrm{T}}(X_i, X_j) \leq r \middle| W_i \leq 1-t \vee W_j \leq 1-t\right)\\ &= \frac{\pi r^2}{n}. \end{aligned} \tag{6.12}$$

Since $\sigma^2 := \mathrm{Var}(W_i) \in o(1)$, there exist $s \in o(1) \cap \omega(\sigma)$. By Chebychev's inequality, it follows that

$$\mathbb{P}(|1 - W_i| \geq s) \leq \frac{\sigma^2}{s^2} \in o(1).$$

This implies

$$\mathbb{P}(W_i \leq 1-s) \in o(1) \qquad \text{and} \qquad \mathbb{P}(W_i \geq 1+s) \in o(1).$$

Now, for all $i \neq j$,

$$\begin{aligned}\mathbb{P}\left(\{X_i, X_j\} \in F_1\right) = &\overbrace{\mathbb{P}\left(\{X_i, X_j\} \in F_1 \middle| W_i, W_j > 1-s\right)}^{\leq \pi r^2 s/n = \pi r^2 o(1)/n \text{ (by Inequality 6.11)}} \overbrace{\mathbb{P}\left(W_i, W_j > 1-s\right)}^{\leq 1}\\ &+ \underbrace{\mathbb{P}\left(\{X_i, X_j\} \in F_1 \middle| W_i \leq 1-s \vee W_j \leq 1-s\right)}_{\leq \pi r^2/n \text{ (by Inequality 6.12)}} \underbrace{\mathbb{P}\left(W_i \leq 1-s \vee W_j \leq 1-s\right)}_{\leq \mathbb{P}(W_i \leq 1-s) + \mathbb{P}(W_i \leq 1-s) \in o(1)}\\ &\leq \pi r^2 o(1) + \pi r^2 o(1) \in o(r^2/n). \end{aligned}$$

We now find an upper bound for $\mathbb{E}[|F_1|]$:

$$\begin{aligned}\mathbb{E}[|F_1|] &= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathbb{P}\left(\{X_i, X_j\} \in F_1\right)\\ &= \sum_{i=1}^{n} \sum_{j=i+1}^{n} o(r^2/n)\\ &\leq n^2 o(r^2/n) = o(nr^2). \end{aligned}$$

Hence, we can choose $s' \in o(nr^2) \cap \omega(\mathbb{E}[|F_1|])$ and apply Markov's inequality, to show that

$$\mathbb{P}(|F_1| \geq s') \leq \frac{\mathbb{E}[|F_1|]}{s'} \in o(1).$$

Thus, $|F_1| < s' \in o(nr^2)$ w.h.p.

We can proceed almost analogously to show $|F_2| \in o(nr^2)$. We will not go into detail here. However, we note that in this case the proof of the equivalent to Inequality 6.12 uses that $W_i < c$ deterministically:

$$\begin{aligned}
\mathbb{P}\left(\{X_i, X_j\} \in F_2 \middle| W_i \geq 1+t \vee W_j \geq 1+t\right) &= \mathbb{P}\left(r\sqrt{W_i W_j} > d_{\mathrm{T}}(X_i, X_j) \geq r \middle| W_i \geq 1+t \vee W_j \geq 1+t\right) \\
&\leq \mathbb{P}\left(r\sqrt{c^2} > d_{\mathrm{T}}(X_i, X_j) \geq r \middle| W_i \geq 1+t \vee W_j \geq 1+t\right) \\
&= \frac{\pi r^2 c}{n} - \frac{\pi r^2}{n} \in \mathcal{O}(\pi r^2/n).
\end{aligned}$$

∎

We have shown that if the requirements of Theorem 6.10 are met, then there exists an toroidal weighted embedding of $\mathcal{G}_{\mathrm{T}}$ with weights $W_1, \ldots, W_n$ with error $(o(r^2 n), o(r^2 n))$. The reason, we call this error sublinear is, because

$$\mathbb{E}\left[E(\mathcal{G}_{\mathrm{T}})\right] = \frac{1}{2}\sum_{i=1}^{n} \mathbb{E}\left[\deg X_i\right] = \frac{1}{2} n(n-1)\frac{\pi r^2}{n} \in \Theta(r^2 n).$$

We note that Theorem 6.10 only can be applied, if W_i and W_j are independent of $d_{\mathrm{T}}(X_i, X_j)$ for all i, j. This might seem like a strong requirement, and it is. However, we note that it seems like $W_i\ W_j$ is, in some sense, almost independent of $d_{\mathrm{T}}(X_i, X_j)$ for many natural weight assignments $W_1, \ldots, W_n$. For instance, if we set $W_i := \deg X_i$, then knowing what $d_{\mathrm{T}}(X_i, X_j)$ is, seems to gives very little information about $\deg X_i$ and $\deg X_j$: Knowing $d_{\mathrm{T}}(X_i, X_j)$ gives information about whether X_i and X_j are adjacent, but not whether any third vertex X_ℓ is adjacent to X_i. A similar behavior seems to be the case for $W_i := \mathrm{hop}_k(X_i)$. We were not able to formalize this intuition of almost independence yet and not able to formulate a more general version of Theorem 6.10. In it's current form, Theorem 6.10 is not applicable on any reasonable weight assignment that is calculated based on properties of $\mathcal{G}_{\mathrm{T}}$. However, Theorem 6.10 shows that W_i having a low variance is a property that is highly relevant for finding embeddings of $\mathcal{G}_{\mathrm{T}}$.

For that reason, we will now compare the variance of $\deg X_i$ and $\mathrm{hop}_k(X_i)$. We begin with $W_i = \deg X_i$. Since W_i and W_j are identically distributed for all i, j, it suffices to only calculate $\mathrm{Var}\left(\deg X_1\right)$. For $i \neq 1$, we define the Bernoulli variable $L_i = \mathbb{1}_{\{\{X_i, X_1\} \in E(\mathcal{G}_{\mathrm{T}})\}}$. Above, we have remarked that $\mathbb{P}\left(L_i = 1\right) = \frac{\pi r^2}{n}$ and thus,

$$\mathrm{Var}\left(L_i\right) = \mathbb{P}\left(L_i = 1\right) - \mathbb{P}\left(L_i = 1\right)^2 \leq \mathbb{P}\left(L_i = 1\right) = \frac{\pi r^2}{n}.$$

Since L_i and L_j are independent for $i \neq j$, we have shown that

$$\mathrm{Var}\left(\deg X_1\right) = \mathrm{Var}\left(\sum_{i=2}^{n} L_i\right) = \sum_{i=2}^{n} \mathrm{Var}\left(L_i\right) \leq (n-1)\frac{\pi r^2}{n} \in \mathcal{O}(r^2).$$

Comparing variances (in the context of Theorem 6.10) is only relevant, if $\mathbb{E}\left[W_i\right] = 1$, however, as mentioned further above, $\mathbb{E}\left[\deg X_1\right] \in \Theta(r^2)$. This can be easily fixed, as we can rescale $\deg X_i$ and instead consider

$$W_i := \frac{\deg X_i}{\mathbb{E}\left[\deg X_i\right]}.$$

We note that $\mathbb{E}[W_1] = 1$ and that if the weights W_i yield an embedding with error $(s_{\text{fn}}, s_{\text{fp}})$, then so does $\deg X_i$. We can now consider the variance of W_1,

$$\operatorname{Var}(W_1) = \operatorname{Var}\left(\frac{\deg X_1}{\mathbb{E}[\deg X_i]}\right) = \frac{\operatorname{Var}(\deg X_1)}{\mathbb{E}[\deg X_i]^2} = \frac{\mathcal{O}(r^2)}{\Theta(r^2)^2} = \mathcal{O}(1/r^2), \tag{6.13}$$

which is in $o(1)$, if $r \in \omega(1)$.

We now consider the variance of the k-hop centrality $\operatorname{hop}_k(X_i)$ on $\mathcal{G}_\text{T}$. In the previous sections, we considered the case $k = \lceil \frac{\operatorname{diam} G}{2} \rceil$. As the calculations get complicated if k is also a random variable that depends on $\mathcal{G}_\text{T}$, we will instead consider the case that $k = \lambda \frac{\sqrt{n}}{r}$ for some constant $0 < \lambda \leq \frac{1}{2}$. This choice of k is motivated by the fact that $\operatorname{diam} \mathcal{G} \in \Theta(\sqrt{n}/r)$ w.h.p. [DMPP14]. We also note, that we only consider values of r for which $r \in \omega(r_c)$, where $r_c = \sqrt{\log n}$ is the connectivity threshold of $\mathcal{G}_\text{T}$, i.e., for all $r \in \omega(r_c)$, the graph $\mathcal{G}_\text{T}$ is connected w.h.p. and disconnected w.h.p. if $r \in o(r_c)$.

For any $X_i \in V(\mathcal{G}_\text{T}(n,r))$, we denote the k-hop centrality of X_i as H_i. The goal of the remainder of this subsection is to show that $\operatorname{Var}(H_i) \in o(n^2)$ for all i. Since H_i and H_j have identical distributions for all i, j, it suffices to show $\operatorname{Var}(H_1) \in o(n^2)$. To prove this, we define the random variable $N_i = \mathbb{1}_{\{X_i \in \operatorname{Hop}_k(X_1)\}} = \mathbb{1}_{\{\operatorname{dist}_{\mathcal{G}_\text{T}}(X_1, X_i) \leq k\}}$ for each $i \in \{1, \ldots, n\}$.

The following Lemma by Diaz et al. about the relation between graph and toroidal distance will be very helpful:

Lemma 6.11 (Theorem 1.1(ii) and Remark 1.1(iv) in [DMPP14]): *If $r \in \omega(r_c)$, then w.h.p., for all i, j,*

$$\operatorname{dist}_{\mathcal{G}_T}(X_i, X_j) \leq \operatorname{dist}_{\mathcal{G}}(X_i, X_j) \leq \left\lceil \frac{d_T(X_i, X_j)}{r}(1 + o(1)) \right\rceil.$$

We define the event

$$E \coloneqq \left\{\text{For all } i, j \in \{1, \ldots, n\}, \operatorname{dist}_{\mathcal{G}_\text{T}}(X_i, X_j) \leq \left\lceil \frac{d_\text{T}(X_i, X_j)}{r}(1 + o(1)) \right\rceil \right\}. \tag{6.14}$$

and note that by Lemma 6.11, E holds with high probability. Thus, $\mathbb{P}(\neg E) \in o(1)$. Note that for any event A with $\mathbb{P}(A) \in \Theta(1)$,

$$\mathbb{P}(E|A) = \frac{\mathbb{P}(A, E)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A) - \mathbb{P}(A, \neg E)}{\mathbb{P}(A)} = 1 - \frac{\mathbb{P}(A, \neg E)}{\mathbb{P}(A)} \geq 1 - \frac{\mathbb{P}(\neg E)}{\mathbb{P}(A)} = 1 - o(1). \tag{6.15}$$

Furthermore, observe that

$$H_1 = \sum_{i=1}^{n} N_i$$

and thus

$$\operatorname{Var}(H_1) = \sum_{i=1}^{n} \operatorname{Var}(N_i) + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \operatorname{Cov}\left(N_i, N_j\right).$$

Hence, we are interested in finding an upper bound for $\operatorname{Cov}\left(N_i, N_j\right)$. We first state some, more general, facts about the distribution of N_i.

Lemma 6.12: *For all $i \in \{1, \ldots, n\}$,*

$$\mathbb{P}(N_i = 1, d_T(X_i, X_1) > rk) = 0$$

and

$$\mathbb{P}\left(N_i = 1 \middle| d_T(X_i, X_1) \leq \frac{rk}{1 + o(1)}, E\right) = 1.$$

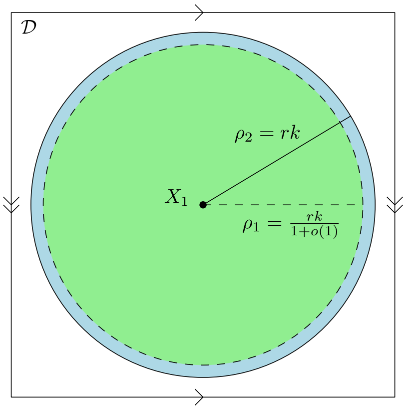


Figure 6.6.: Illustration of the radii ρ_1 and ρ_2. The result of this section heavily relies on the fact that w.h.p. all vertices in the inner disk (green) are contained in $\mathrm{Hop}_k(X_1)$, no vertex outside the outer disk (white) is contained in $\mathrm{Hop}_k(X_1)$ and only very few vertices are in the area between the two circles (blue).

Proof. If $N_i = 1$, then there exist points $X_1 = y_0, y_1, y_2, \ldots, y_k = X_i$ with $d_{\mathrm{T}}(y_j, y_{j-1}) \le r$ for all $j \in \{1, \ldots, k\}$. Applying the triangle inequality yields

$$d_{\mathrm{T}}(X_i, X_1) \le \sum_{j=1}^{k} \underbrace{d_{\mathrm{T}}(y_j, y_{j-1})}_{\le r} \le kr.$$

Thus, the event $\{N_i = 1\} \cap \{d_{\mathrm{T}} X_i - X_1 > rk\} = \emptyset$ does not occur, which implies the first part of the lemma.

If E and $d_{\mathrm{T}}(X_i, X_1) \le \frac{rk}{1+o(1)}$ hold, then

$$\mathrm{dist}_{\mathcal{G}_{\mathrm{T}}}(X_i, X_1) \le \left\lceil \frac{d_{\mathrm{T}}(X_i, X_1)}{r}(1 + o(1)) \right\rceil \le \left\lceil \frac{\frac{rk}{1+o(1)}}{r}(1 + o(1)) \right\rceil = k$$

deterministically and thus, $N_i = 1$. This proves the second part of the lemma. ∎

This motivates us to define $\rho_1 = \frac{rk}{1+o(1)} = rk(1 - o(1))$ and $\rho_2 = rk$. We note that

$$\mathbb{P}\left(d_{\mathrm{T}}(X_i, X_1) \le \rho_1\right) = \frac{\pi\rho_1^2}{n} = \frac{\pi r^2 k^2(1 - o(1))}{n} = \pi\lambda^2(1 - o(1)) \in \Theta(1) \tag{6.16}$$

$$\begin{aligned} \mathbb{P}\left(\rho_1 < d_{\mathrm{T}}(X_i, X_1) \le \rho_2\right) &= \frac{\pi\rho_2^2 - \pi\rho_1^2}{n} = \frac{\pi(r^2k^2 - r^2k^2(1 - o(1)))}{n} \\ &= \frac{\pi r^2 k^2 o(1)}{n} = \pi\lambda^2 o(1) = o(\pi\lambda^2) = o(1) \end{aligned} \tag{6.17}$$

$$\mathbb{P}\left(d_{\mathrm{T}}(X_i, X_1) \le \rho_2\right) = \frac{\pi\rho_2^2}{n} = \frac{\pi r^2 k^2}{n} = \pi\lambda^2 \in \Theta(1). \tag{6.18}$$

As a consequence

$$\begin{aligned} \mathbb{P}\left(N_i = 1\right) &\ge \mathbb{P}\left(N_i = 1, E, d_{\mathrm{T}}(X_i, X_1) \le \rho_1\right) \\ &= \underbrace{\mathbb{P}\left(N_i = 1 | E, d_{\mathrm{T}}(X_i, X_1) \le \rho_1\right)}_{=1,\text{ by Lemma 6.12}} \underbrace{\mathbb{P}\left(E | d_{\mathrm{T}}(X_i, X_1) \le \rho_1\right)}_{=1-o(1)\text{ by Inequality 6.15}} \underbrace{\mathbb{P}\left(d_{\mathrm{T}}(X_i, X_1) \le \rho_1\right)}_{=\pi\lambda^2(1-o(1)),\text{ by Equation 6.16}} \\ &\ge \pi\lambda^2(1 - o(1))^2 = \pi\lambda^2(1 - o(1)) = \pi\lambda^2 - o(1) \end{aligned} \tag{6.19}$$

and

$$\begin{aligned}\mathbb{P}(N_i = 1) &= \mathbb{P}(N_i = 1, d_{\mathrm{T}}(X_i, X_1) \le \rho_2) + \underbrace{\mathbb{P}(N_i = 1, d_{\mathrm{T}}(X_i, X_1) > \rho_2)}_{=0 \text{ by Lemma 6.12}} \\ &\le \mathbb{P}(d_{\mathrm{T}}(X_i, X_1) \le \rho_2) = \pi\lambda^2 \end{aligned} \tag{6.20}$$

for all $i \in \{2, \dots, n\}$.

These observations allows us to calculate the covariance of N_i and N_j for $i \neq j$.

Lemma 6.13: *For all* $1 \neq i \neq j \neq 1$, $Cov\left(N_i, N_j\right) \le o(1)$.

Proof. By the definition of covariance

$$\begin{aligned}\mathrm{Cov}\left(N_i, N_j\right) &= \mathbb{E}\left[N_i N_j\right] - \mathbb{E}\left[N_i\right]\mathbb{E}\left[N_j\right] \\ &= \mathbb{P}\left(N_i = N_j = 1\right) - \mathbb{P}(N_i = 1)^2 . \end{aligned} \tag{6.21}$$

By law of total probability and union bound (verify that all cases are considered), we observe that

$$\begin{aligned}\mathbb{P}\left(N_i = N_j = 1\right) \le\ & \mathbb{P}\left(N_i = N_j = 1, \neg E\right) \\ &+ 2\mathbb{P}\left(N_i = N_j = 1, \rho_1 \le d_{\mathrm{T}}(X_i, X_1) \le \rho_2, E\right) \\ &+ 2\mathbb{P}\left(N_i = N_j = 1, d_{\mathrm{T}}(X_i, X_1) > \rho_2, E\right) \\ &+ \mathbb{P}\left(N_i = N_j = 1, d_{\mathrm{T}}(X_i, X_1) \le \rho_1, d_{\mathrm{T}}(X_i, X_1) \le \rho_1, E\right)\end{aligned}$$

We evaluate these four terms separately. For the first term,

$$\mathbb{P}\left(N_i = N_j = 1, \neg E\right) \le \mathbb{P}(\neg E) \in o(1)$$

holds. For the second term, we use Equation 6.17, to show that

$$\mathbb{P}\left(N_i = N_j = 1, \rho_1 \le d_{\mathrm{T}}(X_i, X_1) \le \rho_2, E\right) \le \mathbb{P}(\rho_1 \le d_{\mathrm{T}}(X_i, X_1) \le \rho_2) \in o(1).$$

The third term is equal to 0 by Lemma 6.12 and the fourth term can be bound from above by

$$\mathbb{P}(d_{\mathrm{T}}(X_i, X_1) \le \rho_1, d_{\mathrm{T}}(X_i, X_1) \le \rho_1),$$

which, by Equation 6.16 and the independence of $d_{\mathrm{T}}(X_i, X_1)$ and $d_{\mathrm{T}}(X_j, X_1)$, is equal to

$$\left(\pi\lambda^2(1 - o(1))\right)^2 = \pi^2\lambda^4(1 - o(1)).$$

So

$$\mathbb{P}\left(N_i = N_j = 1\right) \le o(1) + o(1) + 0 + \pi^2\lambda^4(1 - o(1)) = \pi^2\lambda^4 \pm o(1).$$

Inserting this, together with Inequality 6.19, into Equation 6.21, yields

$$\begin{aligned}\mathrm{Cov}\left(N_i, N_j\right) &\le \pi^2\lambda^4 + o(1) - (\pi\lambda^2 - o(1))^2 \\ &= \pi^2\lambda^4 + o(1) - (\pi^2\lambda^4 - o(1)) = \pm o(1) \le o(1).\end{aligned}$$

∎

We are now able to estimate the variance and the expected value of H_i (see next two lemmata):

Lemma 6.14: *For all $\ell \in \{1,\dots,n\}$, $Var(H_\ell) \in o(n^2)$*

Proof. We note that for all $i \in \{1,\dots,n\}$,

$$\mathrm{Var}(N_i) = \underbrace{\mathbb{E}\left[N_i^2\right]}_{\leq 1} - \underbrace{\mathbb{E}\left[N_i\right]^2}_{\geq 0} \leq 1$$

holds and that $N_1 = 1$ deterministically (every vertex is always in it's own k-hop neighborhood) and thus,

$$\mathrm{Cov}(N_1, N_i) = \mathbb{E}[\underbrace{N_1}_{=1} N_i] - \underbrace{\mathbb{E}[N_1]}_{=1}\mathbb{E}[N_i] = 0.$$

By these observations and Lemma 6.13, it follows that

$$\begin{aligned}\mathrm{Var}(H_\ell) = \mathrm{Var}(H_1) &= \sum_{i=1}^{n} \underbrace{\mathrm{Var}(N_i)}_{\leq 1} + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \underbrace{\mathrm{Cov}\left(N_i, N_j\right)}_{\leq o(1) \text{ or } =0} \\ &\leq n \cdot 1 + n^2 o(1) = n + o(n^2) = o(n^2).\end{aligned}$$

∎

Lemma 6.15: *For all $\ell \in \{1,\dots,n\}$, $\mathbb{E}[H_\ell] \in \Theta(n)$*

Proof. By Equations 6.19 and 6.20, it follows that

$$\Theta(1) \ni \pi\lambda^2 - o(1) \leq \mathbb{P}\left(N_j = 1\right) \leq \pi\lambda^2 \in \Theta(1)$$

and thus $\mathbb{P}\left(N_j = 1\right) \in \Theta(1)$ for all $j \in \{2,\dots,n\}$. Now

$$\mathbb{E}[H_\ell] = \mathbb{E}[H_1] = \sum_{j=1}^{n} \mathbb{E}\left[N_j\right] = n\Theta(1) = \Theta(n).$$

∎

As for the degree centrality, we note that $\mathbb{E}[H_i] \neq 1$. To allow comparison, we consider the normalized k-hop centrality $W_i := \frac{H_i}{\mathbb{E}[H_i]}$. Note that $\mathbb{E}[W_i] = 1$ and

$$\mathrm{Var}(W_i) = \frac{\mathrm{Var}(H_i)}{\mathbb{E}[H_i]} = \frac{o(n^2)}{\Theta(n)^2} = o(1).$$

In summary, we have shown that if $0 < \lambda \leq \frac{1}{2}$ is constant, $k = \lambda\frac{\sqrt{n}}{r}$ and $r \in \omega(r_c)$, then the variance of the normalized k-hop centrality of a vertex in $\mathcal{G}_\mathrm{T}(n,r)$ is in $o(1)$. Similarly, we have shown that the normalized degree centrality of a vertex in $\mathcal{G}_\mathrm{T}(n,r)$ has variance $\mathcal{O}(1/r^2)$ (see Equation 6.13). The last expression is equal to $o(1/\log n)$, if $r \in \omega(r_c) = \omega(\sqrt{\log n})$. We can note that the latter bound $o(1/\log n)$ is smaller then the bound $o(1)$ for the variance of the normalized k-hop centrality. However, we remark that the upper bound $o(1)$ might be further decreased by a deeper analysis. We also remark that $o(1/\log n)$ and $o(1)$ are very similar bounds.

Additionally, we note that we only considered the toroidal RGG $\mathcal{G}_\mathrm{T}$. For the degree centrality, the boundary effects on the normal RGG $\mathcal{G}$ make the analysis a little harder for vertices near the boundary of $\mathcal{D}$. For the k-hop centrality, the boundary effects not only make the analysis much harder, but also affect much more vertices. It seems likely that no 2-dimensional weighted embedding of $\mathcal{G}$ with weights $\mathrm{hop}_k(\cdot)$ and sublinear error exists.

In this section, we were unable to establish definitive evidence for the superiority of degree centrality over k-hop centrality as a weight assignment. However, we discussed some indicative observations supporting this notion. We also note that it remains possible that neither of the two centrality measures is well-suited for the graphs $\mathcal{G}_\mathrm{T}$ and $\mathcal{G}$.

7. Conclusion

In this thesis, we have established several theoretical results on weighted embeddings. In Chapter 3, we presented multiple theorems that provide a clearer understanding of the structure and properties of weighted embeddings of trees. Additionally, we have gained some insights into the properties of weighted embeddings of grids.

Building on those insights, we proposed an alternative to assigning the weight of a vertex in an embedding based on it's degree - the (smooth) k-hop centrality: We assign the weight w_v of a vertex $v \in V(G)$ to

$$\begin{aligned} w_v &:= (\text{s-hop}(v))^\beta \\ &:= \left(\left| \left\{ u \in V(G) \mid \text{dist}_G(u,v) \leq \left\lceil \frac{\text{diam}\, G}{2} \right\rceil \right\} \right| \cdot \left| \left\{ u \in V(G) \mid \text{dist}_G(u,v) \leq \left\lceil \frac{\text{diam}\, G}{2} \right\rceil + 1 \right\} \right| \right)^\beta \end{aligned}$$

for a constant $\beta \geq 4/d$ (compare Definition 5.5).

We proceeded by analyzing the quality of weighted embeddings with weights $(\text{s-hop}(\cdot))^\beta$ and compared it to embeddings with weights $\deg(\cdot)^\beta$. Table 7.1 gives an overview of this comparison.

In particular, using k-hop centrality outperforms degree-based weighting on complete trees and yields comparable results on grids. The analysis on arbitrary graphs and grid graphs was inconclusive: For both weight assignment methods, there exist graphs that can not be embedded under some conditions. However, particularly on grid graphs it seems like counterexamples for the k-hop centrality are much rarer as ones for degree-based weights. For RGGs it seems as degree-based weights are more suitable than k-hop based weights, however no conclusive evidence for this claim was found.

In summary, our modification appears to yield embeddings that are at least as good as, if not better than, those obtained with degree-based weights across all graph classes we considered, except RGGs. However, we have not examined geometric inhomogeneous random graphs (GIRGs) or real-world networks, where degree-based weights have been shown to perform well in experiments [BHKM24]. Whether replacing the first step of the WEmbed algorithm with a k-hop-based weight assignment is beneficial depends on how well it performs on these two graph classes. Therefore, experimental research in this direction would be highly valuable.

Another caveat to keep in mind is that all our results assume that, given the right weights, the embedding positions can be determined effortlessly. However, in practice, WEmbed relies on gradient descent to optimize the positions, and it is not guaranteed to always succeed. Analyzing whether it does is an interesting problem. This question is particularly relevant for embeddings of grids, as the proof of Theorem 4.4 carefully partitions the graph into multiple subgraphs and embeds them separately. It seems unlikely that a machine learning algorithm could replicate this process.

Furthermore, we note that we used trees as *the* example for a graph with a homogeneous degree distribution that cannot be embedded with homogeneous weights. However, hyperbolic tilings are another class of graphs with that property. As they can be trivially embedded in

Table 7.1.: Comparison of the existence of weighted embeddings with weights set to degree centrality $(\deg v)^\beta$ or (smooth) k-hop centrality $\text{s-hop}(v)^\beta$ on different classes of graphs, for sufficiently large instances and sufficiently large β.

	Degree centrality $(\deg v)^\beta$	(Smooth) k-hop centrality $\text{s-hop}(v)^\beta$
Complete trees	No embedding with sublinear error for all graphs (Variant of Corollary 3.2)	Perfect 1-dimensional embeddings (Theorem 5.4)
Grids	2-dimensional embeddings with total error $\Theta(\sqrt{n})$ (Corollary 4.5)	2-dimensional embeddings with total error $\Theta(\sqrt{n})$ (Section 5.2)
Arbitrary trees	See complete trees above	Graph exists with no embedding with sublinear error (Theorem 6.3)
Grid graphs (induced subgraphs of grids)	Simple/basic graph exists with no embedding with sublinear error (Theorem 6.9)	Complex graph exists with no embedding with sublinear error (Theorem 6.7)
GIRGs and real-world networks	Low-dimensional embeddings of high quality (empirically) [BHKM24]	Unknown

the hyperbolic space it seems likely that they can also be perfectly embedded in a weighted setting. Nevertheless, we were unable to find such embeddings for hyperbolic tilings. Whether such embeddings exist, and whether using k-hop-based weights can yield them, remains an open question.

On another vein, we note that calculating the k-hop neighborhood of all vertices in a graph using Breadth-First Search or Depth-First Search has time complexity $\Theta(nm + n^2)$. This can be computational infeasible for big real-world graphs. Hence, it might be useful to have a weight assignment that can be calculated more easily, but behaves similar to the k-hop centrality.

Lastly, in Section 6.1 we identified a tree for which no suitable embedding with k-hop-based weights exists. The proof relied on the fact that this tree had a highly 'path-like' structure (caterpillar). It would be interesting to investigate whether there exists another counterexample that is more 'tree-like' or whether a positive result can be found for 'tree-like' trees. In either case, it remains true that not all trees can be embedded with k-hop-based weights, nor can all trees be embedded with degree-based weights. As discussed at the end of Section 6.1, it seems promising to find a middle ground between these two approaches, e.g., by considering an weight assignment that depends on $\text{hop}_k(\cdot)$ for all $k \in \{1, 2, \ldots, \operatorname{diam} G\}$.

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A. Appendix

A.1. Proof of Lemma 4.1

Lemma A.1 (Lemma 4.1): *For all $a \in \mathbb{N}_+$, there exists a perfect 1-dimensional weighted embedding of the grid $\Gamma_{a,2}$.*

Proof. We show that $\psi = (p_v, w_v)_{v \in V(\Gamma_{a,2})}$, with $w_{(x,y)} := 2^{-x}$ and $p_{(x,y)} := (-1)^y 2^{-2x-1}$ for all $(x, y) \in V(\Gamma_{a,2})$, is a perfect 1-dimensional weighted embedding of $\Gamma_{a,2}$. See Figure 4.1 for an illustration of ψ.

Let $\{(x_1, y_1), (x_2, y_2)\} \in E(\Gamma_{a,2})$, W.l.o.g. $x_1 \le x_2$ and $y_1 \le y_2$. Now, either $x_1 = x_2, y_1 = 0, y_2 = 1$ or $x_2 = x_1 + 1, y_1 = y_2$ holds. In the former case follows

$$\operatorname{dist}(\psi_{(x_1,0)}, \psi_{(x_1,1)}) = \frac{2^{-2x_1-1} - (-2^{-2x_1-1})}{2^{-x_1} 2^{-x_1}} = 1.$$

In the second case follows

$$\operatorname{dist}(\psi_{(x_1,y_1)}, \psi_{(x_1+1,y_1)}) = \frac{2^{-2(x_1+1)-1} - 2^{-2x_1-1}}{2^{-x_1} 2^{-x_1-1}} \le \frac{2^{-2x_1} - 2^{-2x_1-1}}{2^{-2x_1-1}} = 1.$$

Now let $(x_1, y_2), (x_2, y_2) \in V(\Gamma_{a,2})$, with $\{(x_1, y_2), (x_2, y_2)\} \notin E(\Gamma_{a,2})$. W.l.o.g. assume $x_1 \le x_2$ and $y_1 \le y_2$. Now, either $y_1 \ne y_2, x_2 \ge x_1 + 1$ or $y_1 = y_2, x_2 \ge x_1 + 2$. In the first case

$$\operatorname{dist}(\psi_{(x_1,0)}, \psi_{(x_2,1)}) = \frac{2^{-2x_1-1} - \overbrace{(-2^{-2x_2-1})}^{<0}}{2^{-x_1} 2^{-x_2}} > \frac{2^{-2x_1-1}}{2^{-x_1} 2^{-(x_1+1)}} = 1$$

holds and in the second case follows

$$\operatorname{dist}(\psi_{(x_1,y_1)}, \psi_{(x_2,y_1)}) = \frac{\overbrace{2^{-2x_1-1} - 2^{-2x_2-1}}^{>2^{-2x_1-2}}}{2^{-x_1} \underbrace{2^{-x_2}}_{\le 2^{-(x_2+2)}}} > \frac{2^{-2x_1-2}}{2^{-2x_1-2}} = 1.$$

For the last inequality, we used

$$2^{-2x_1-1} - 2^{-2x_2-1} \ge 2^{-2x_1-1} - 2^{-2(x_1+1)-1} > 2 \cdot 2^{-2x_1-2} - 2^{-2x_1-2} = 2^{-2x_1-2}.$$

Thus, we have shown that ψ is perfect. ∎

A.2. Proof of Lemma 5.3

Lemma A.2 (Lemma 5.3): *For all $h > i \ge 0$ and $b \ge 2$,*

$$\frac{\text{s-hop}(v_i)}{\text{s-hop}(v_{i+1})} \ge \begin{cases} \frac{23}{20} & \text{if } b = 2 \\ \frac{1}{2}b & \text{if } b \ge 3 \end{cases}$$

holds.

Proof. We will use Lemma 5.2 multiple times in this proof. Furthermore, we will use that $-h+i \leq -1$ follows from $i < h$.

We will show the claim for $i = 0$ first:

$$\begin{aligned}
\frac{\text{s-hop}(v_0)}{\text{s-hop}(v_1)} = \frac{\text{hop}_h(v_0)}{\text{hop}_h(v_1)} &= \frac{b^{h-0/2}(b+1) - b^{h-0} - 1}{2b^{h-(1-1)/2} - b^{h-1} - 1} \\
&= \frac{b^h(b+1) - b^h - 1}{2b^h - b^{h-1} - 1} = \frac{b^{h+1} - 1}{2b^h - b^{h-1} - 1} \\
&= \underbrace{\frac{b^2x - 1}{2bx - x - 1}}_{g(x) :=}
\end{aligned}$$

Here, we defined $x := b^{h-1}$. We observe that for all $x \geq 1$

$$\begin{aligned}
\frac{d}{dx} g(x) &= \frac{b^2(2bx - x - 1) - (b^2x - 1)(2b - 1)}{2bx - x - 1} \\
&= \frac{2b^3x - b^2x - b^2 - 2b^3x + b^2x + 2b - 1}{2bx - x - 1} \\
&= \frac{-b^2 + 2b - 1}{2bx - x - 1} < 0.
\end{aligned}$$

Thus g is monotonically decreasing (for $x \geq 1$) and

$$\begin{aligned}
\frac{\text{s-hop}(v_0)}{\text{s-hop}(v_1)} = g(b^{h-1}) &\geq \lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{b^2x - 1}{2bx - x - 1} \\
&= \lim_{x\to\infty} \frac{x(b^2 - x^{-1})}{x(2b - 1 - x^{-1}))} = \lim_{x\to\infty} \frac{b^2 - x^{-1}}{2b - 1 - x^{-1})} \\
&= \frac{b^2}{2b - 1} \geq \begin{cases} \frac{4}{3} \geq \frac{23}{20} & \text{if } b = 2 \\ \frac{b^2}{2b} = \frac{b}{2} & \text{if } b \geq 3 \end{cases}.
\end{aligned}$$

We now show the claim for all even $i \geq 2$ (in that case $i-1$ and $i+1$ are odd):

$$\begin{aligned}
\frac{\text{s-hop}(v_i)}{\text{s-hop}(v_{i+1})} &= \frac{\text{hop}_h(v_{i-1})}{\text{hop}_h(v_{i+1})} = \frac{2b^{h-(i-1-1)/2} - b^{h-(i-1)} - 1}{2b^{h-(i+1-1)/2} - b^{h-(i+1)} - 1} \\
&= \frac{2b^{h-i/2+1} - b^{h-i+1} - 1}{2b^{h-i/2} - b^{h-i-1} - 1} = \frac{b^{h-i/2+1}\left(2 - b^{-i/2} - b^{-h+i/2-1}\right)}{b^{h-i/2+1}\left(2b^{-1} - b^{-i/2-2} - b^{-h+i/2-1}\right)} \\
&= \underbrace{\frac{2}{B}}_{C_0:=} - \underbrace{\frac{b^{-i/2}}{B}}_{C_1:=} - \underbrace{\frac{b^{-h+i/2-1}}{B}}_{C_2:=}.
\end{aligned}$$

Here, we set $B := 2b^{-1} - b^{-i/2-2} - b^{-h+i/2-1}$. Now,

$$\begin{aligned}
C_0 &\geq \frac{2}{2b^{-1}} = b, \\
C_1 &= \frac{1}{2b^{i/2-1} - b^{-2} - b^{-h+i-1}} \leq \frac{1}{2b^{2/2-1} - b^{-2} - b^{-1-1}} \\
&= \frac{1}{2 - 2b^{-2}} \leq \frac{1}{2 - 2 \cdot 2^{-2}} = \frac{2}{3} \qquad \text{and} \\
C_2 &= \frac{b^{-h+i/2-1}}{B} \leq \frac{b^{-(i+1)+i/2-1}}{B} = b^{-2} \underbrace{\frac{b^{-i/2}}{B}}_{C_1} \leq \frac{2b^{-2}}{3}.
\end{aligned}$$

Thus,

$$\frac{\text{s-hop}(v_i)}{\text{s-hop}(v_{i+1})} \geq b - \frac{2}{3} - \frac{2b^{-2}}{3} = b - b\left(\underbrace{\frac{2b^{-1}}{3} + \frac{2b^{-3}}{3}}_{\leq \frac{1}{3}+\frac{1/4}{3}=\frac{5}{12}}\right) \geq \frac{7}{12}b \geq \frac{1}{2}b.$$

In the case $b = 2$, we observe that $\frac{7}{12}b = \frac{7}{6} \geq \frac{23}{20}$.

We will now show the claim for all odd i in a very similar way. Again, by Lemma 5.2 ($i-1$ and $i+1$ are even and $i \geq 1$),

$$\begin{aligned}\frac{\text{s-hop}(v_i)}{\text{s-hop}(v_{i+1})} = \frac{\text{hop}_h(v_{i-1})}{\text{hop}_h(v_{i+1})} &= \frac{b^{h-(i-1)/2}(b+1) - b^{h-(i-1)} - 1}{b^{h-(i+1)/2}(b+1) - b^{h-(i+1)} - 1} \\ &= \frac{b^{h-i/2+3/2}\left(1 + b^{-1} - b^{-i/2-1/2} - b^{-h+i/2-3/2}\right)}{b^{h-i/2+3/2}\left(b^{-1} + b^{-2} - b^{-i/2-5/2} - b^{-h+i/2-3/2}\right)} \\ &= \underbrace{\frac{1+b^{-1}}{B'}}_{:=C'_0} - \underbrace{\frac{b^{-i/2-1/2}}{B'}}_{:=C'_1} - \underbrace{\frac{b^{-h+i/2-3/2}}{B'}}_{:=C'_2},\end{aligned}$$

where $B' := b^{-1} + b^{-2} - b^{-i/2-5/2} - b^{-h+i/2-3/2}$. Now

$$\begin{aligned} C'_0 &\geq \frac{1+b^{-1}}{b^{-1}+b^{-2}-b^{-1/2-5/2}} \geq \frac{1+b^{-1}}{b^{-1}+b^{-2}-b^{-3}} \geq \begin{cases} \frac{1+2^{-1}}{2^{-1}+2^{-2}-2^{-3}} = \frac{12}{5} & \text{if } b = 2 \\ \frac{1+b^{-1}}{b^{-1}+b^{-2}} = b & \text{if } b \geq 3 \end{cases}, \\ C'_1 &= \frac{1}{b^{i/2-1/2} + b^{i/2-3/2} - b^{-2} - b^{-h+i-1}} \leq \frac{1}{b^{1/2-1/2} + b^{1/2-3/2} - b^{-2} - b^{-1-1}} \\ &= \frac{1}{1+b^{-1}-2b^{-2}} \leq \frac{1}{1+b^{-1}-bb^{-2}} = 1 \qquad \text{and} \\ C'_2 &= \frac{b^{-h+i/2-3/2}}{B'} \leq \frac{b^{-(i+1)+i/2-3/2}}{B'} = b^{-2}\underbrace{\frac{b^{-i/2-1/2}}{B'}}_{C'_1} \leq b^{-2}. \end{aligned}$$

If $b = 2$, this implies

$$\frac{\text{s-hop}(v_i)}{\text{s-hop}(v_{i+1})} \geq \frac{12}{5} - 1 - 2^{-2} = \frac{23}{20}$$

and if $b \geq 3$, we obtain

$$\frac{\text{s-hop}(v_i)}{\text{s-hop}(v_{i+1})} \geq b - 1 - b^{-2} = b - b(\underbrace{b^{-1} + b^{-3}}_{\leq 3^{-1}+3^{-3}}) = b - \frac{10}{27}b = \frac{17}{27}b \geq \frac{1}{2}b.$$

∎