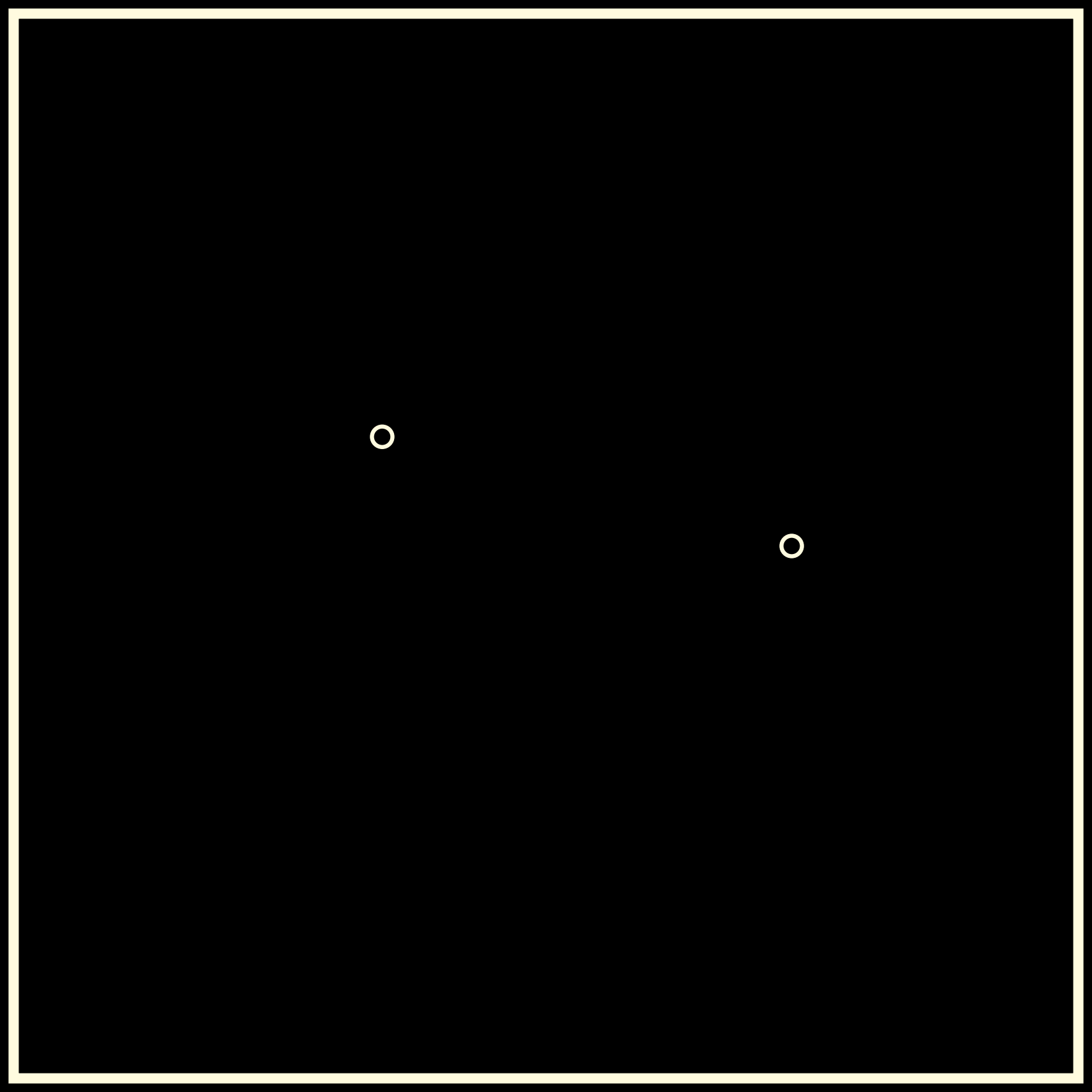
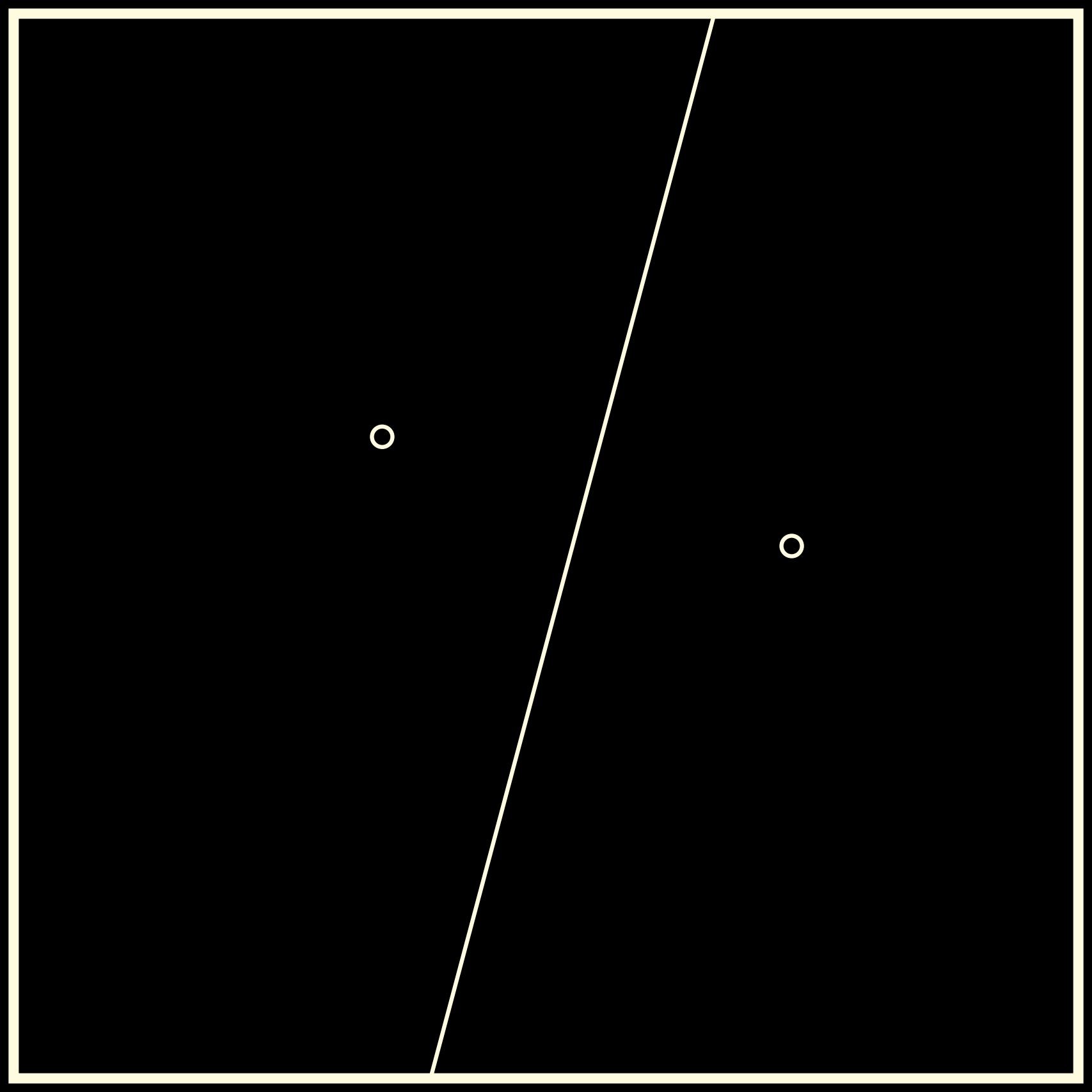
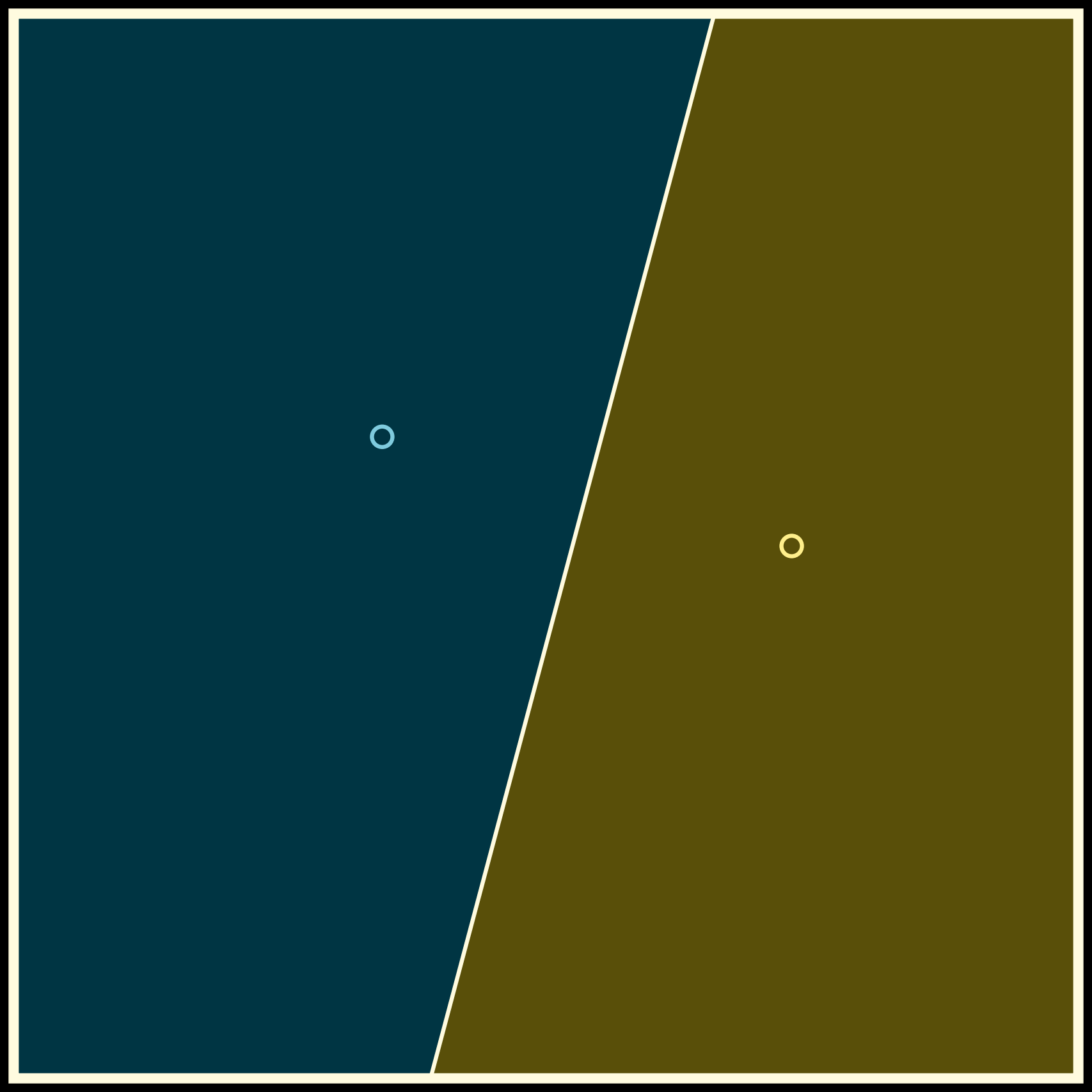
…, and the Complexity of Voronoi Diagrams

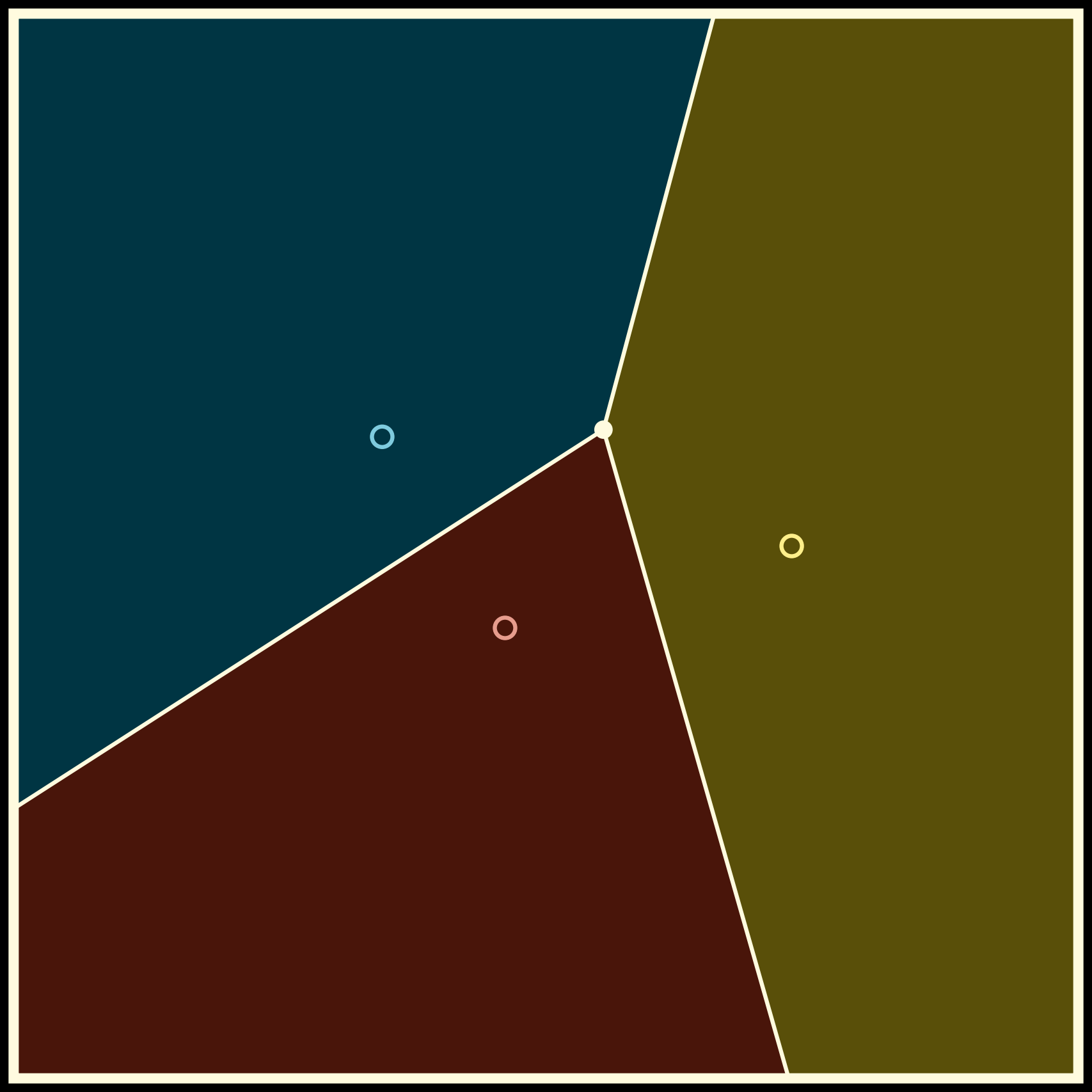
Thomas, Tobias, Andreas, Jordi, Ralf

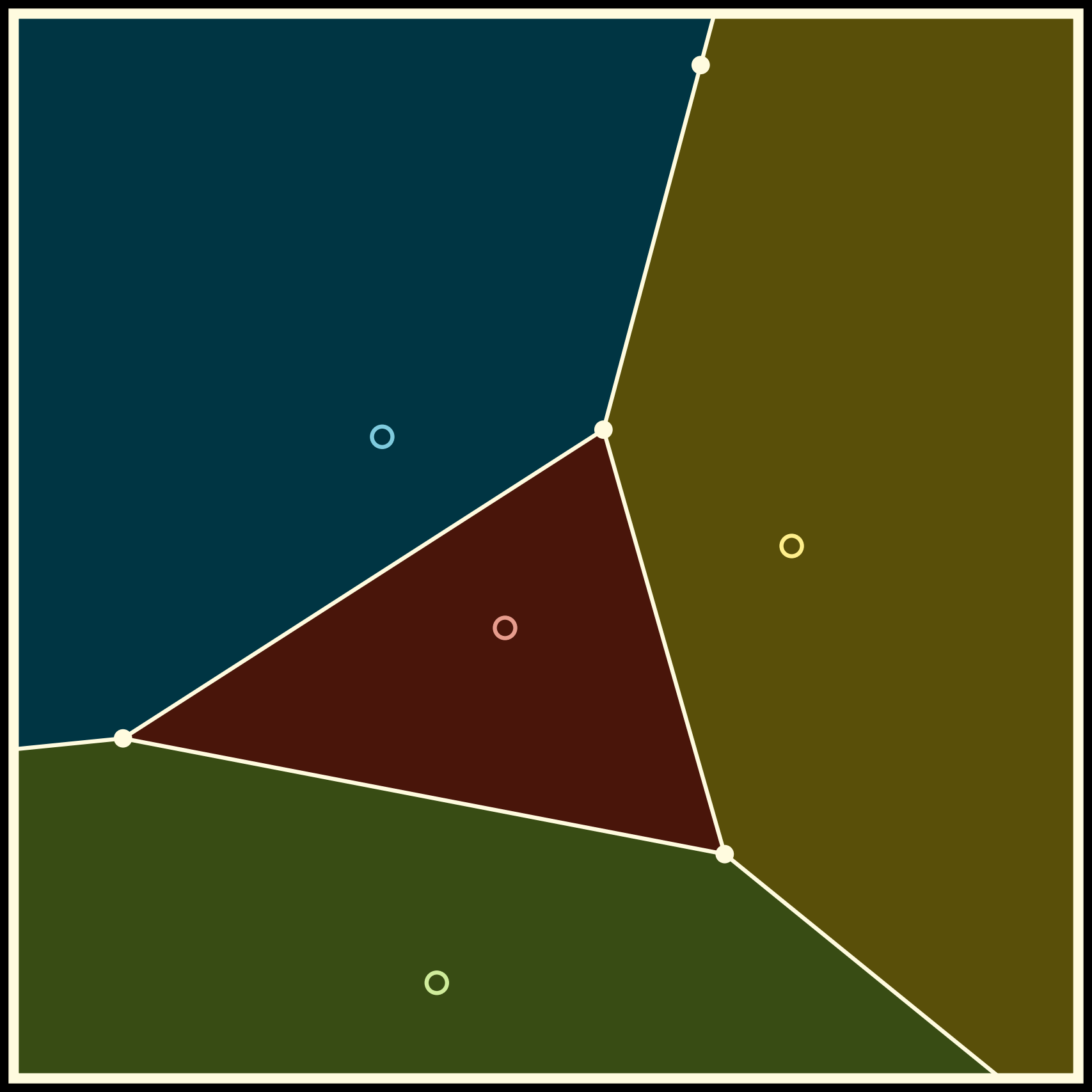
site

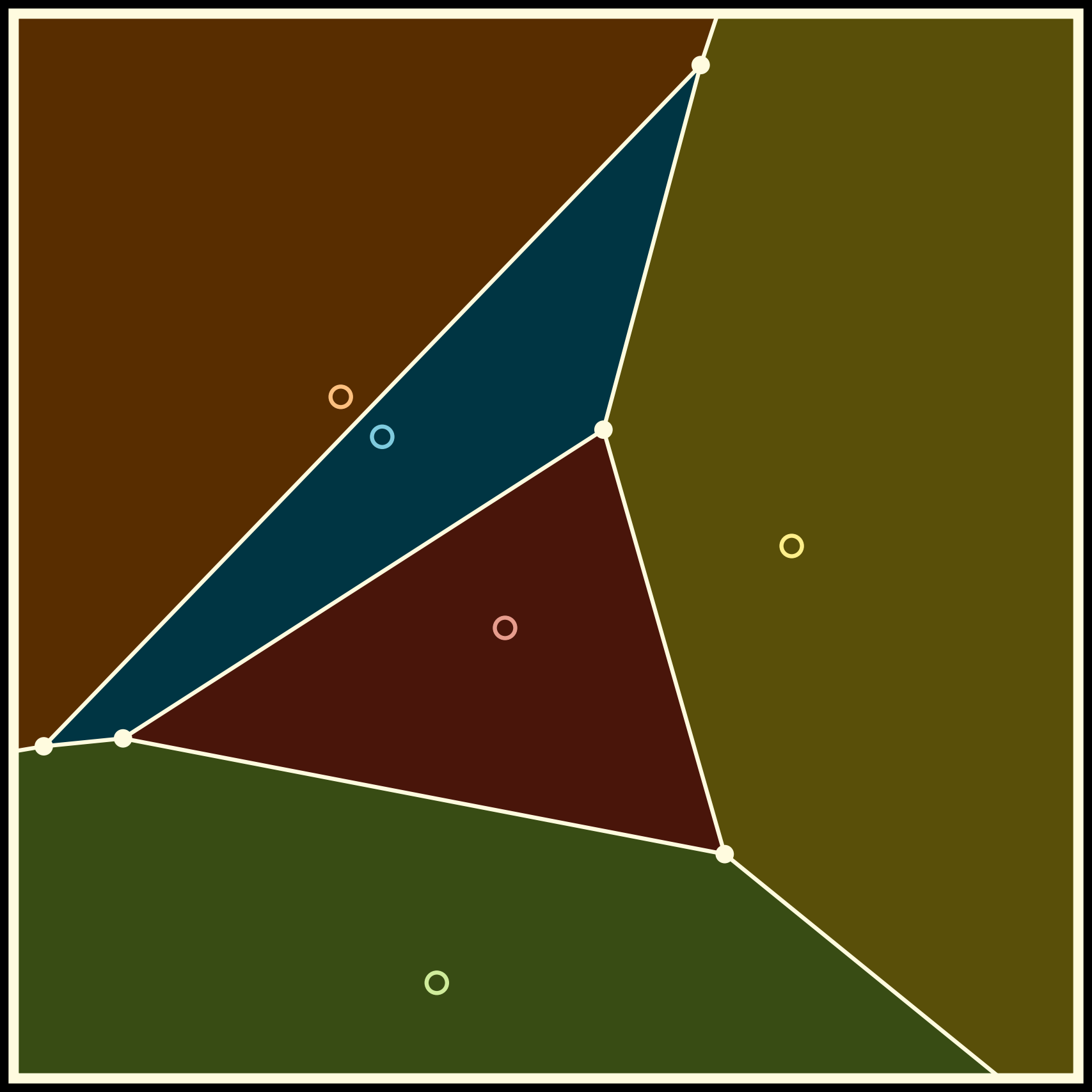


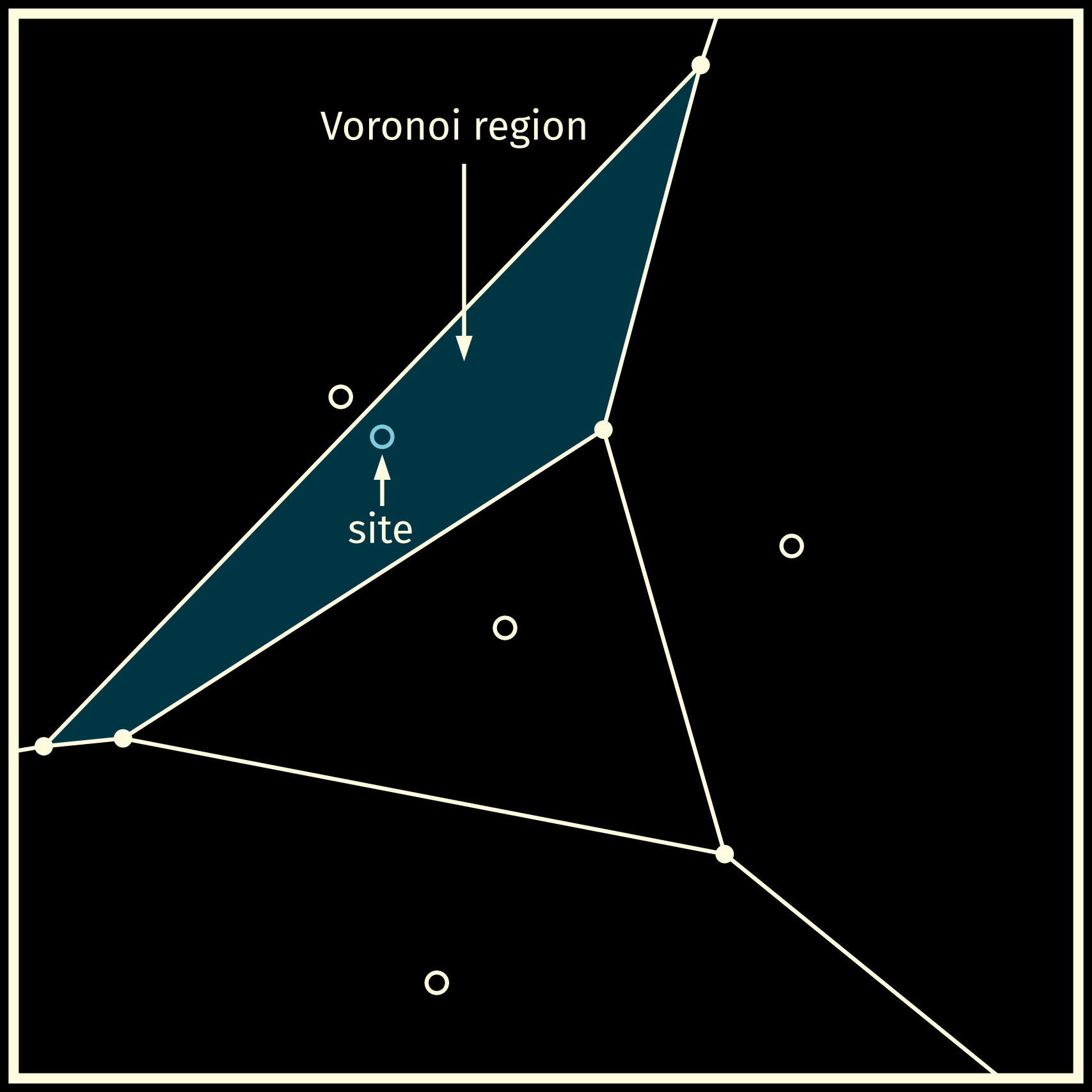











Voronoi region
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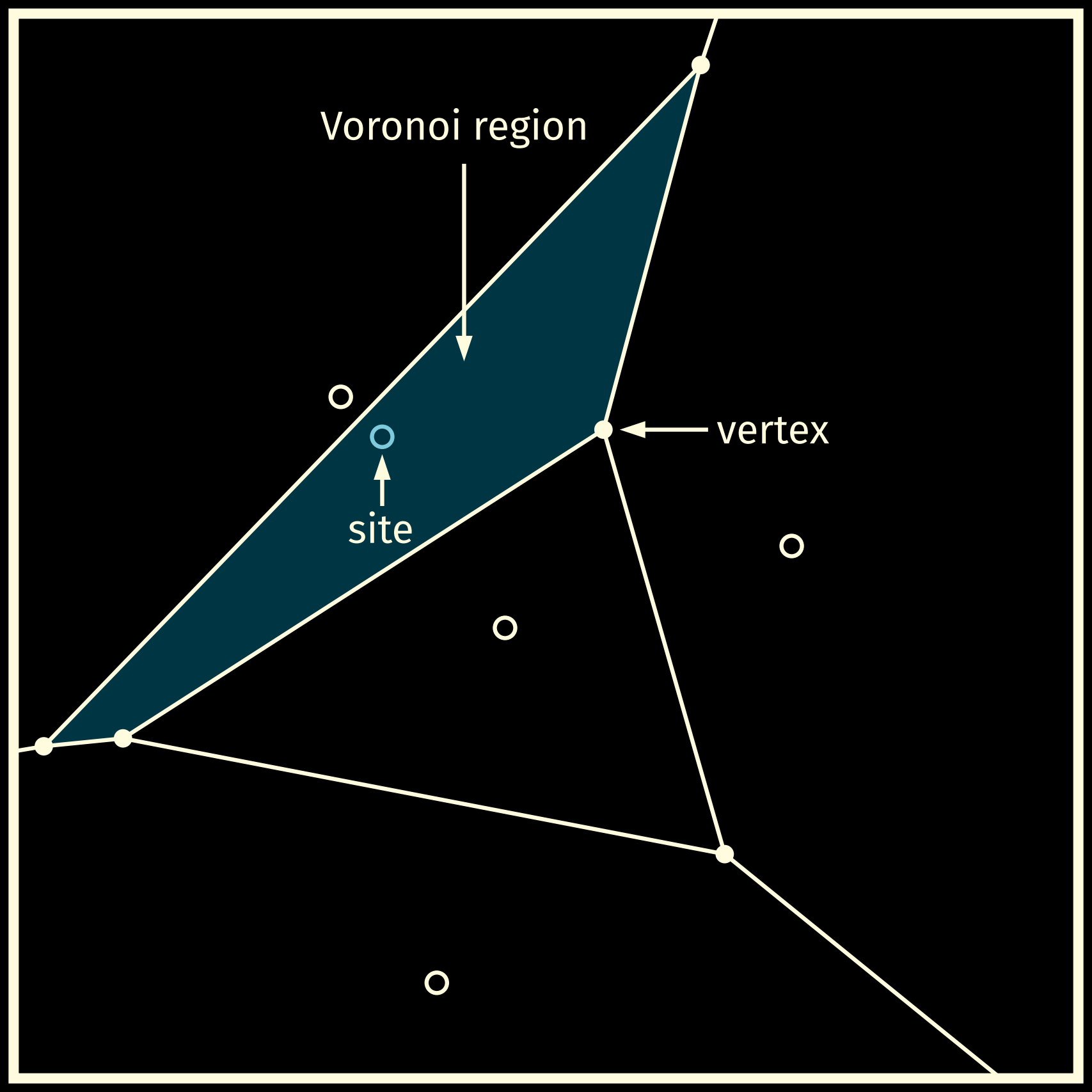
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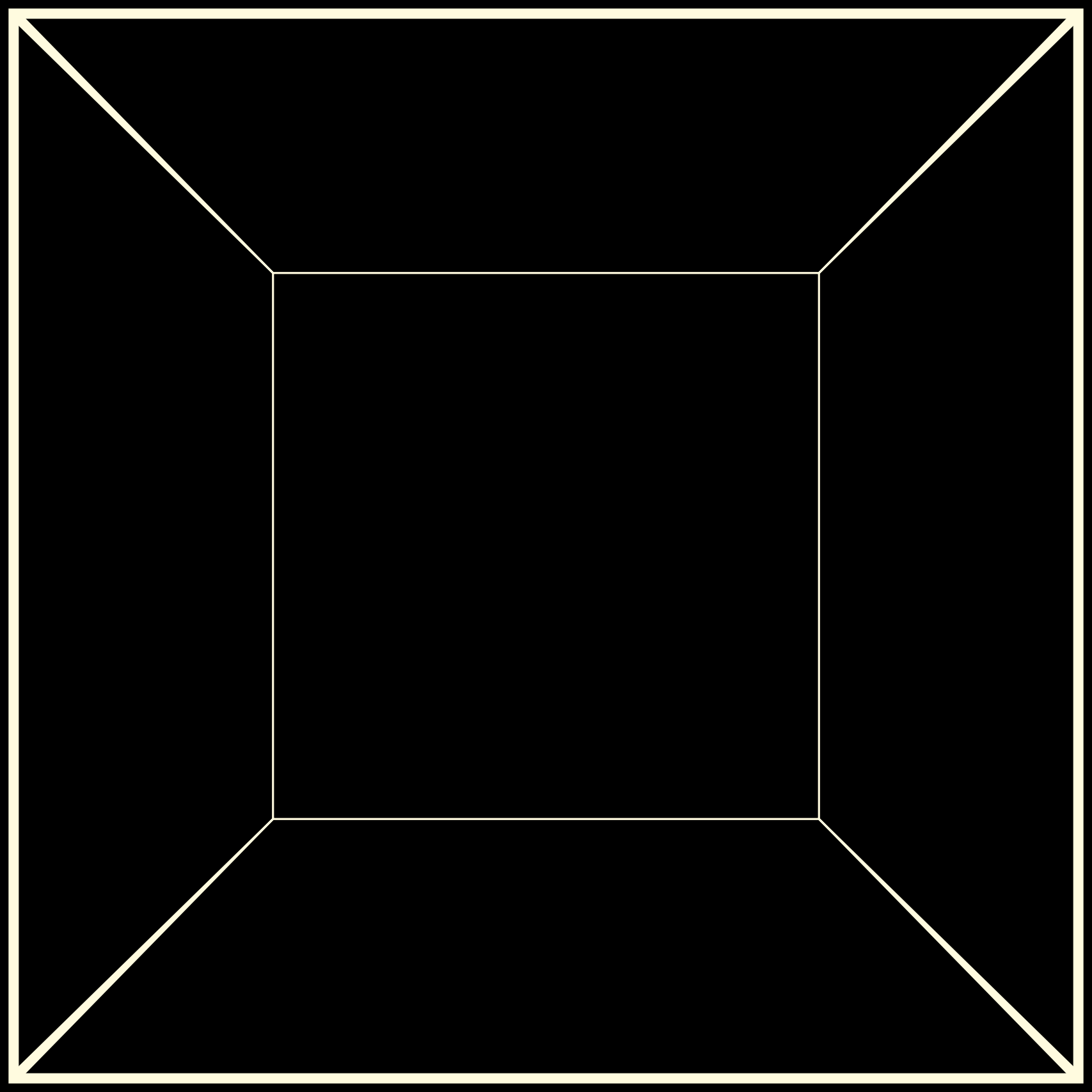

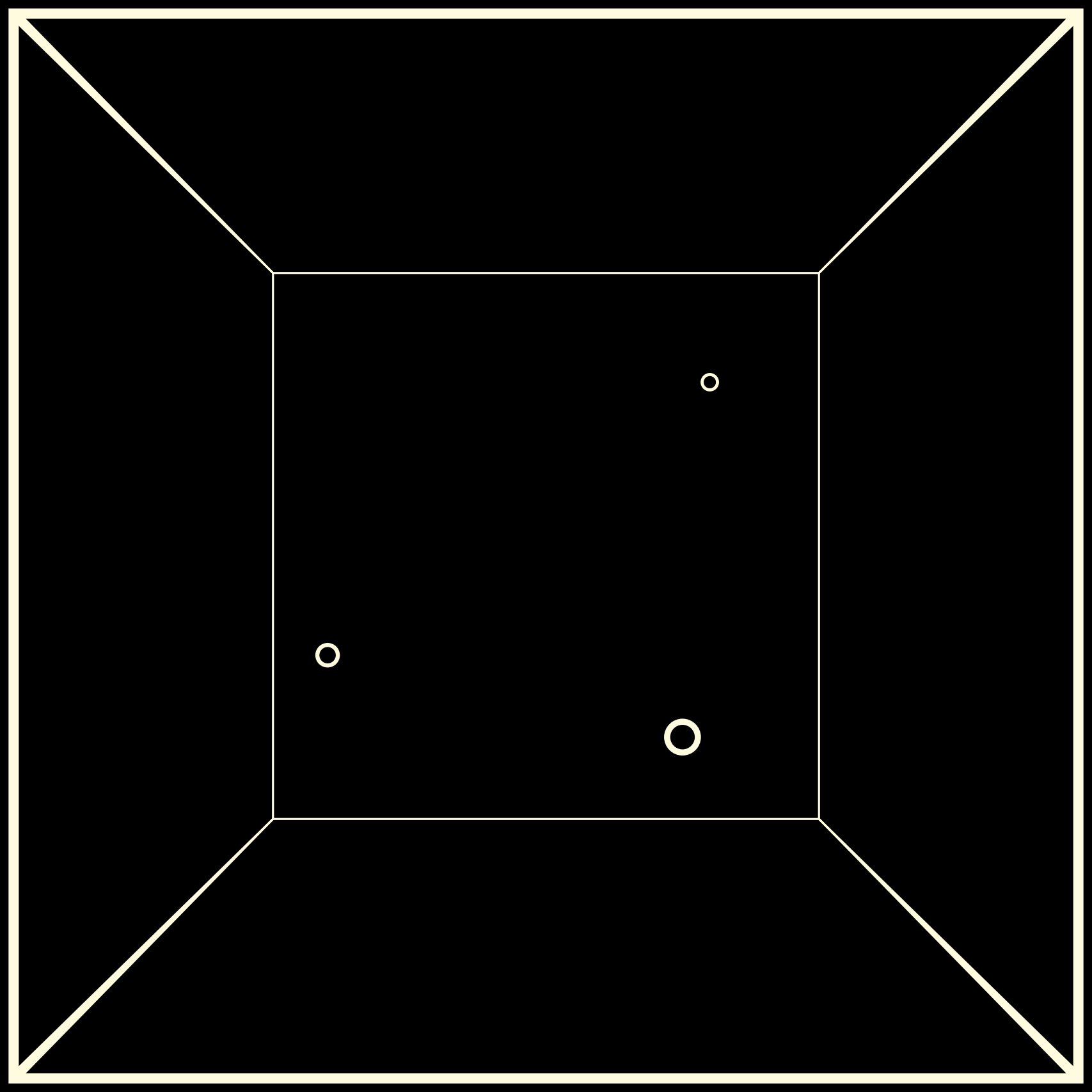

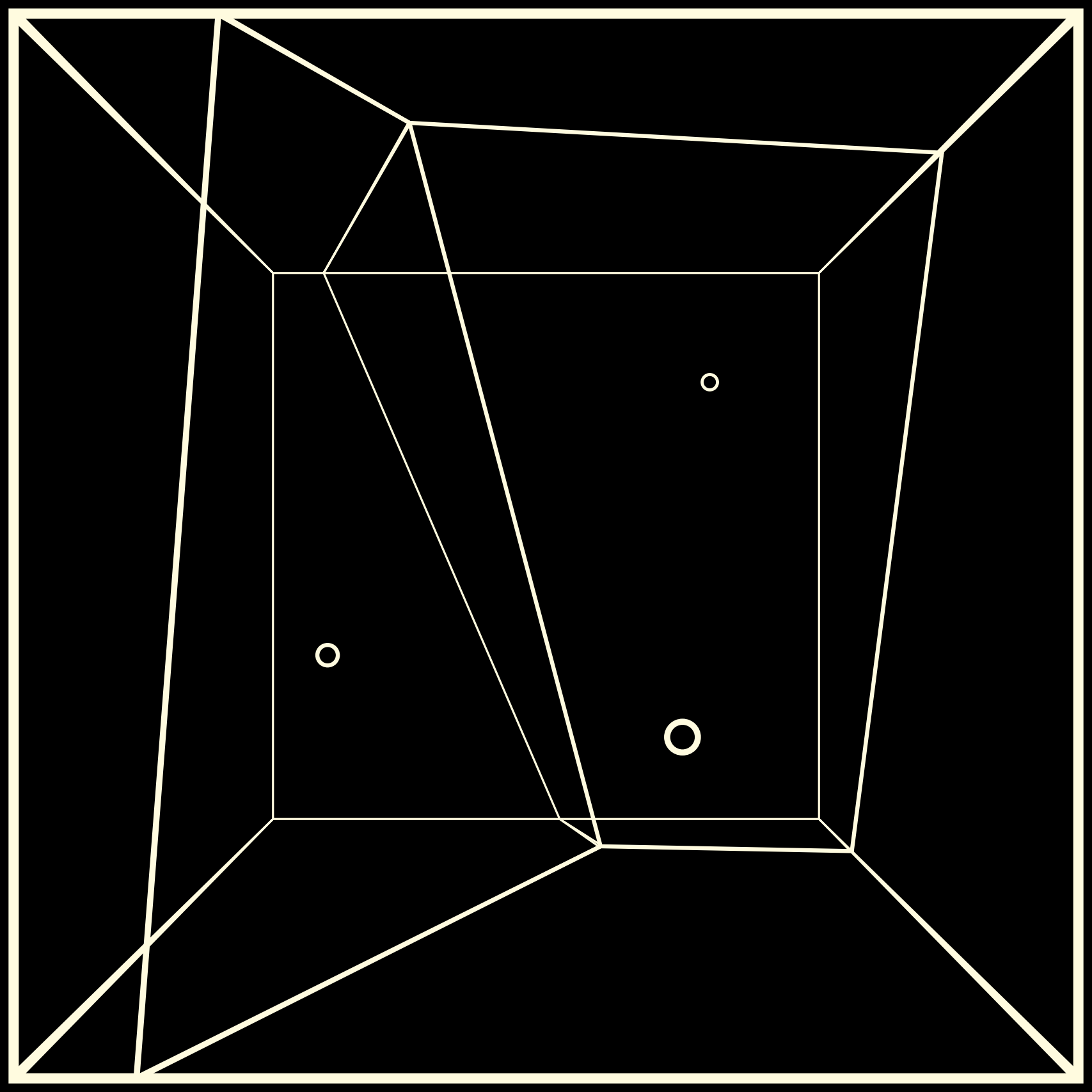

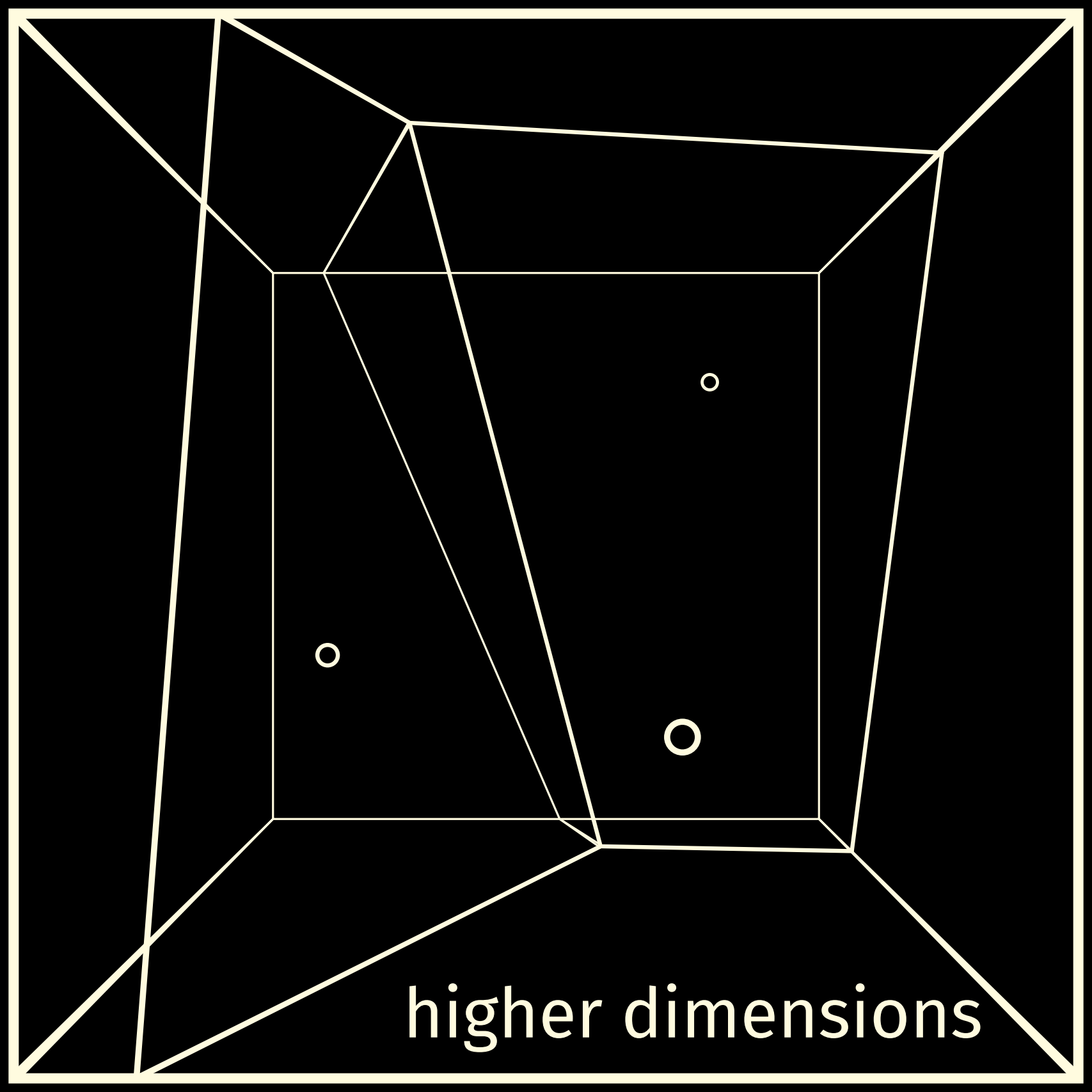
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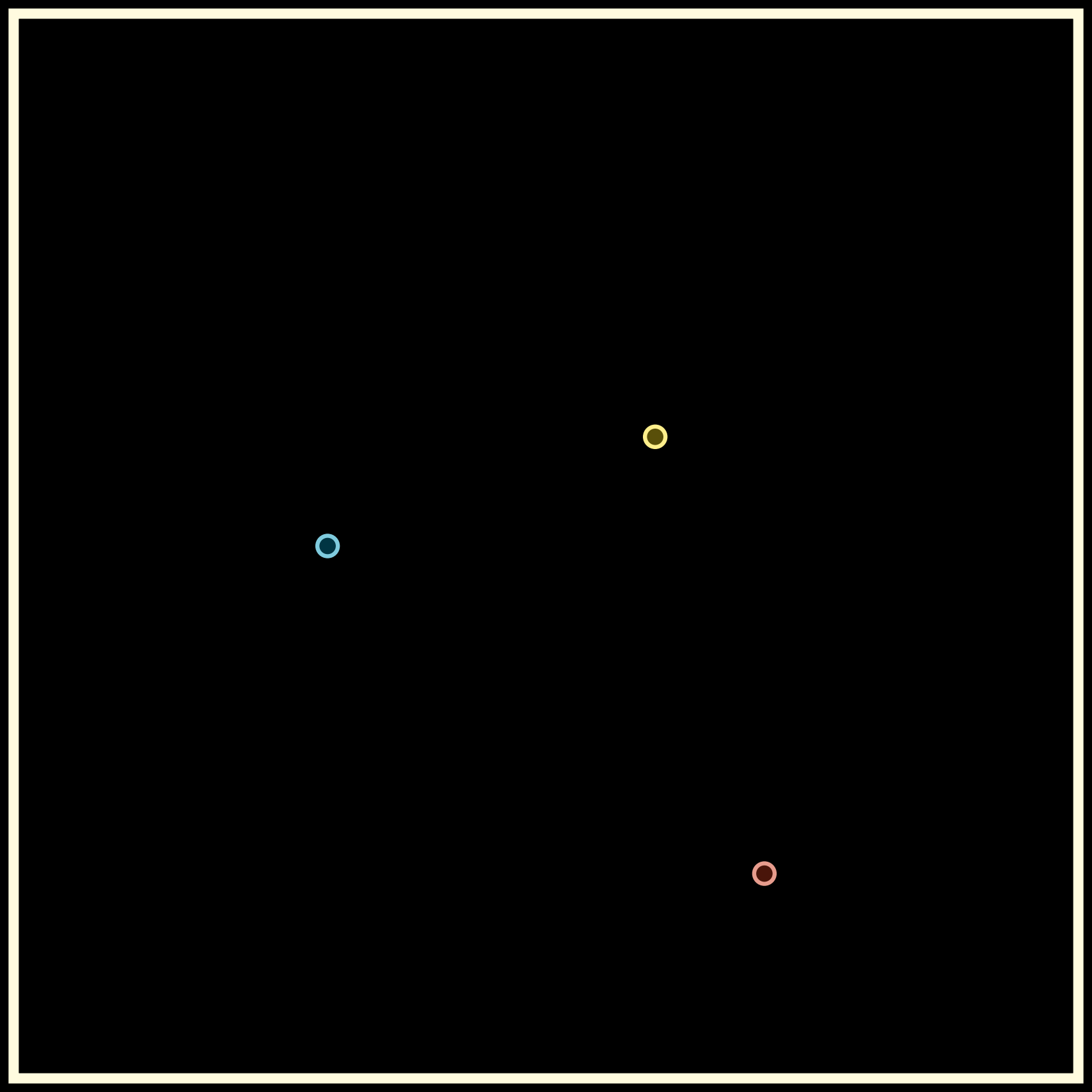

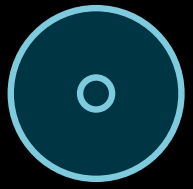

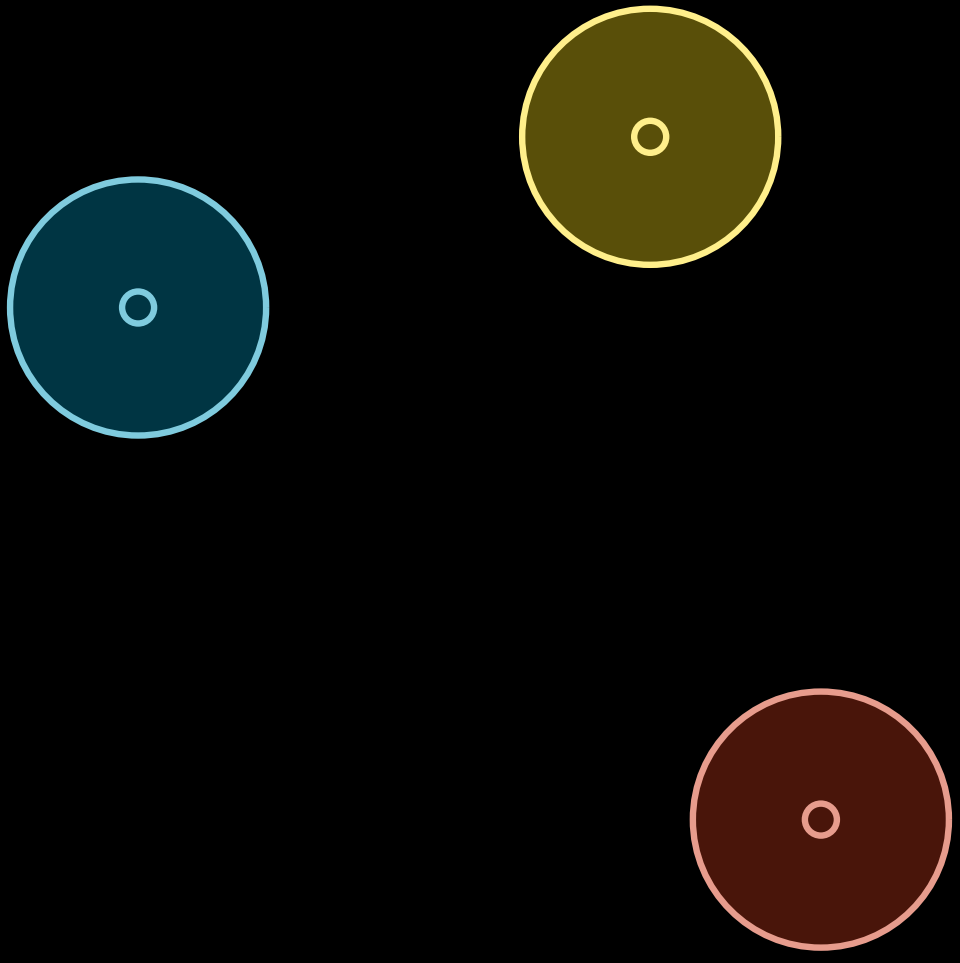

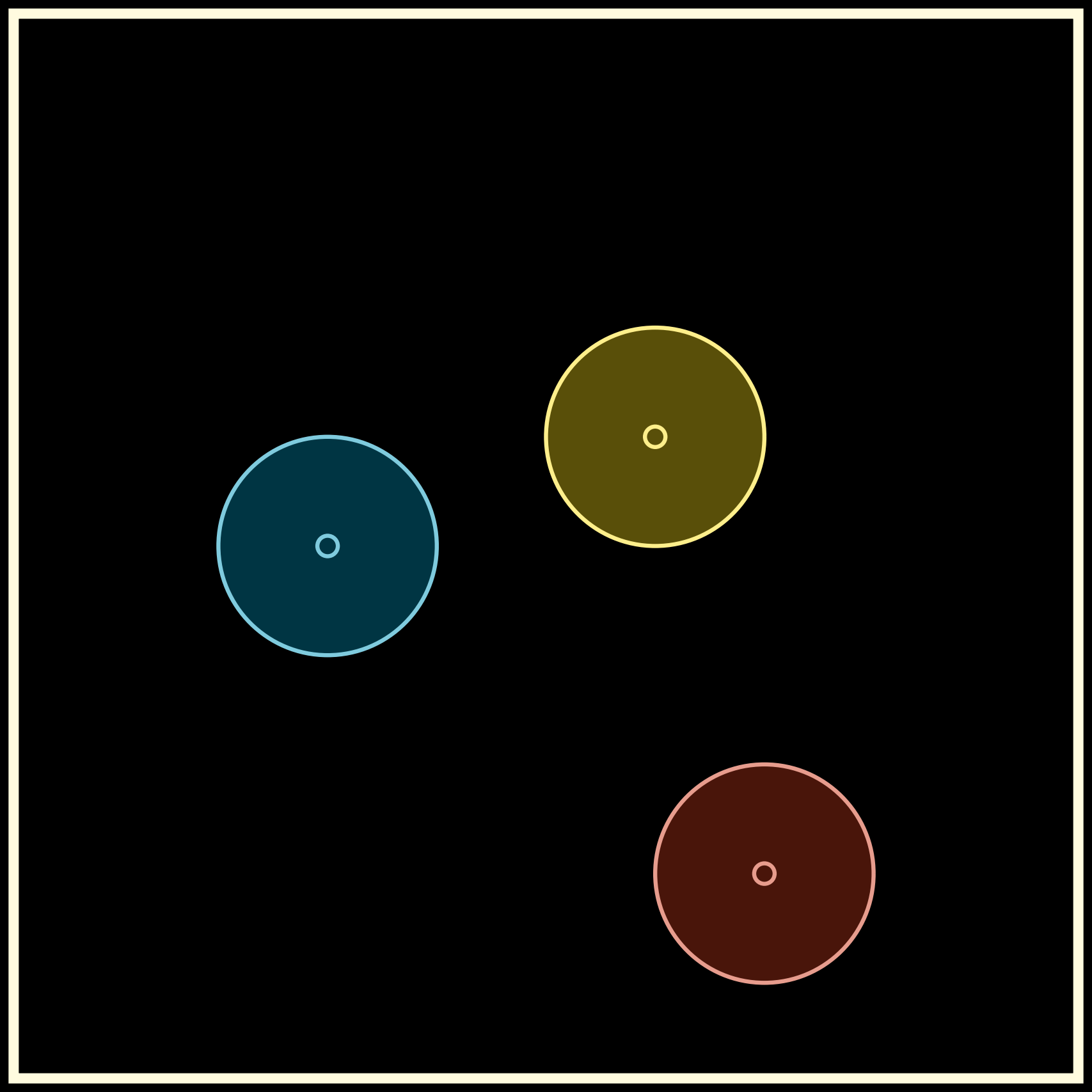

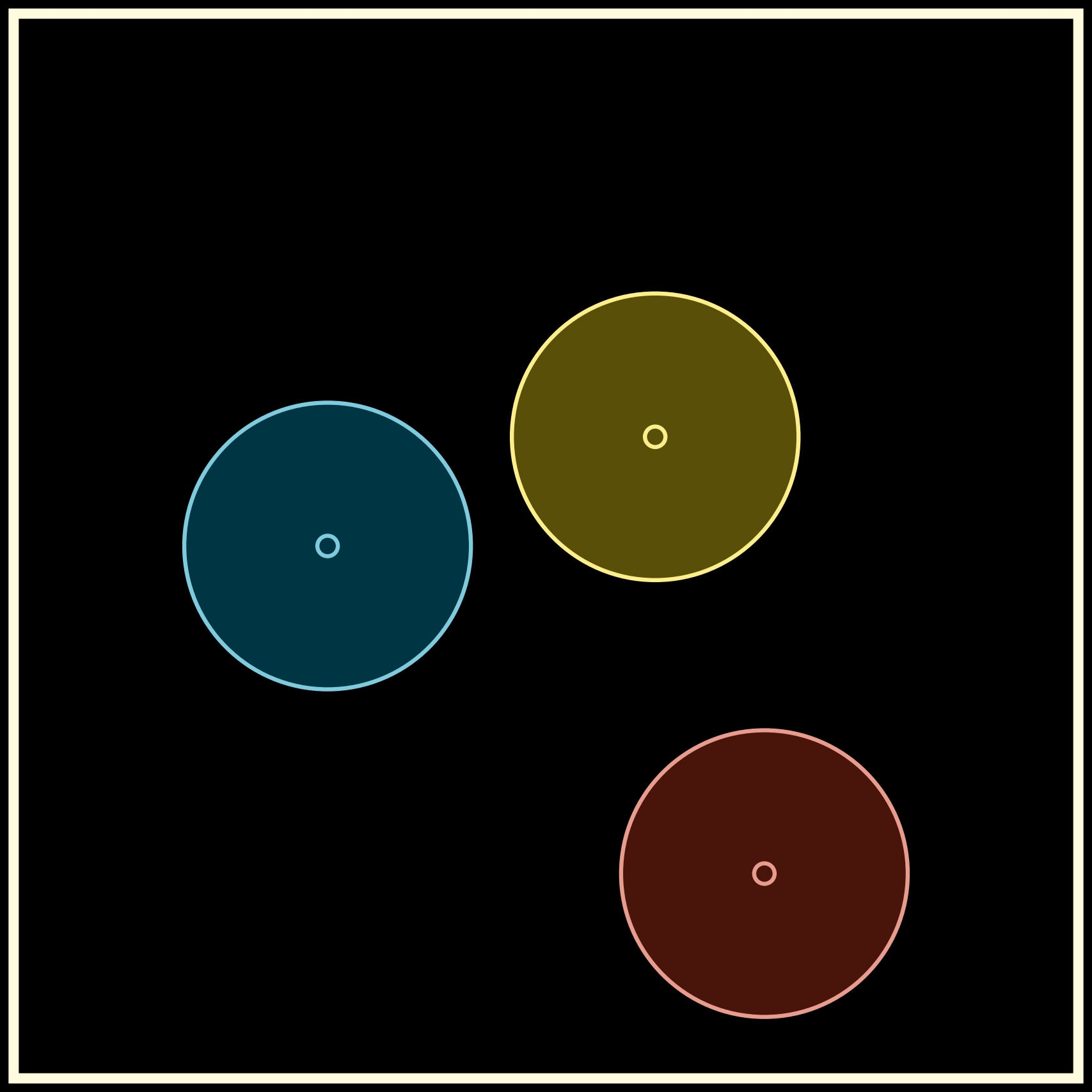

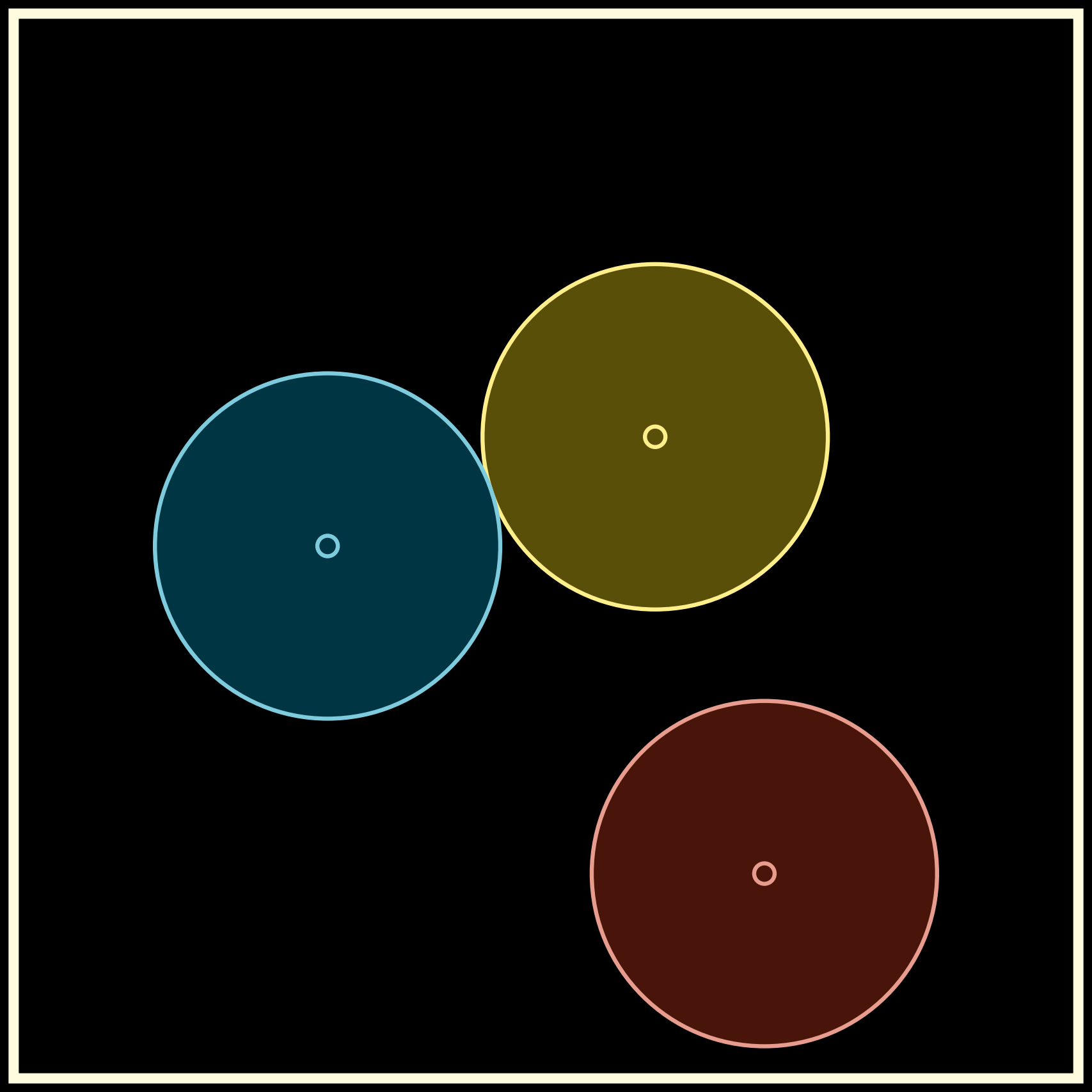

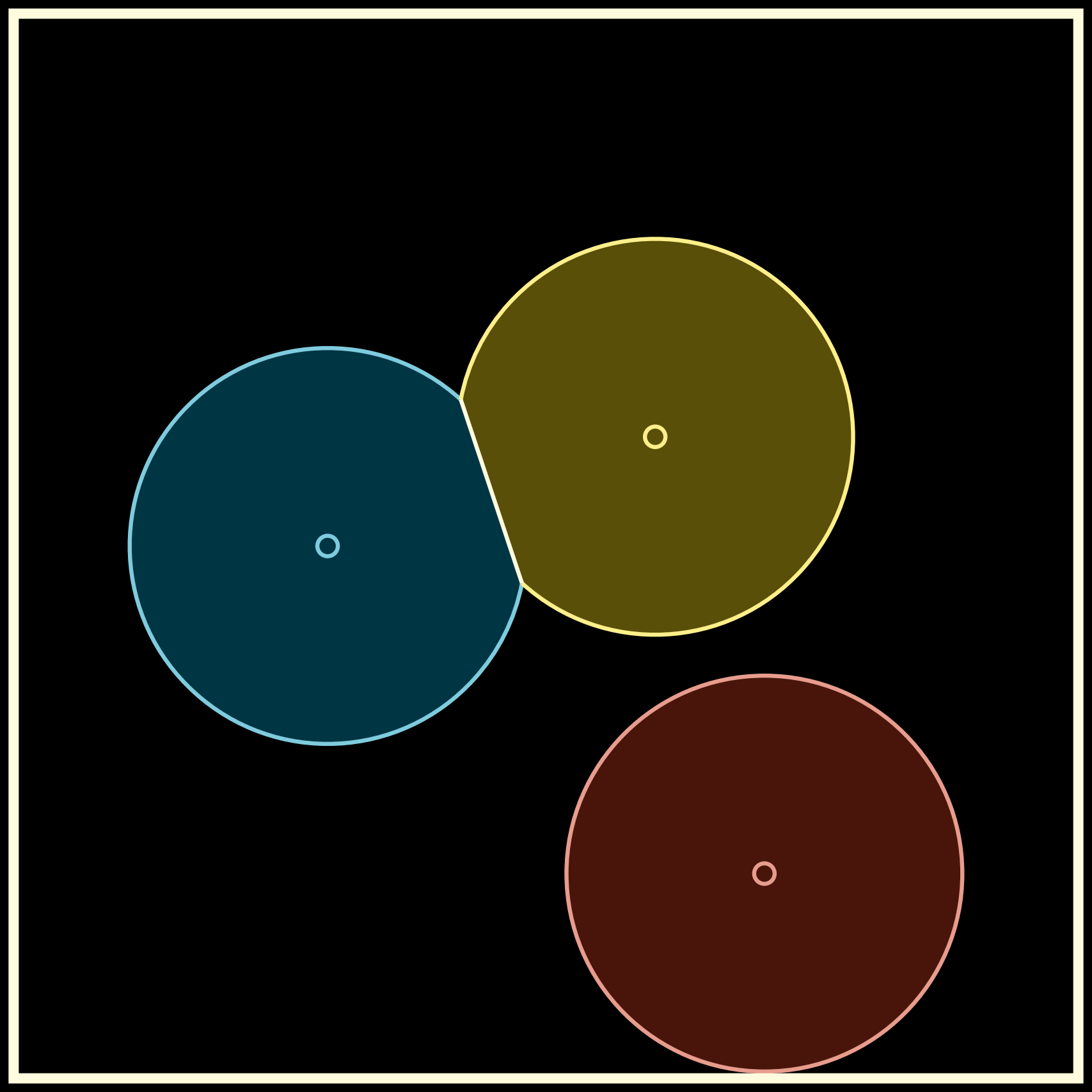

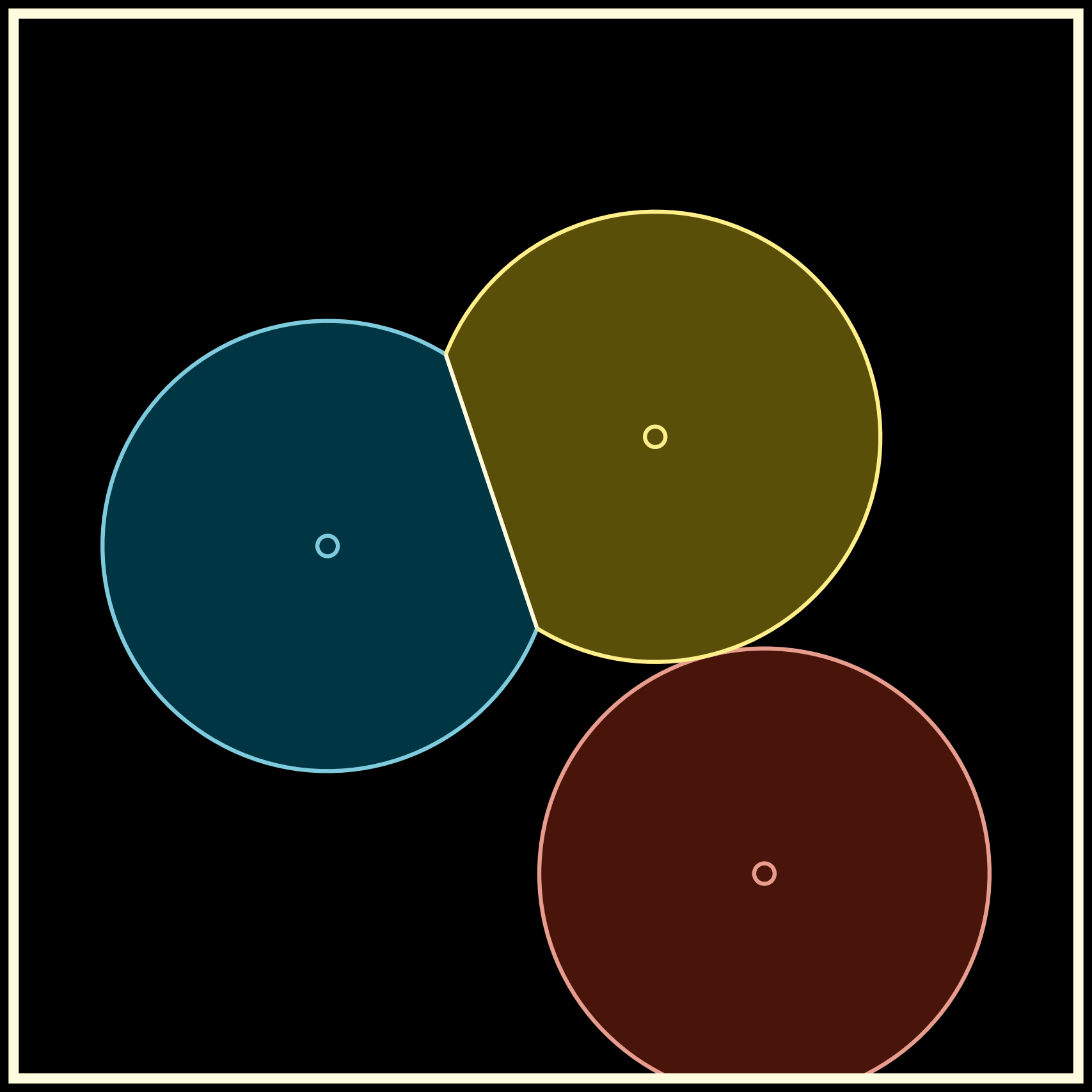

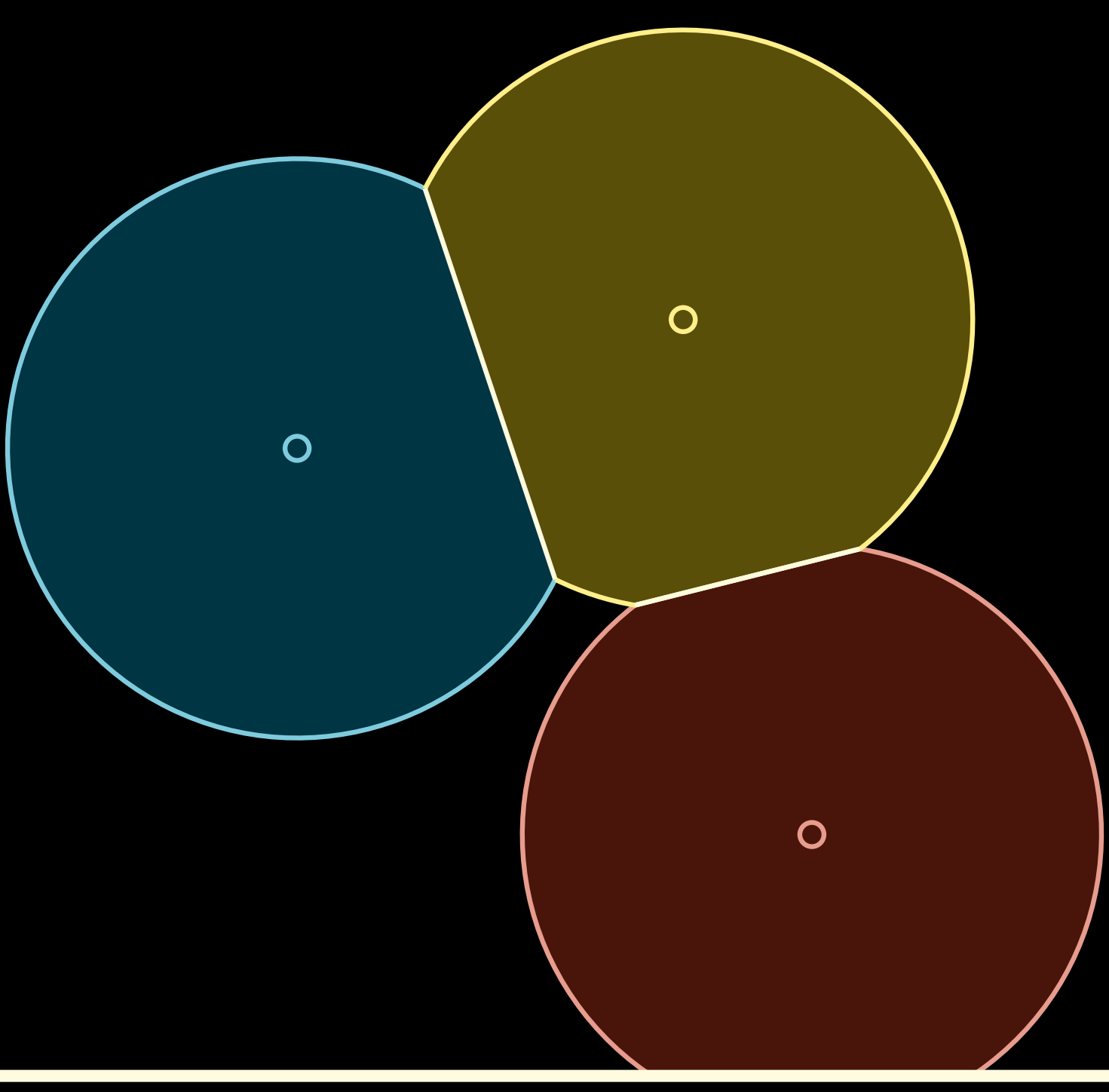

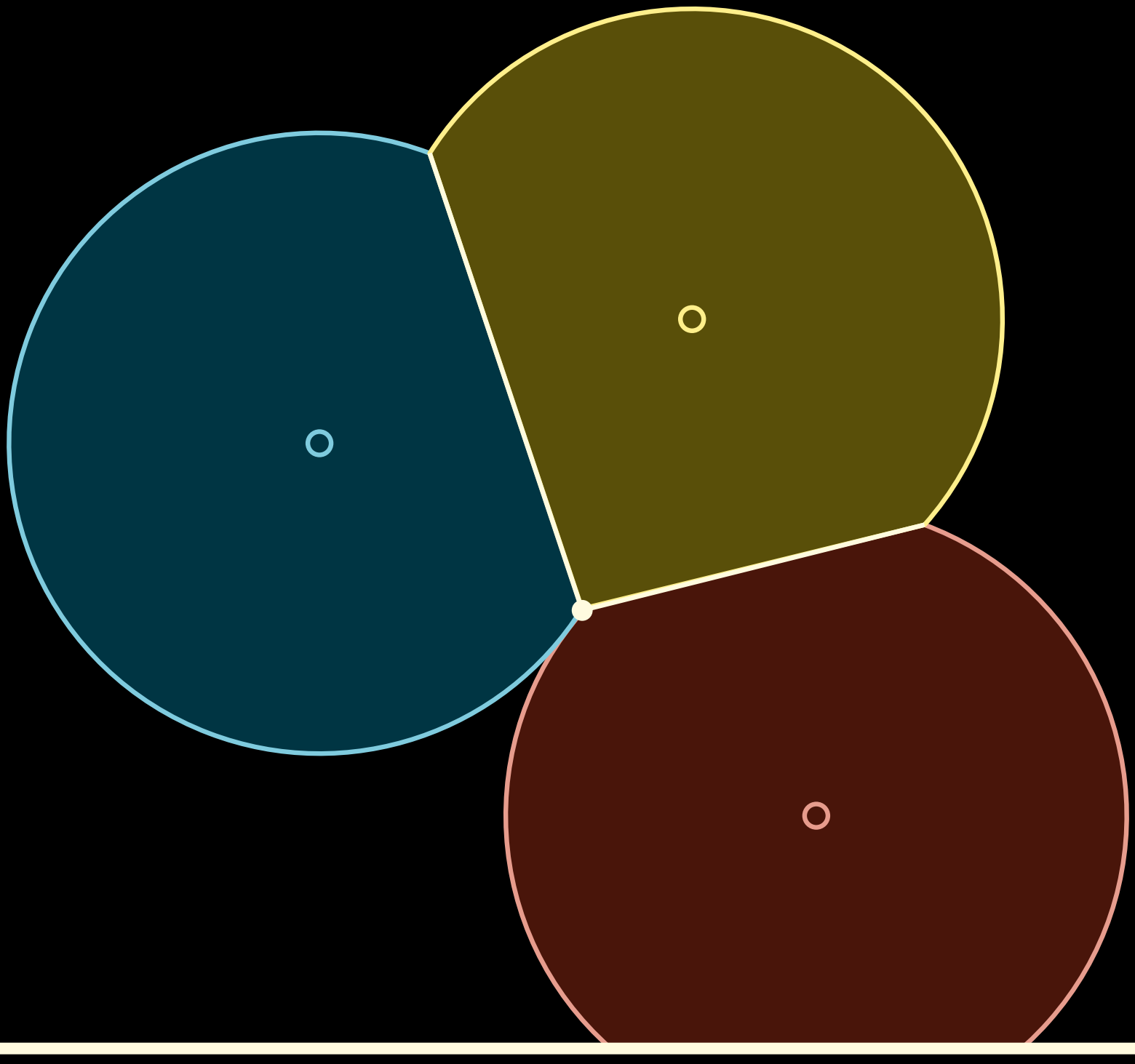

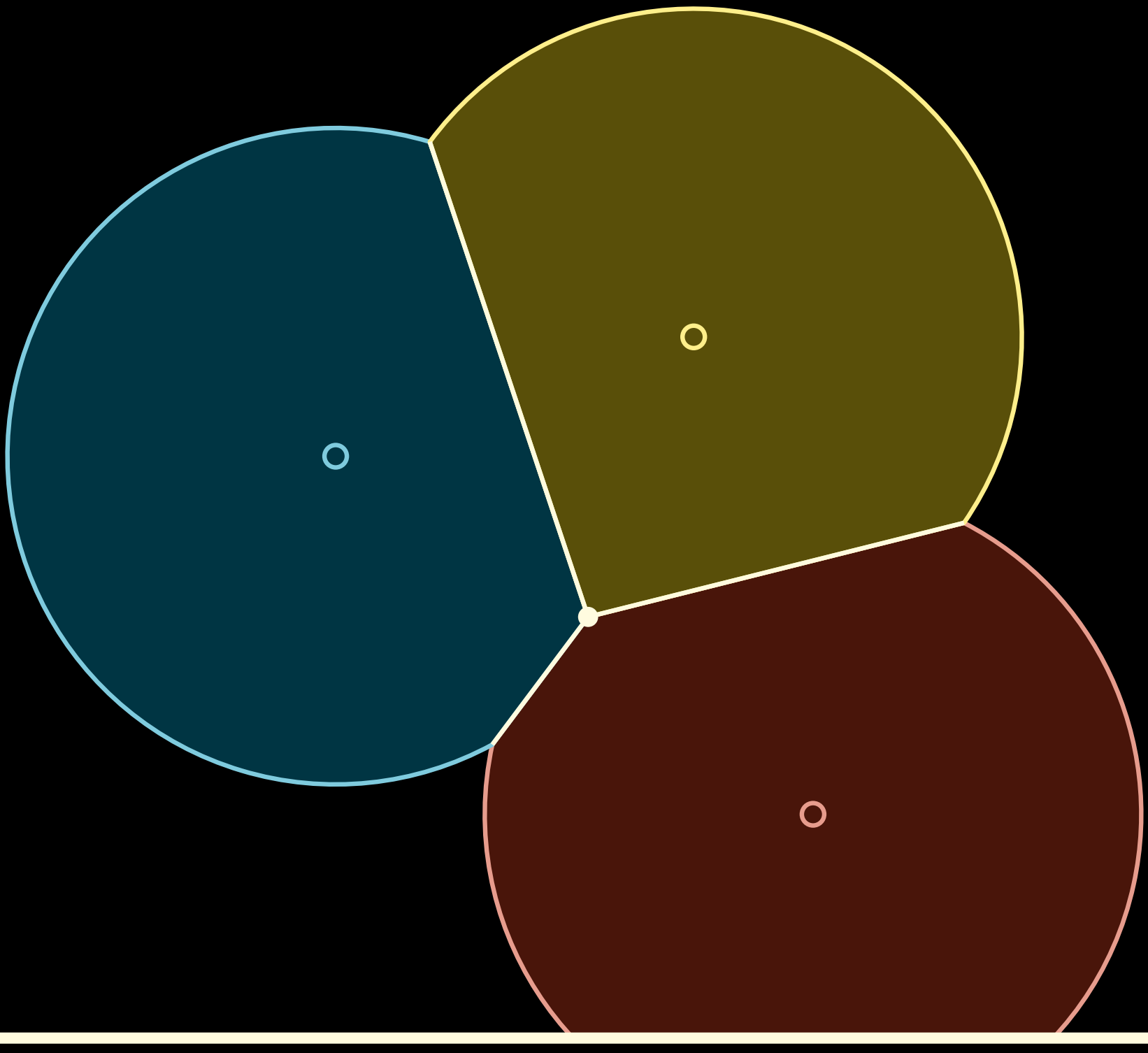

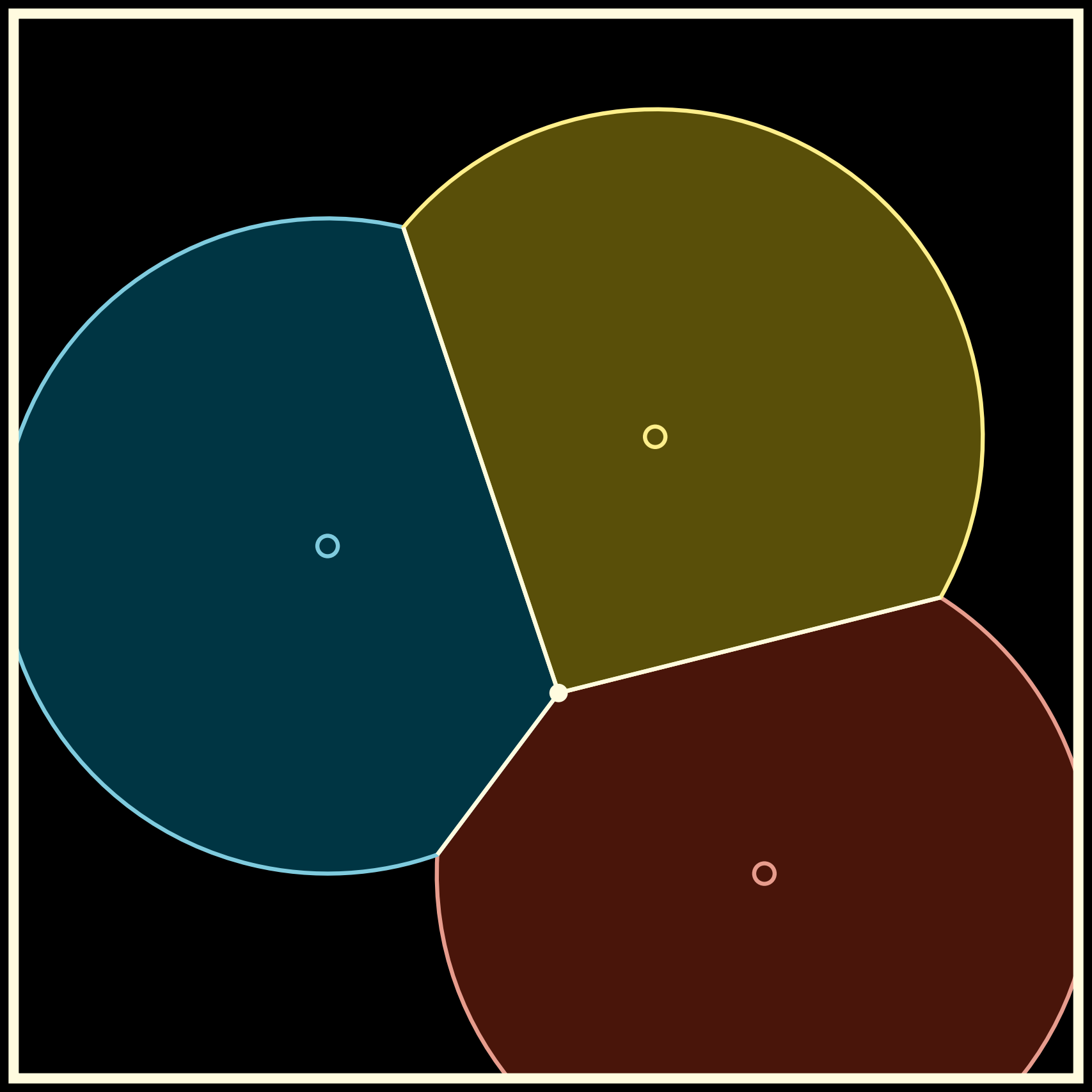

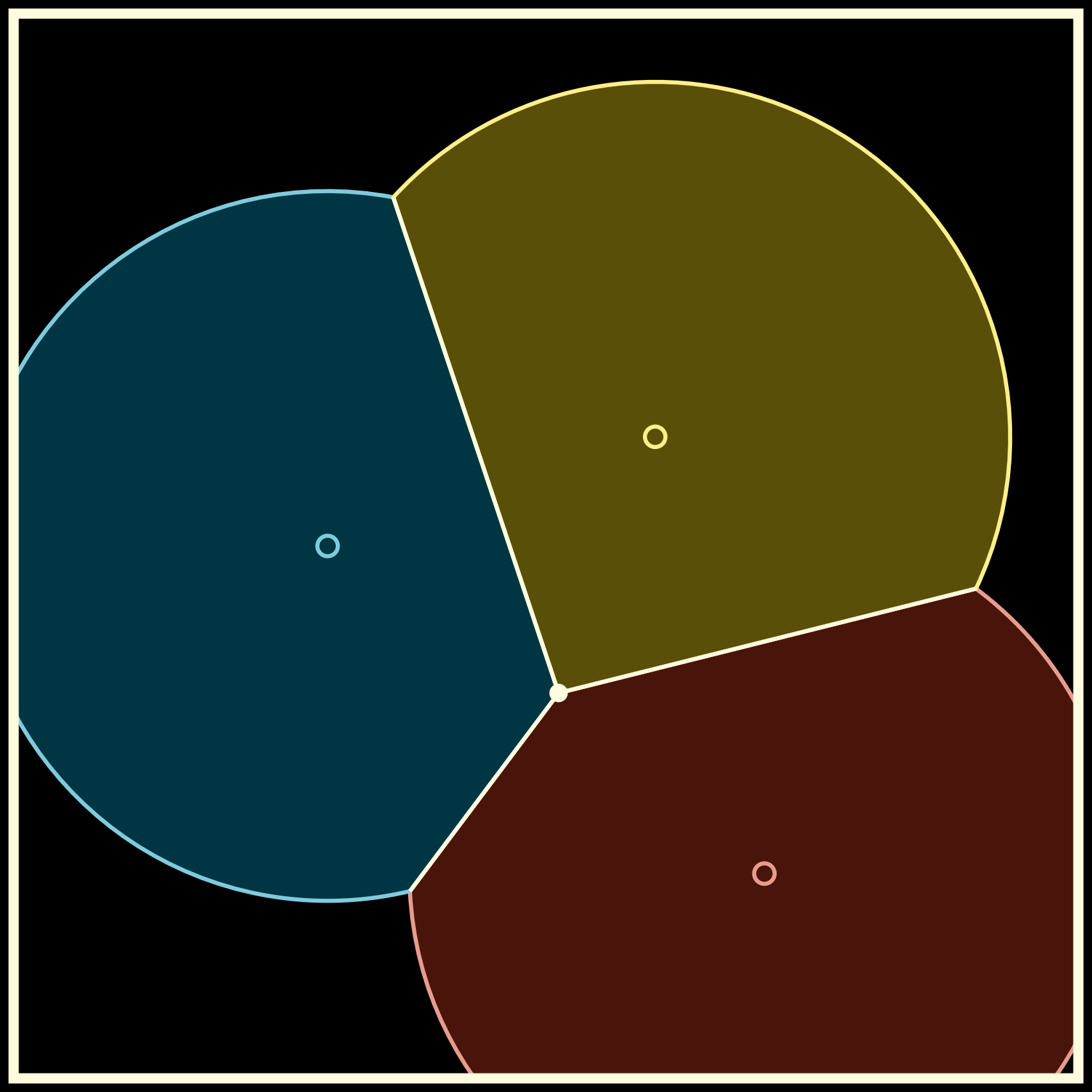

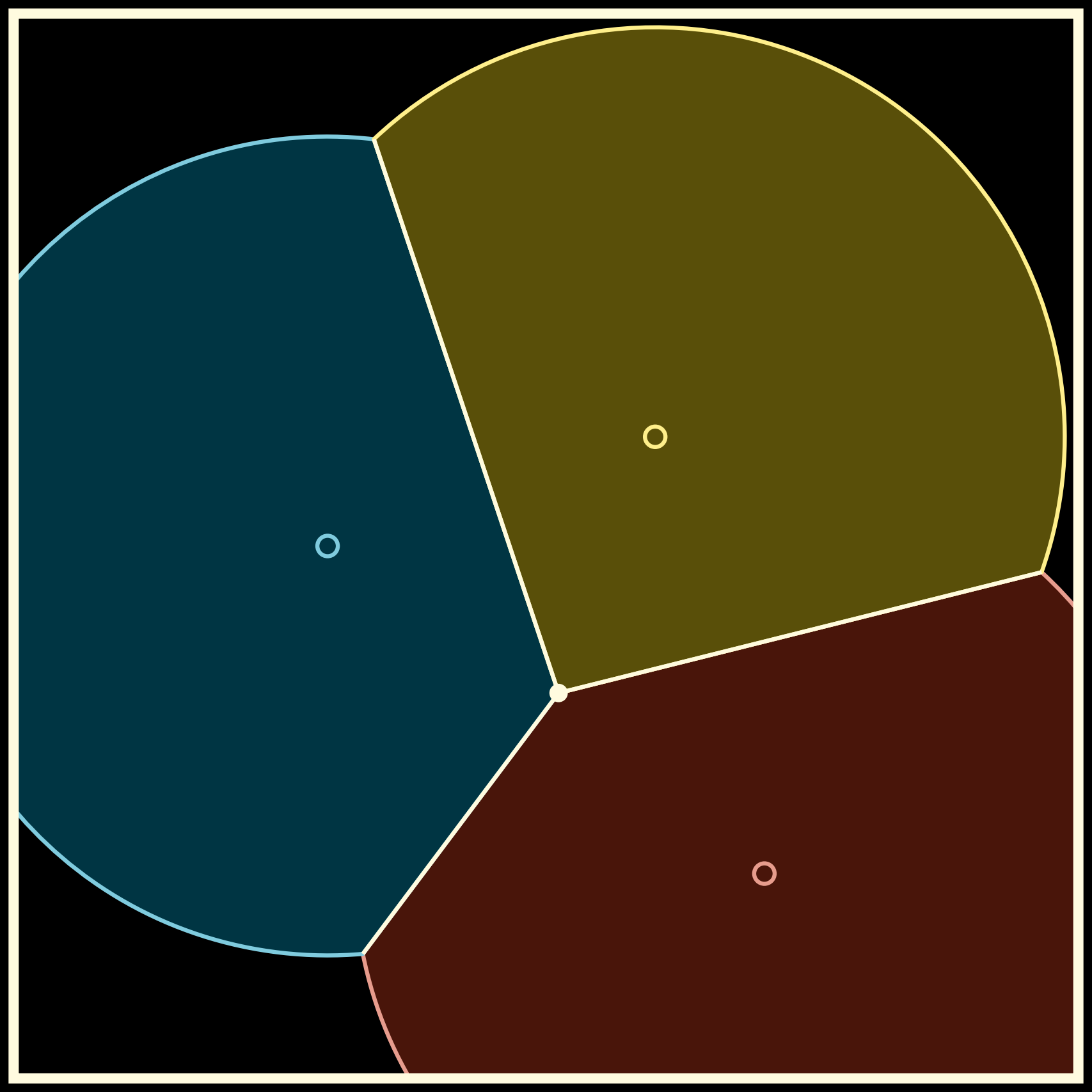

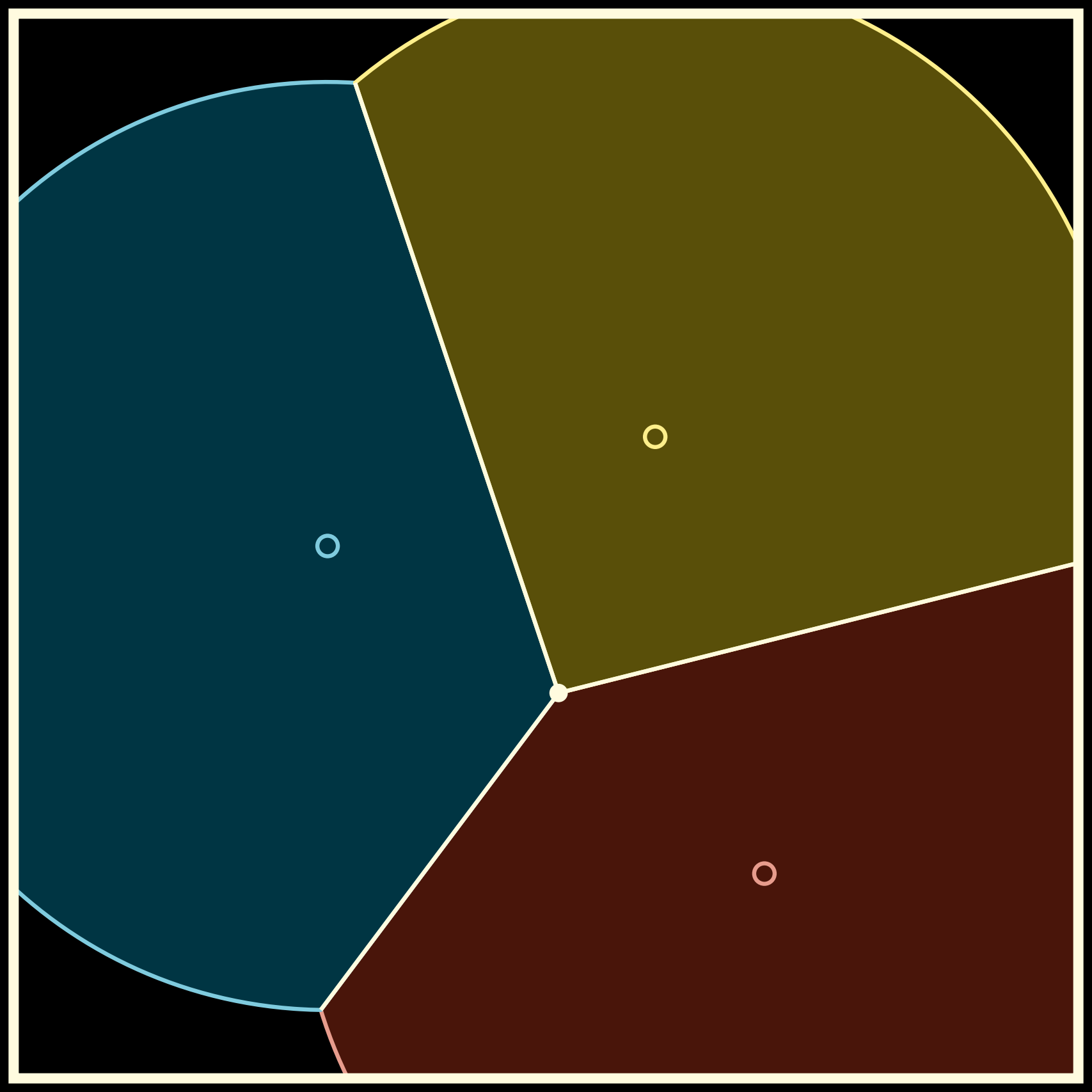

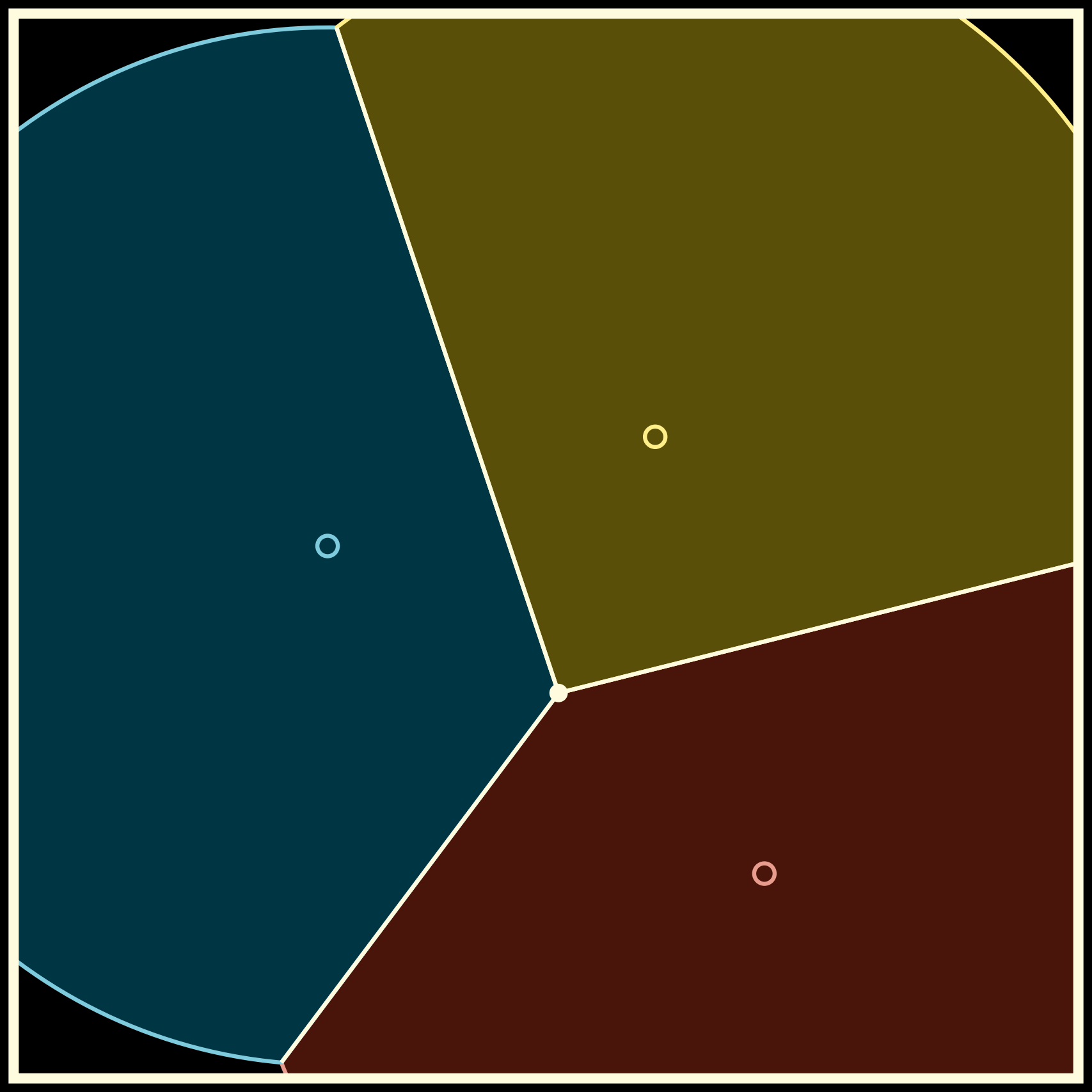

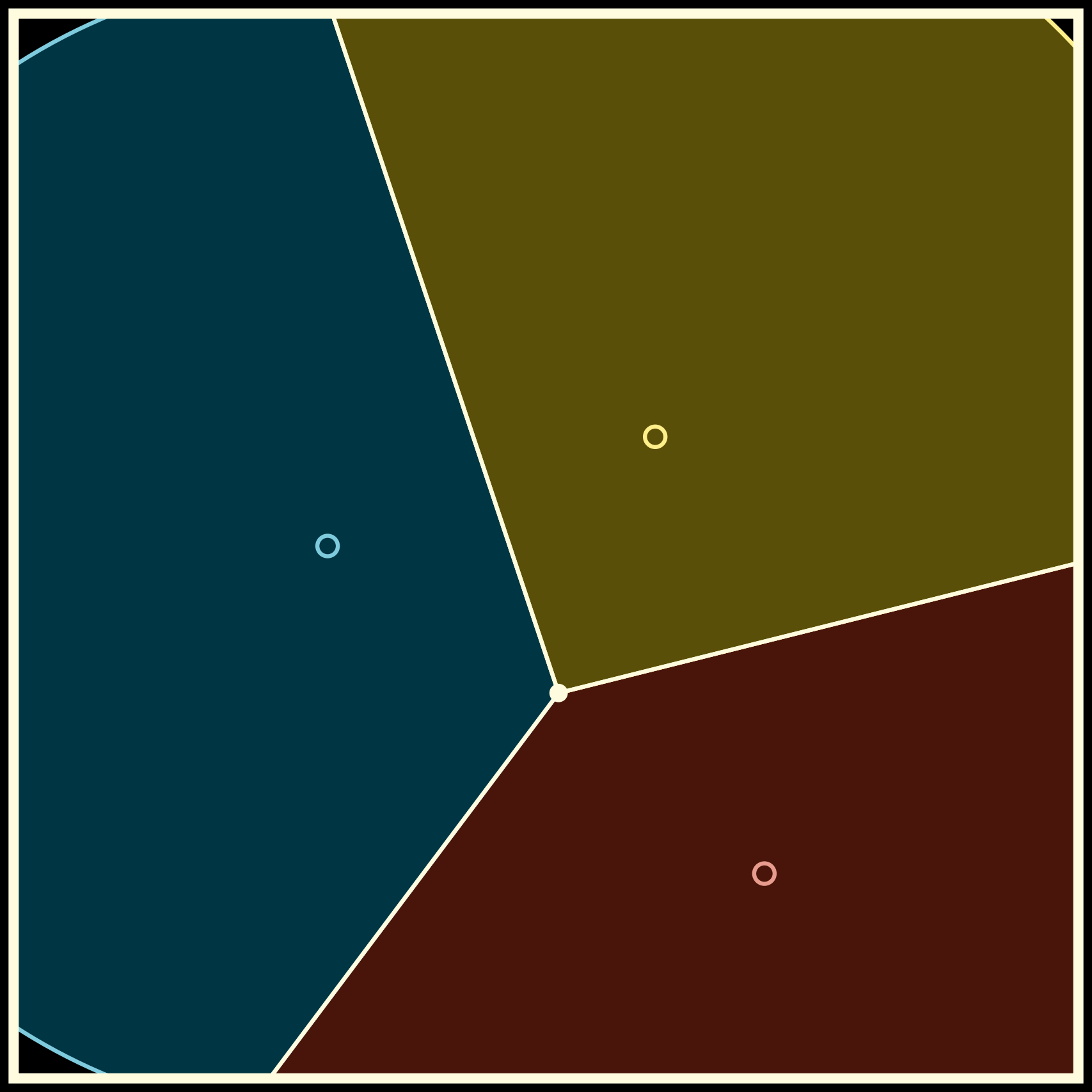

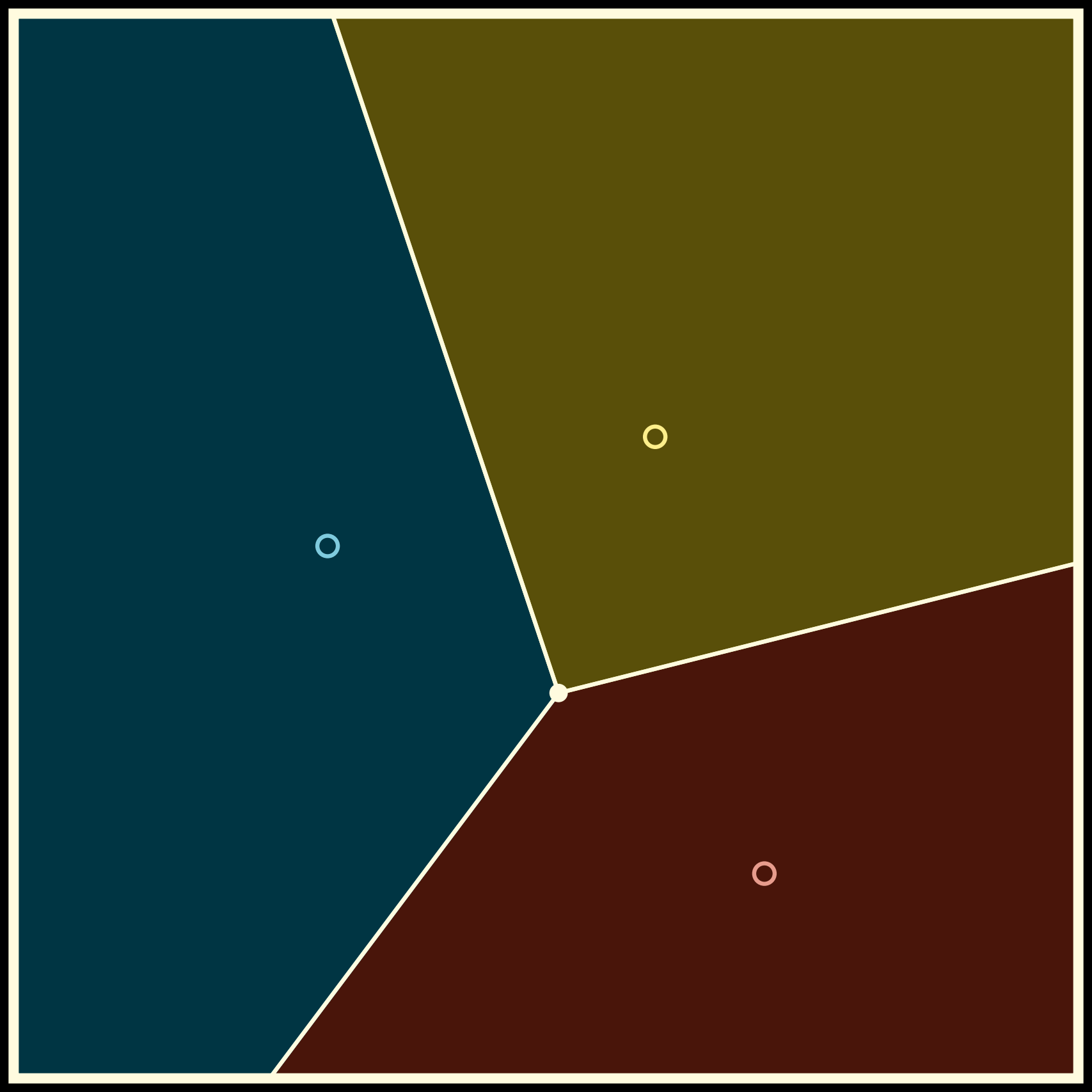

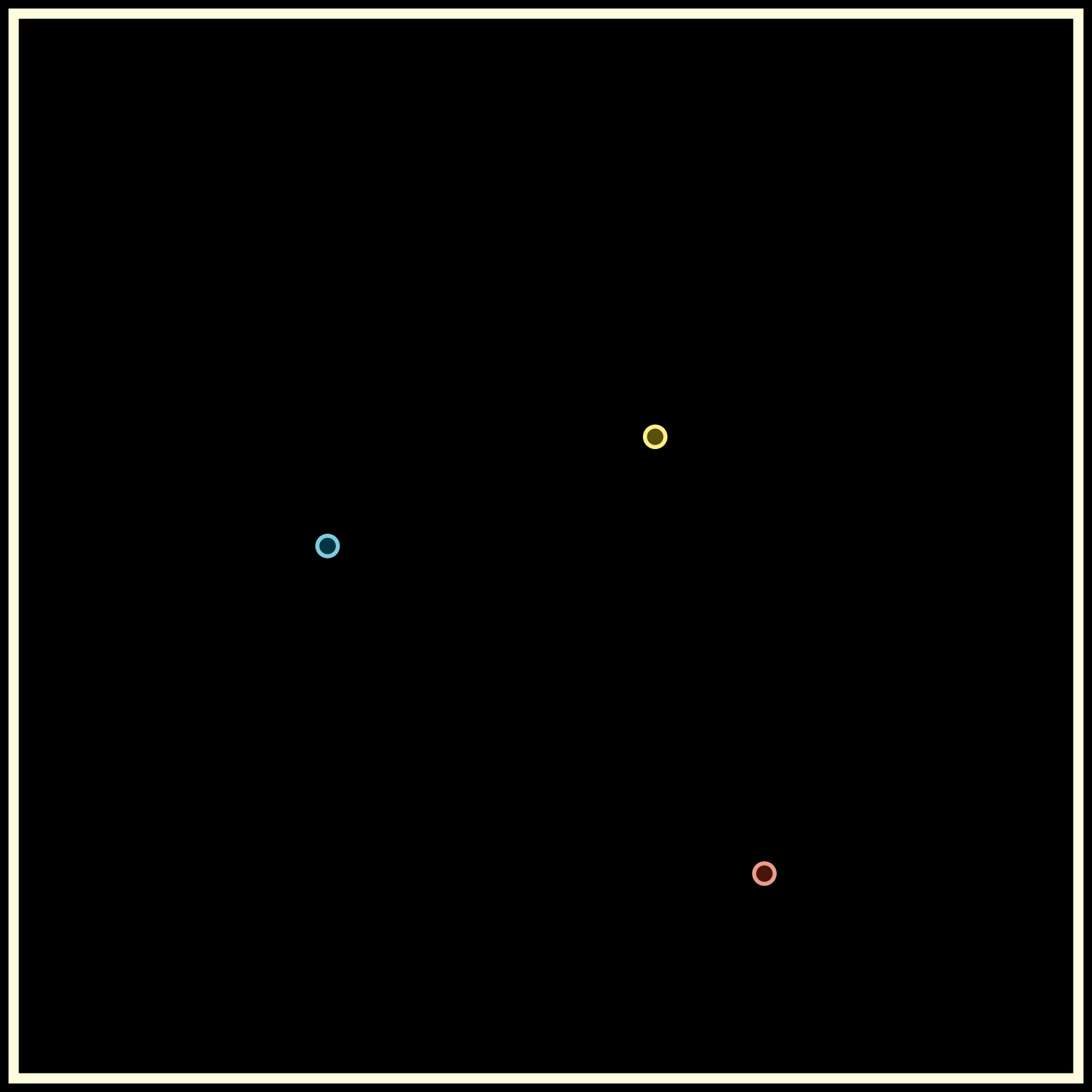

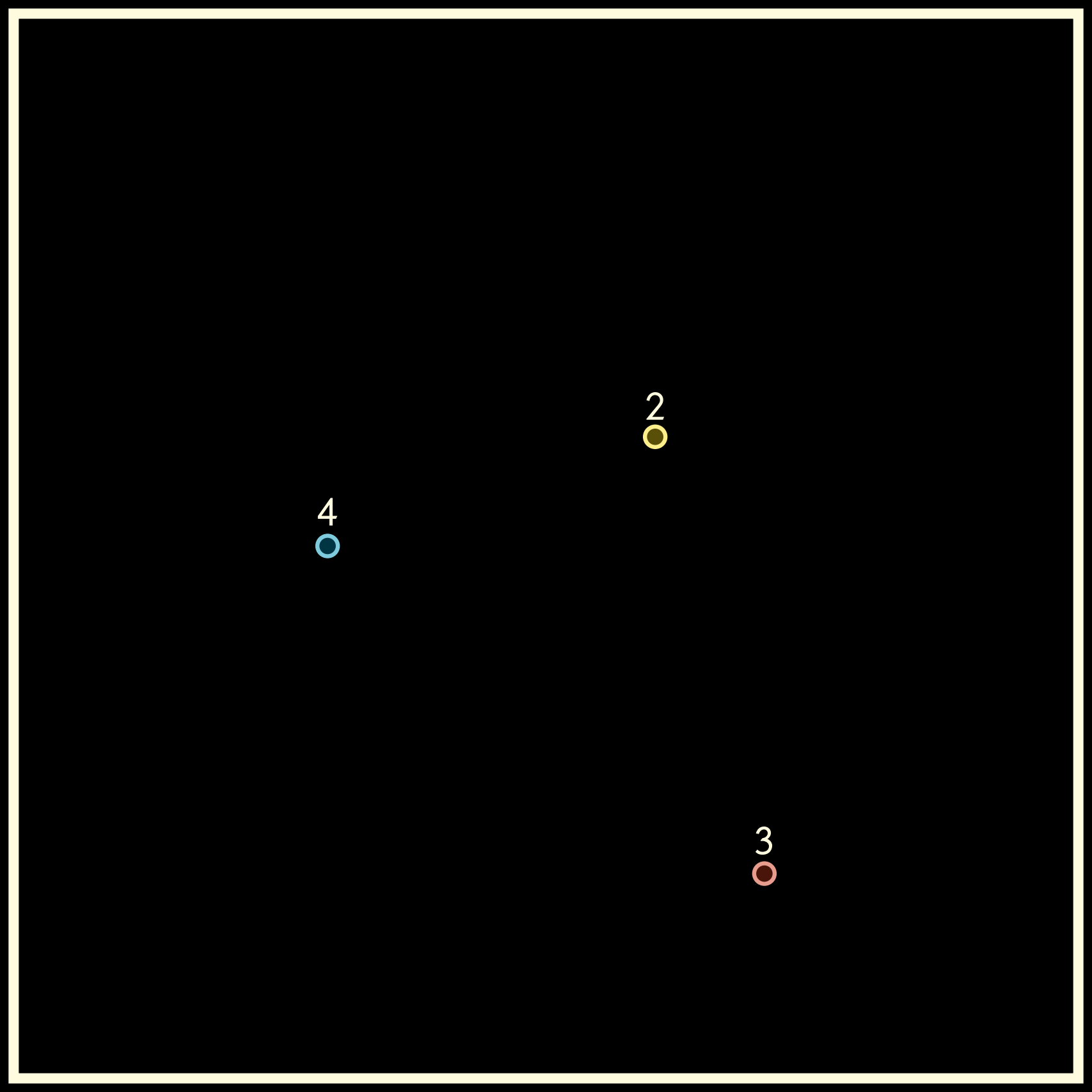
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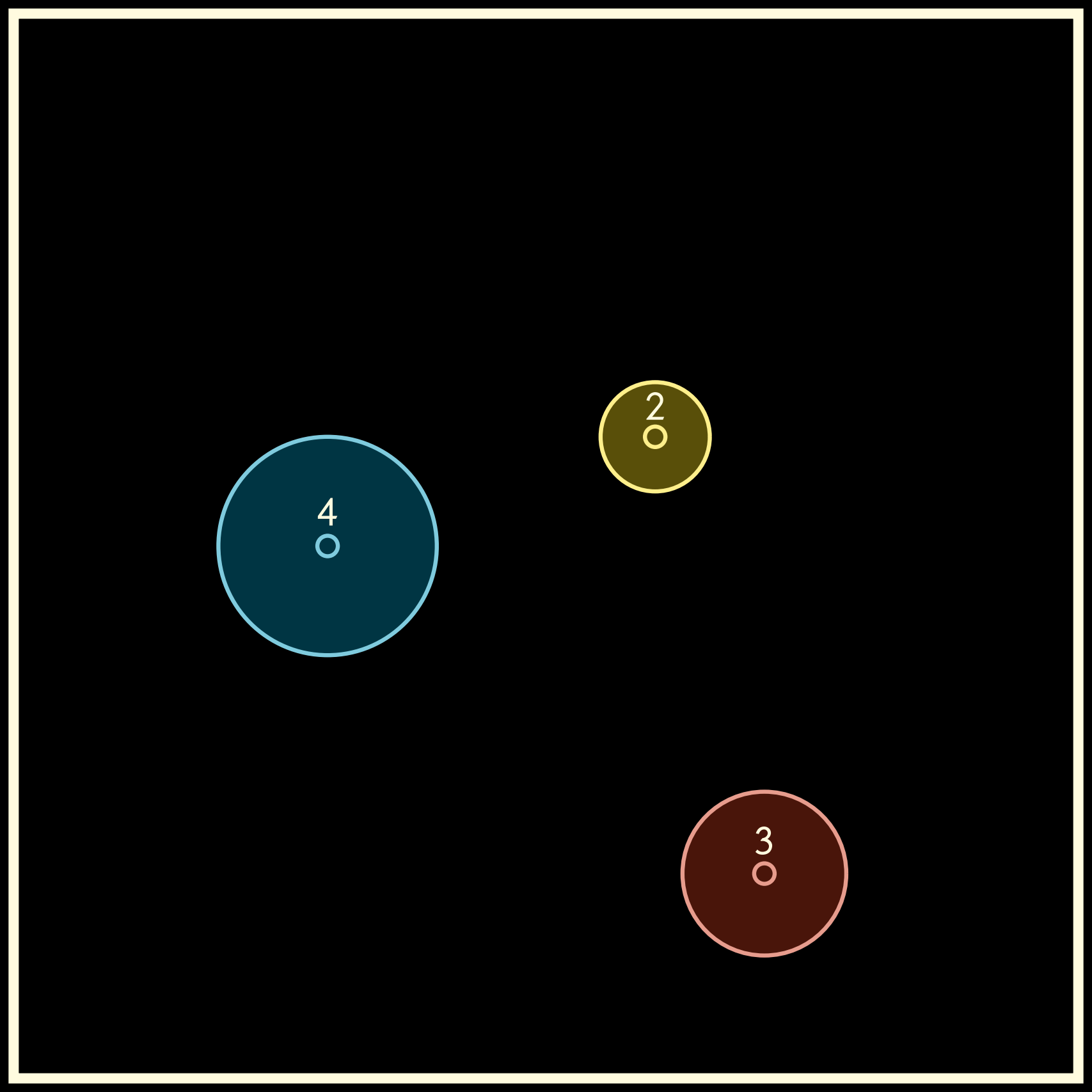
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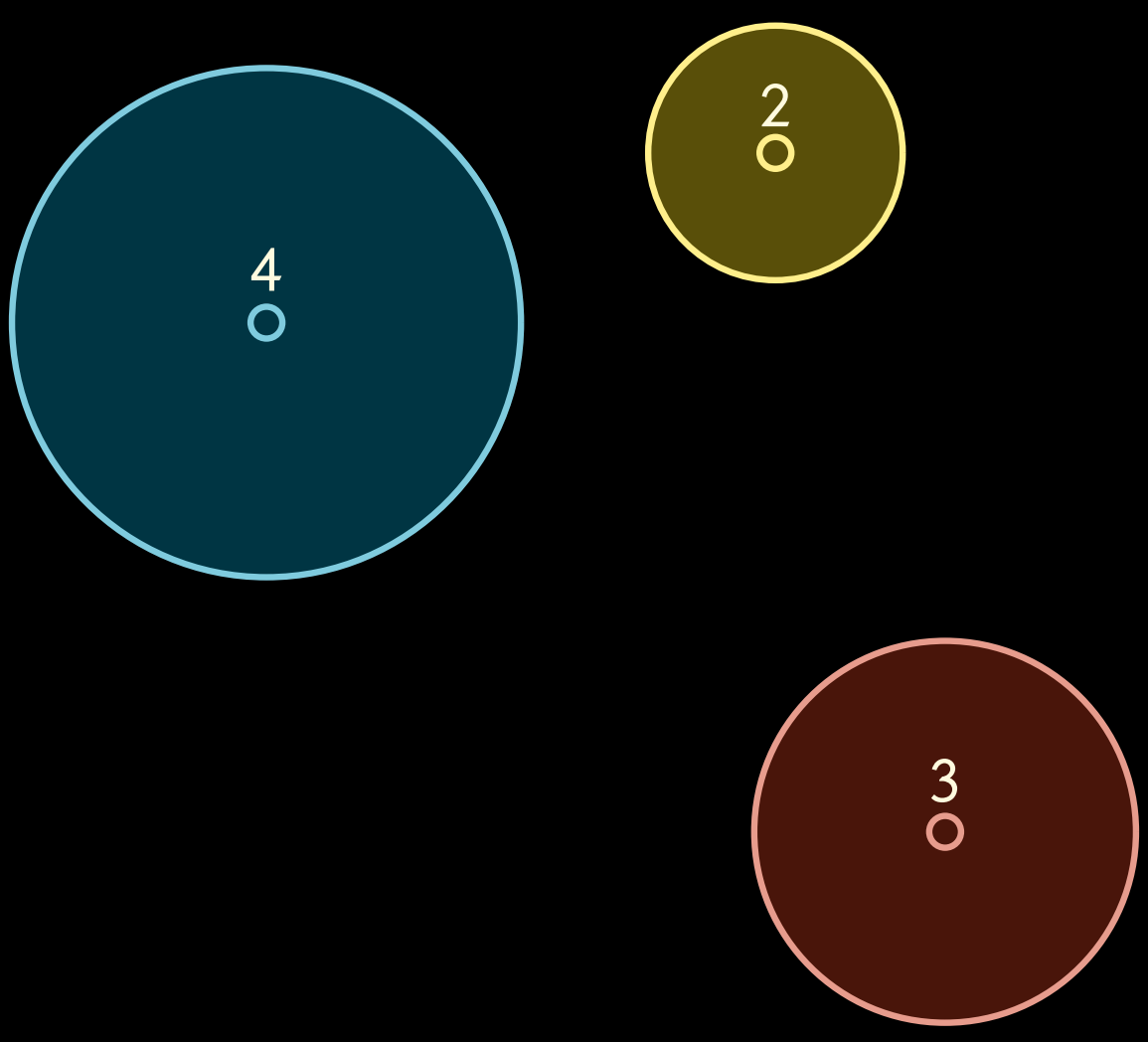
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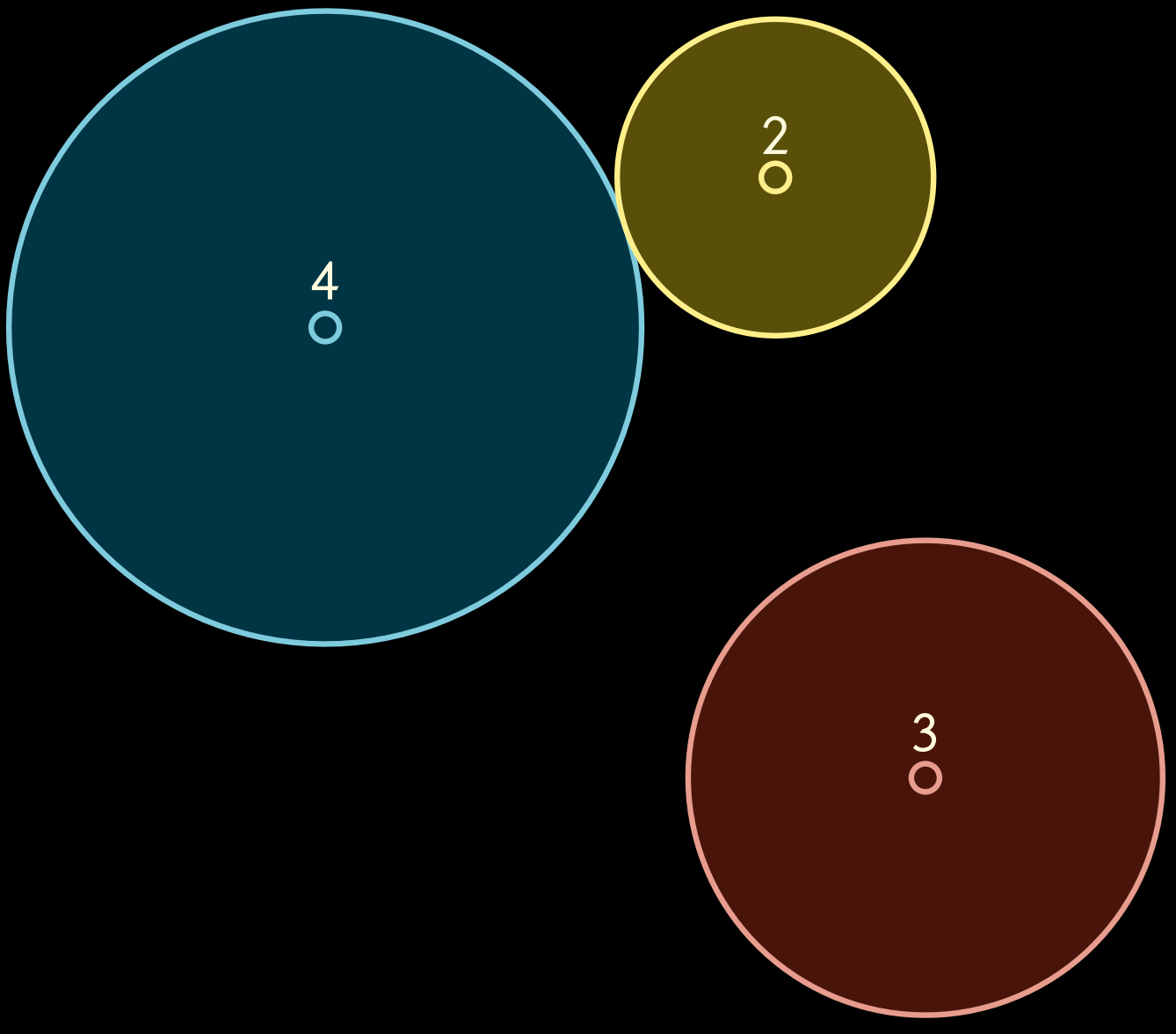
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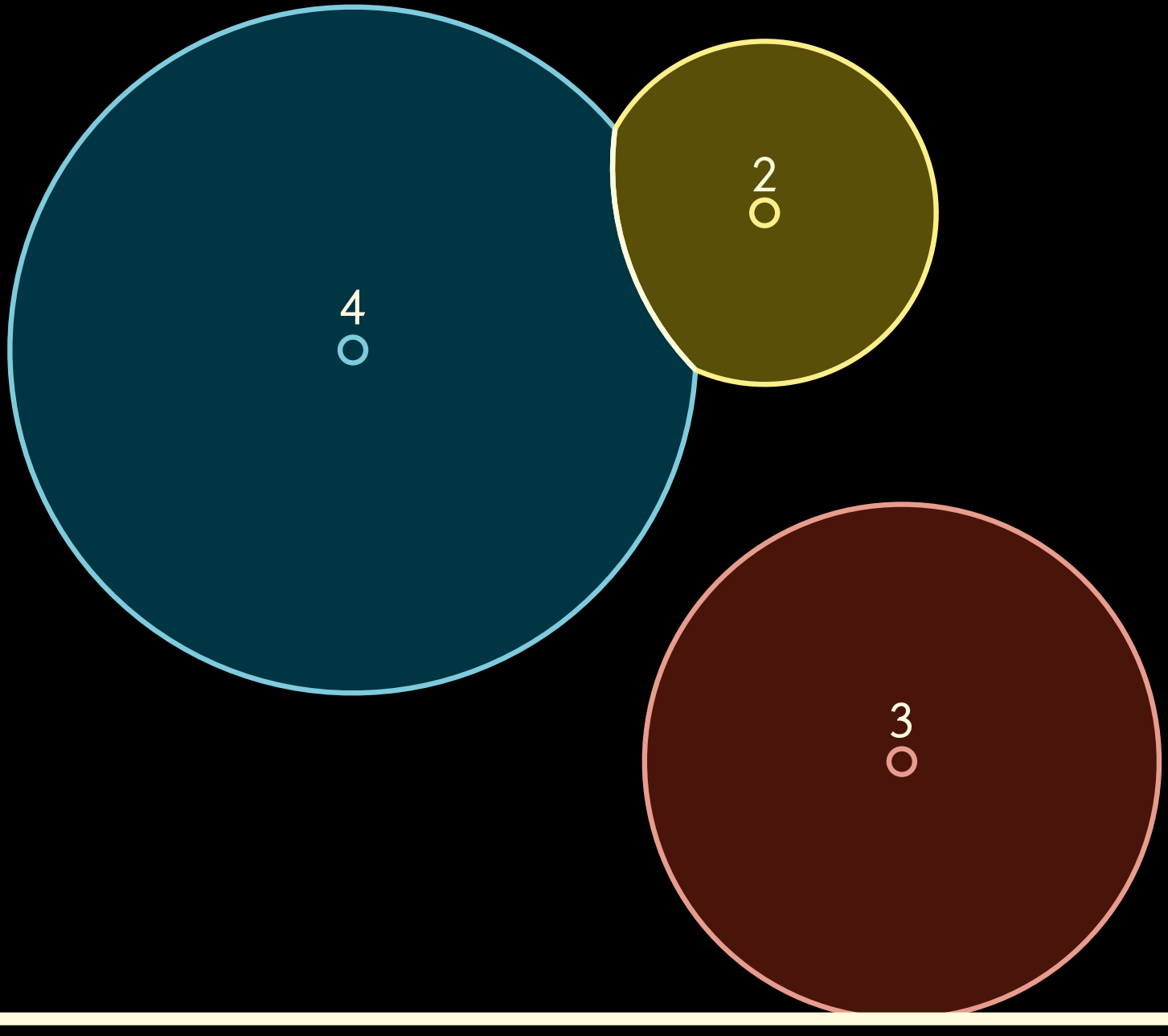
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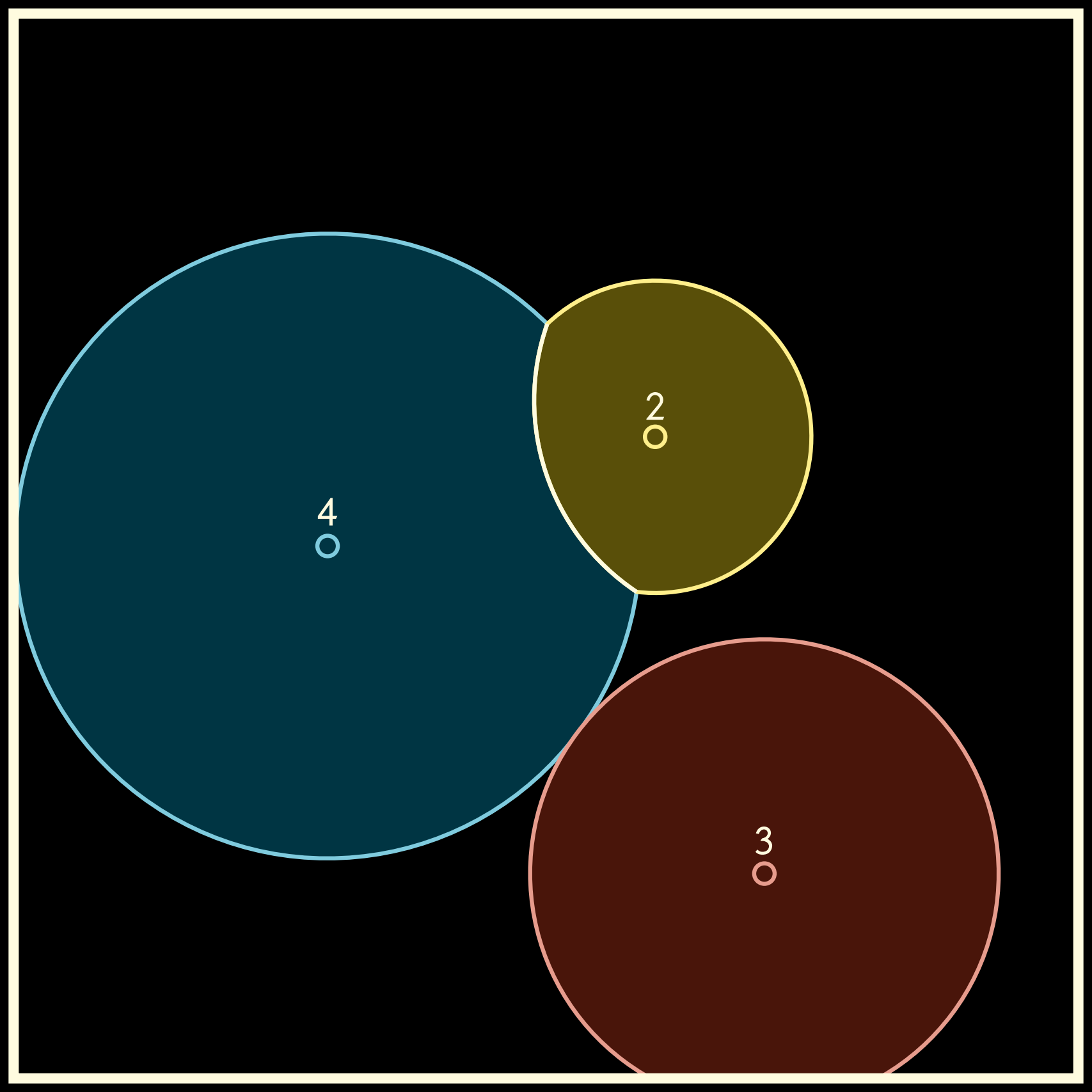
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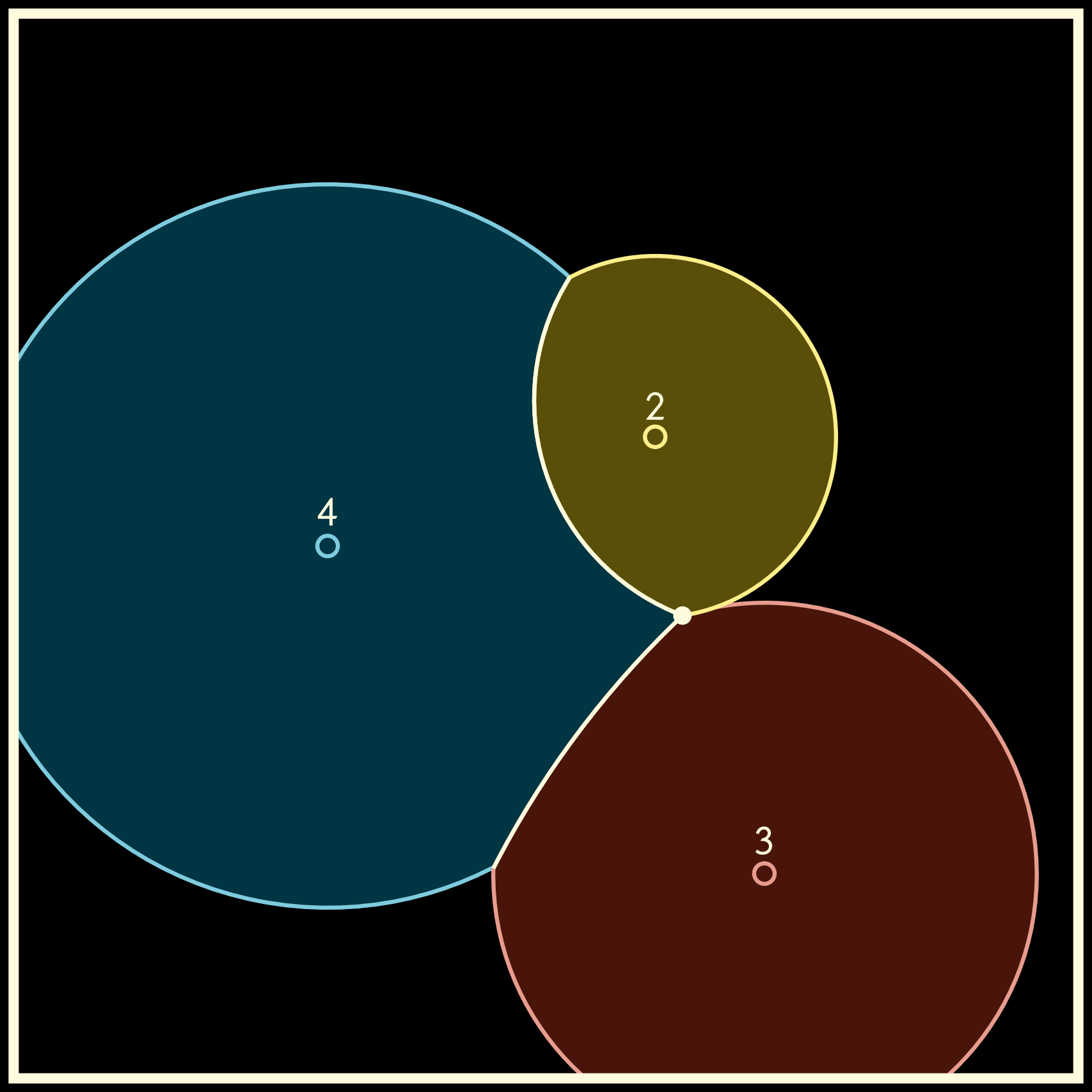
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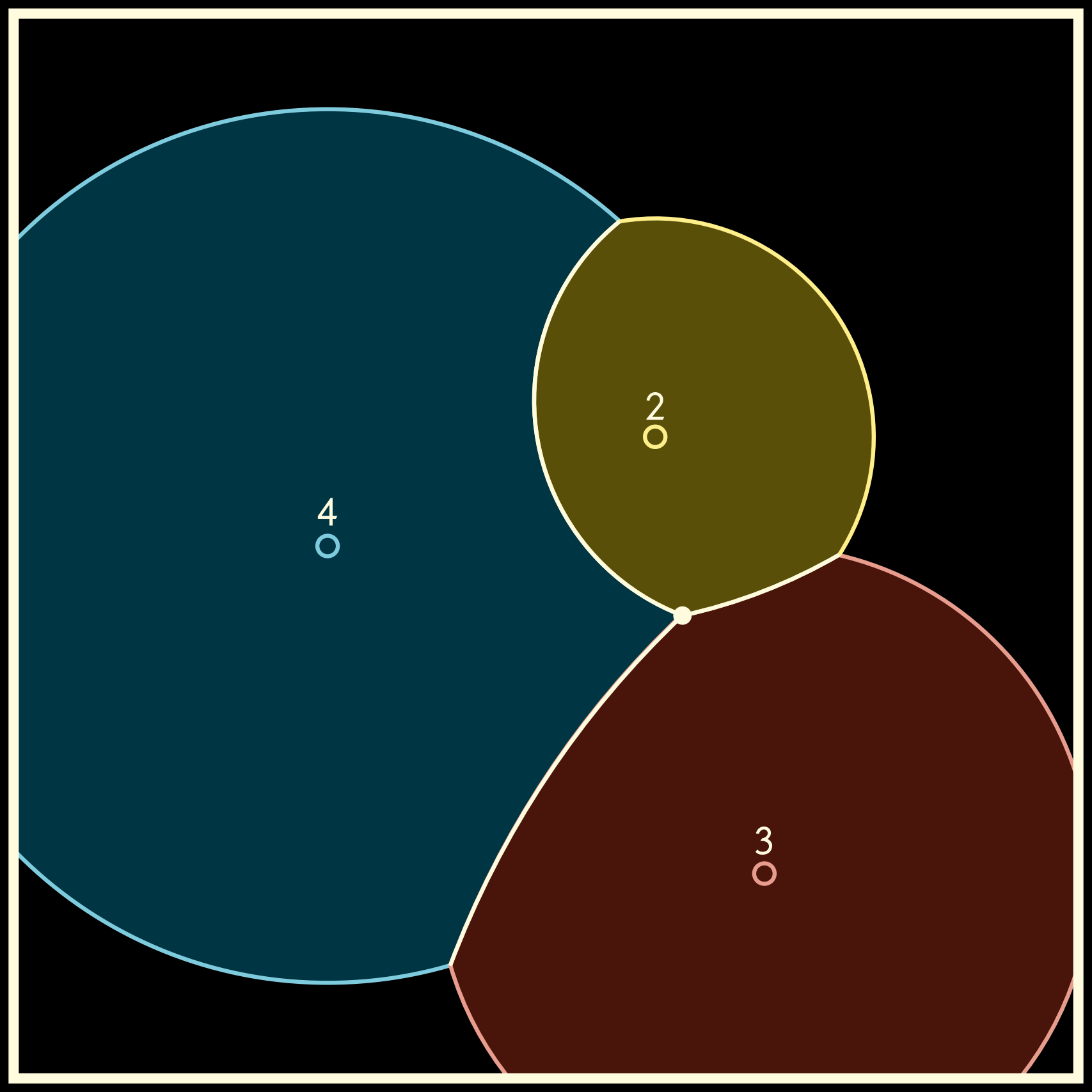
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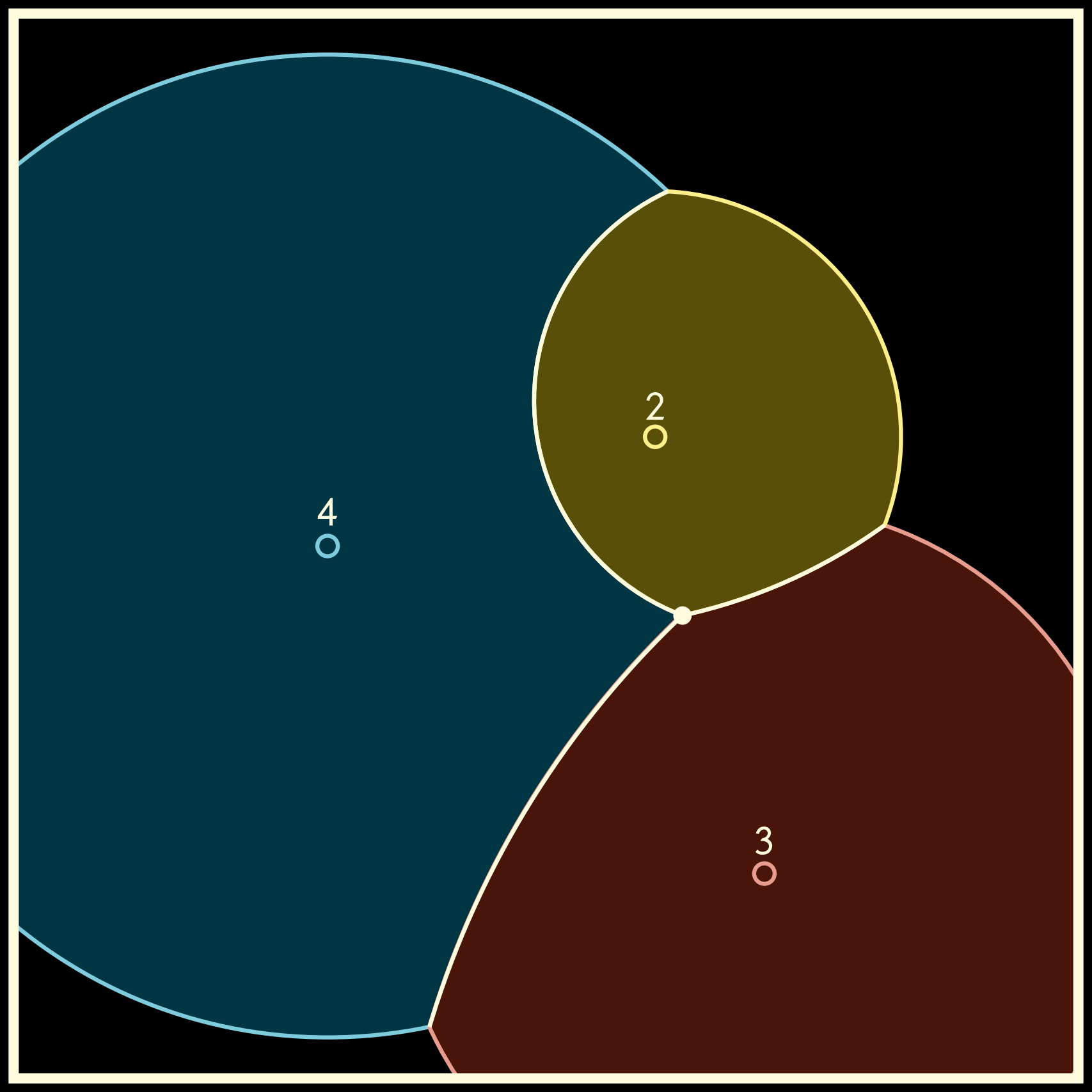
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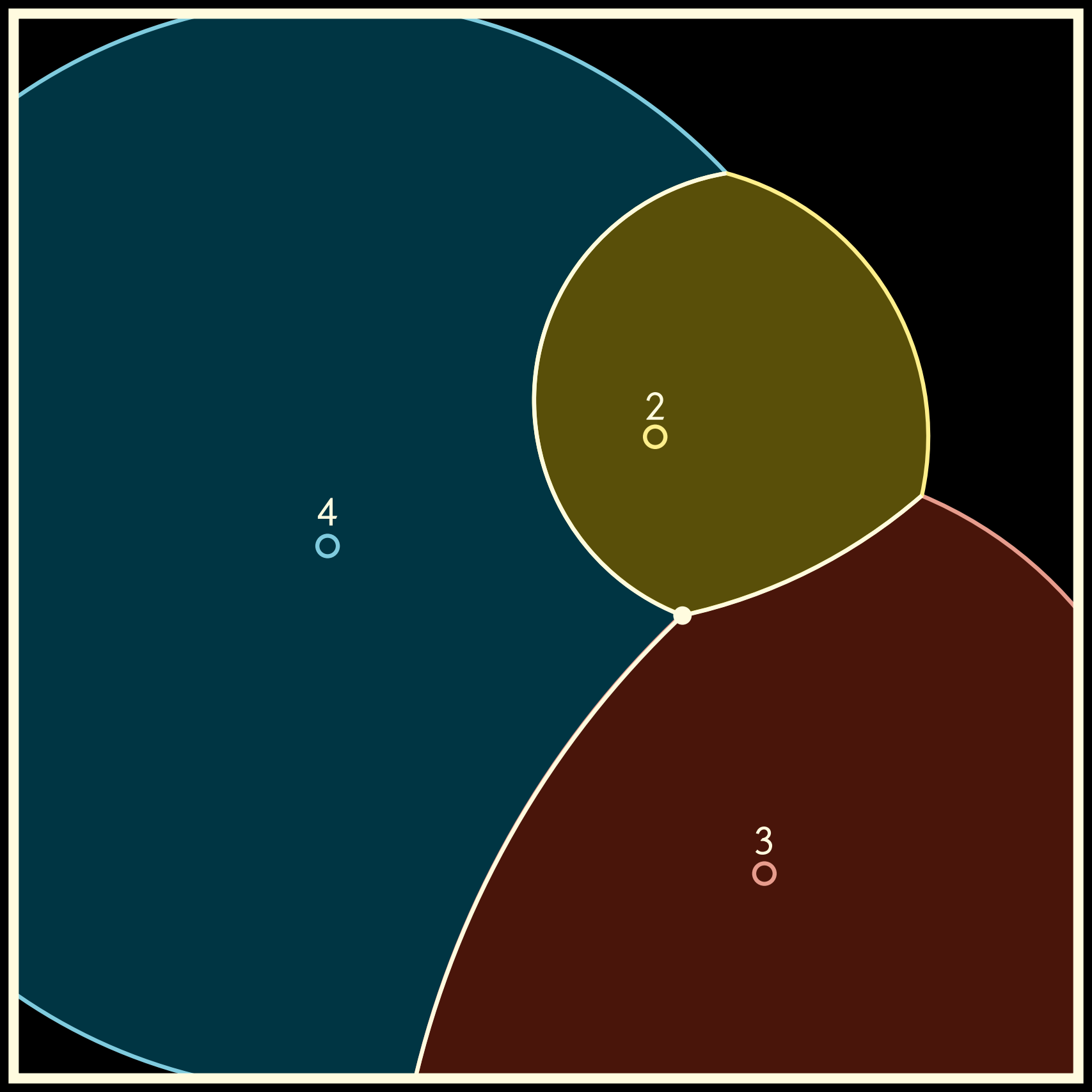
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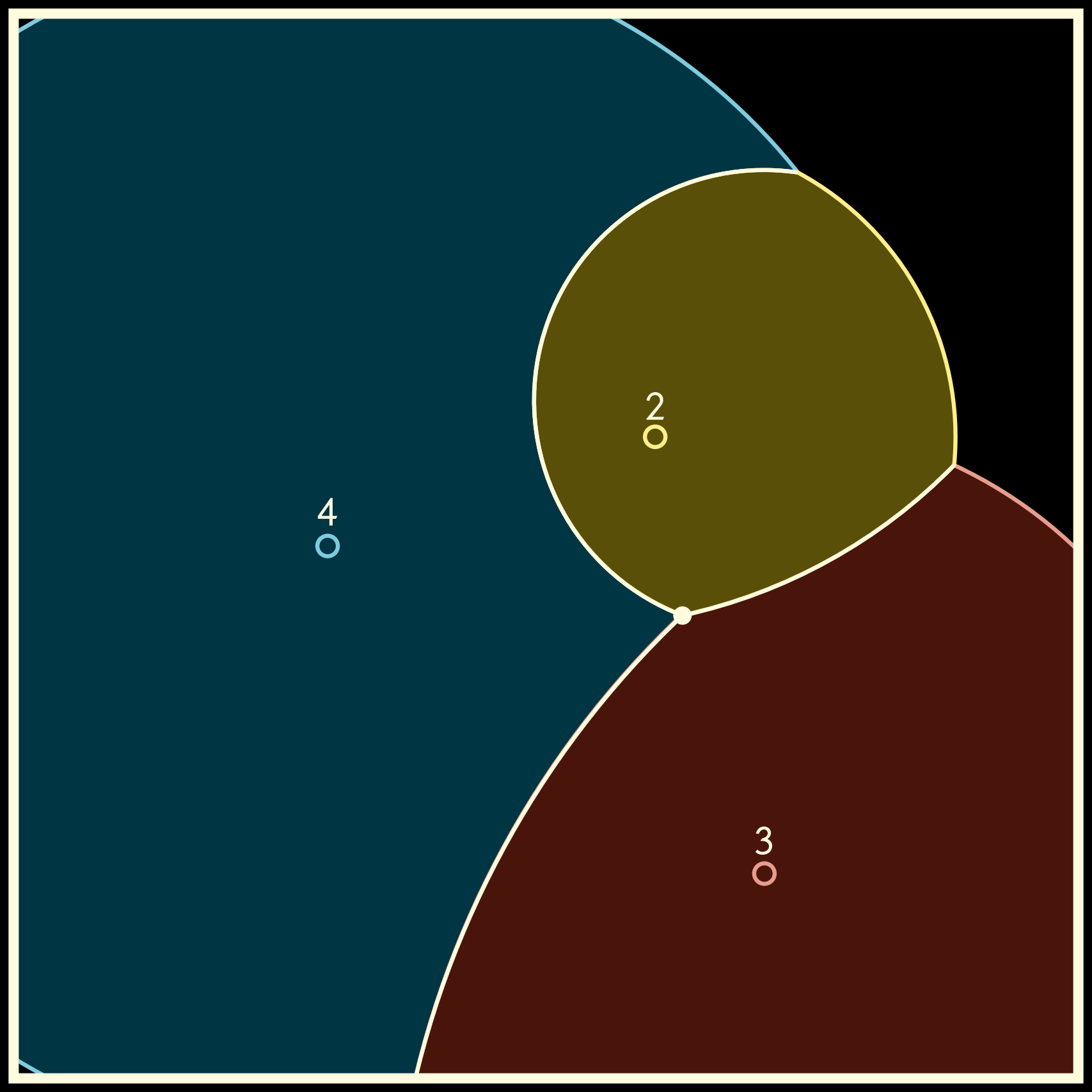
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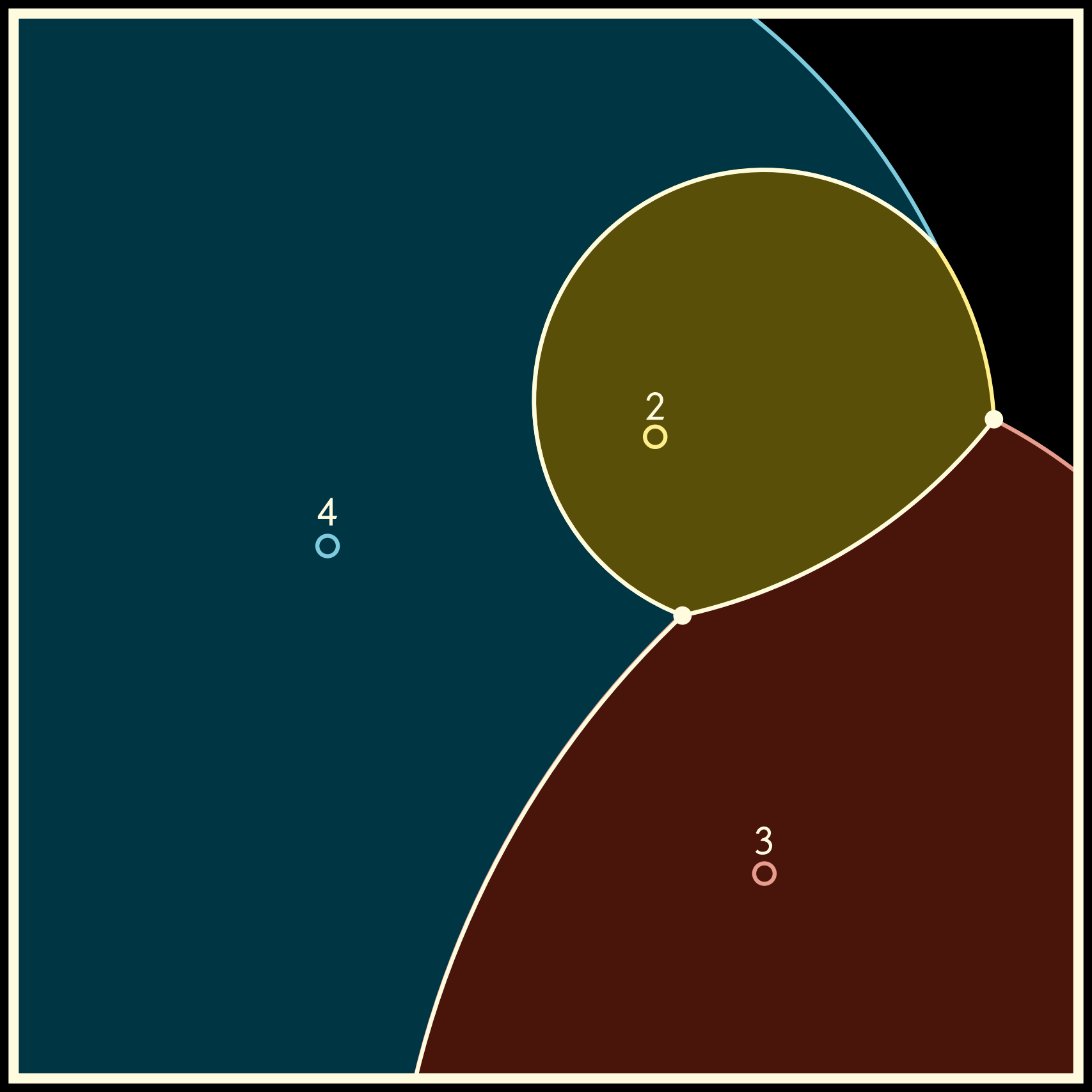
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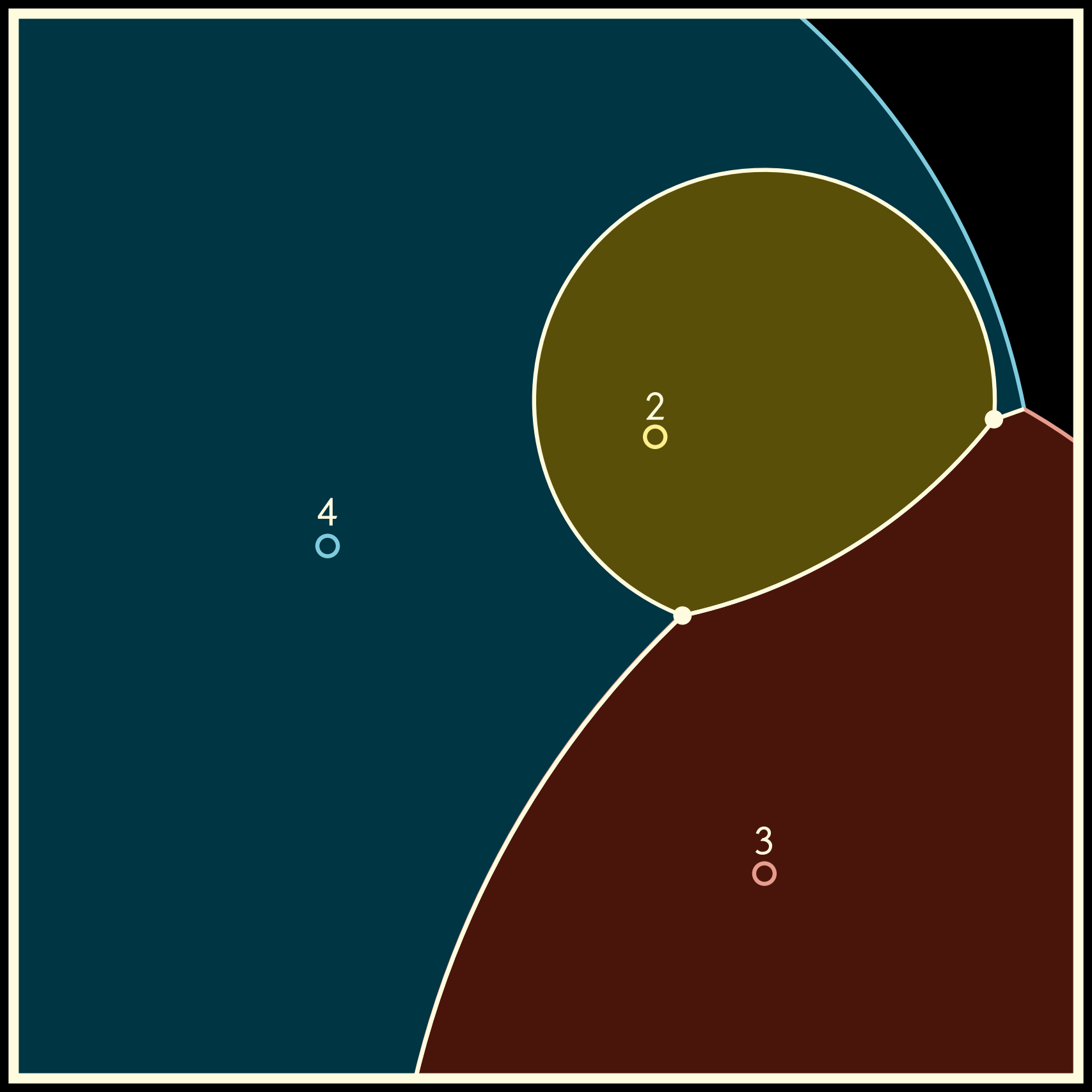
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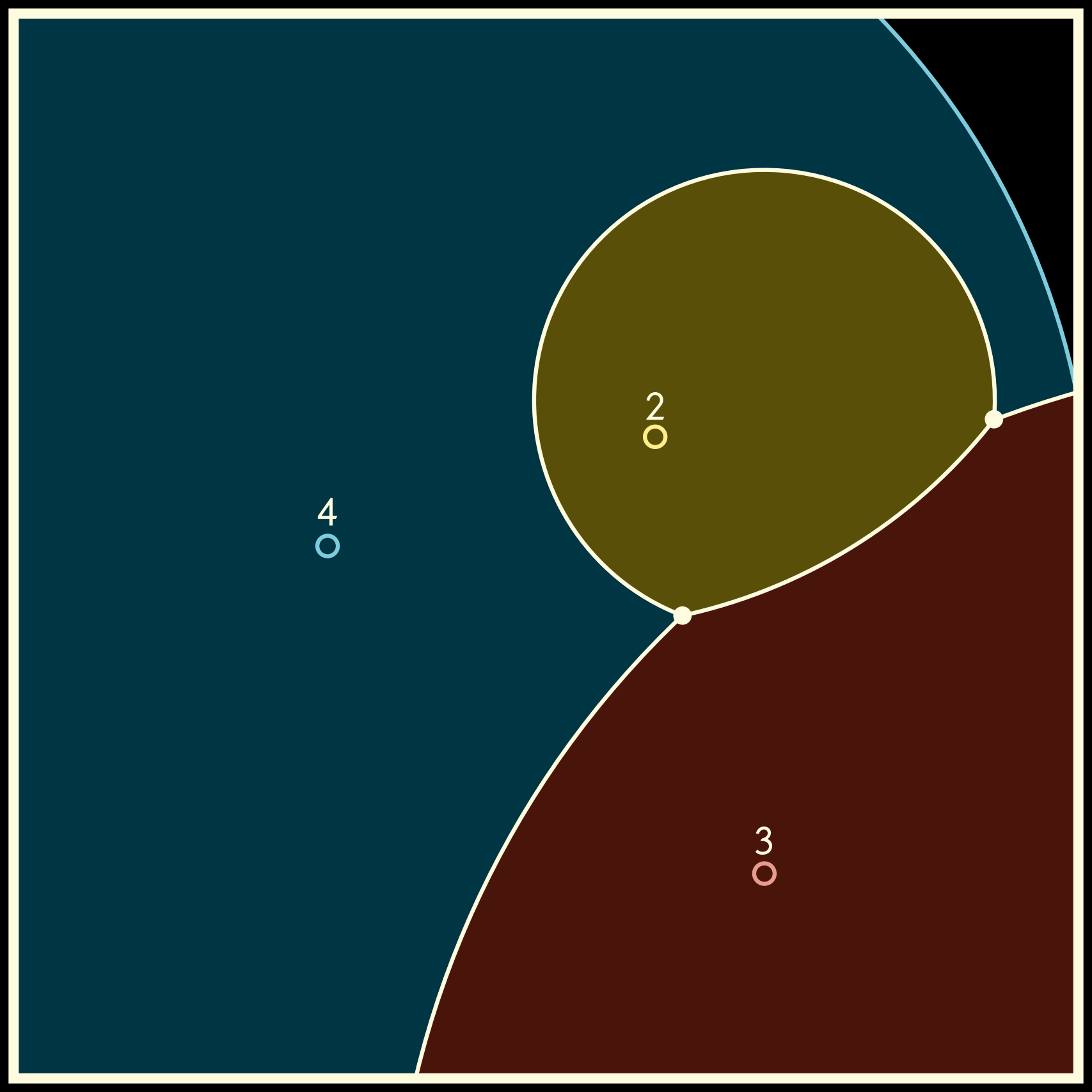
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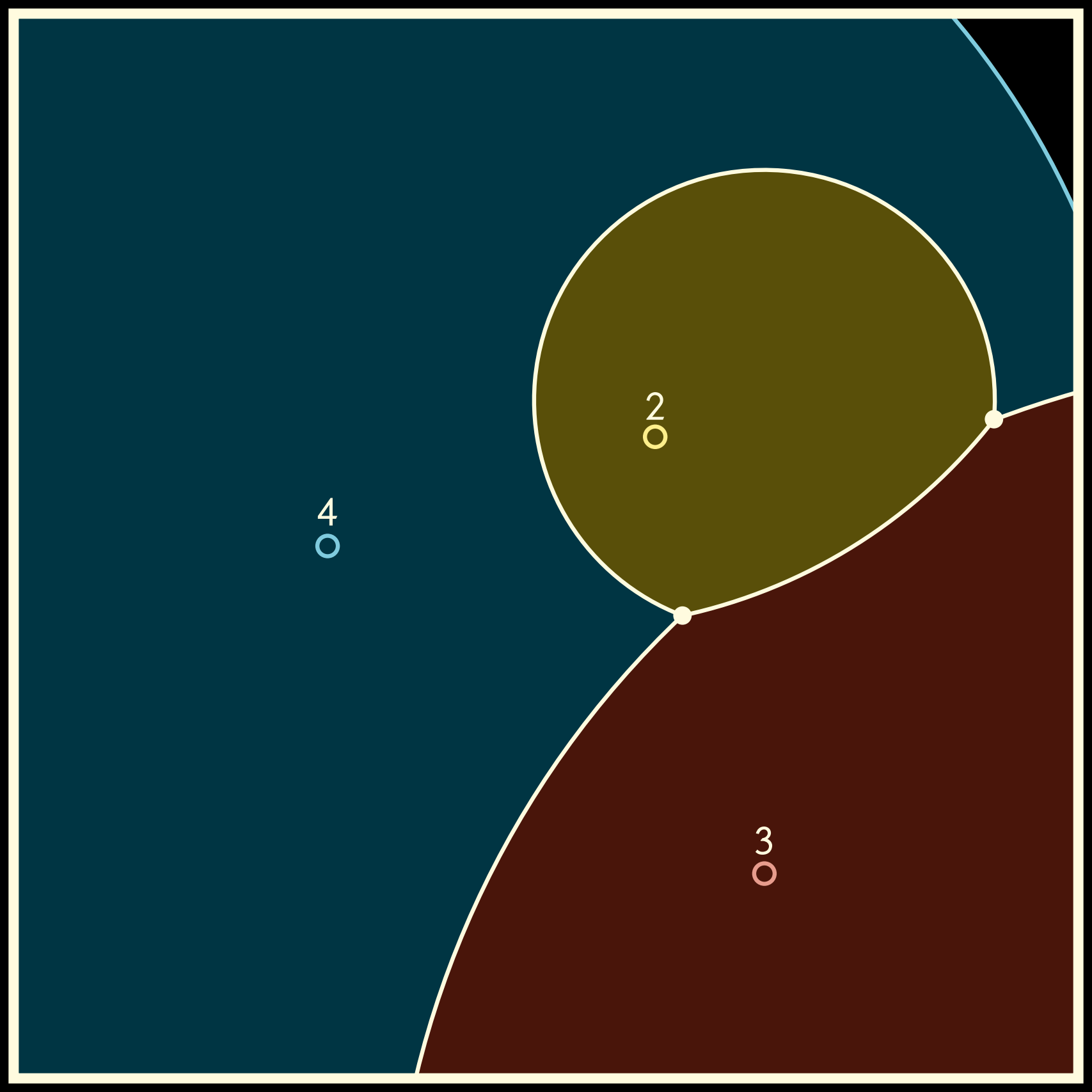
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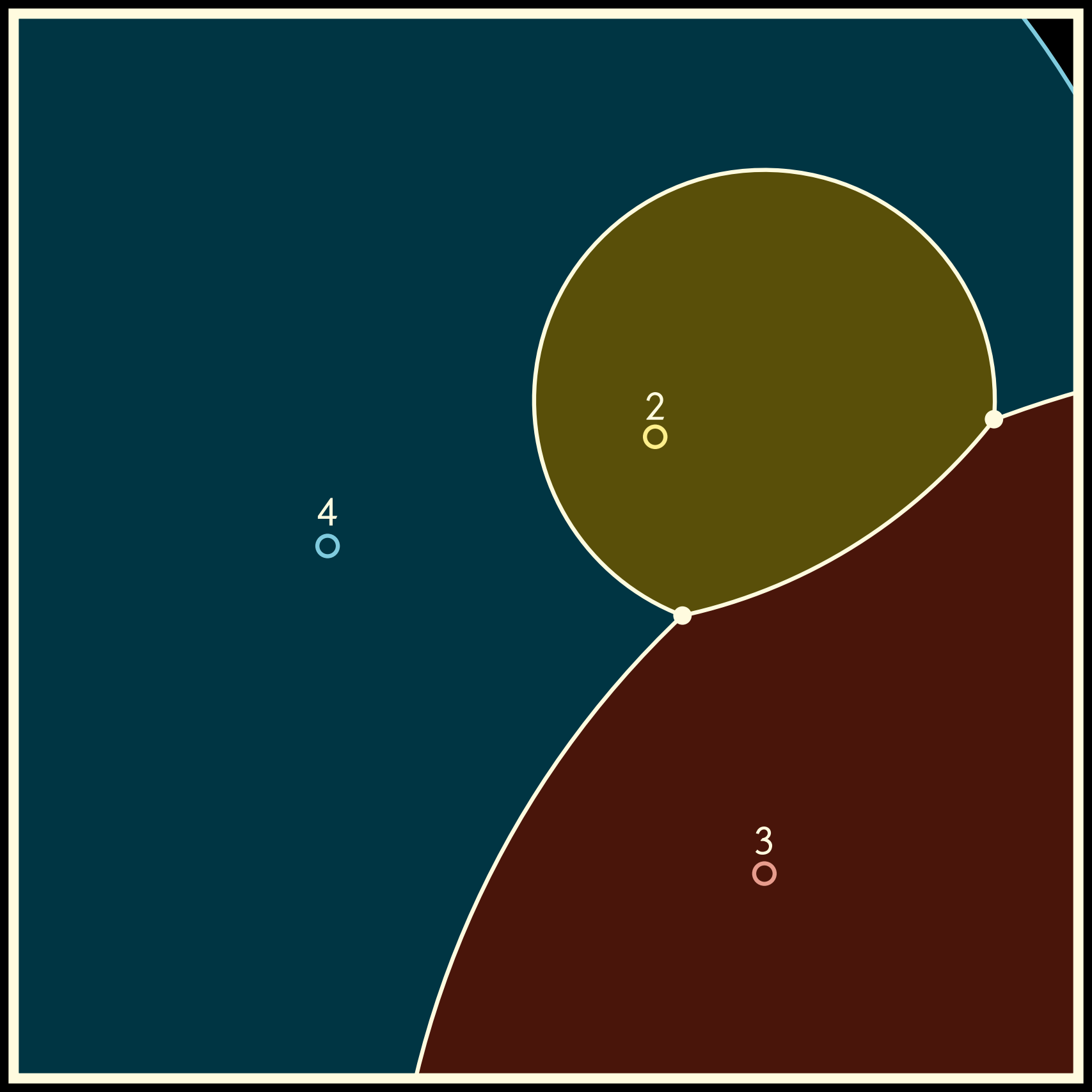
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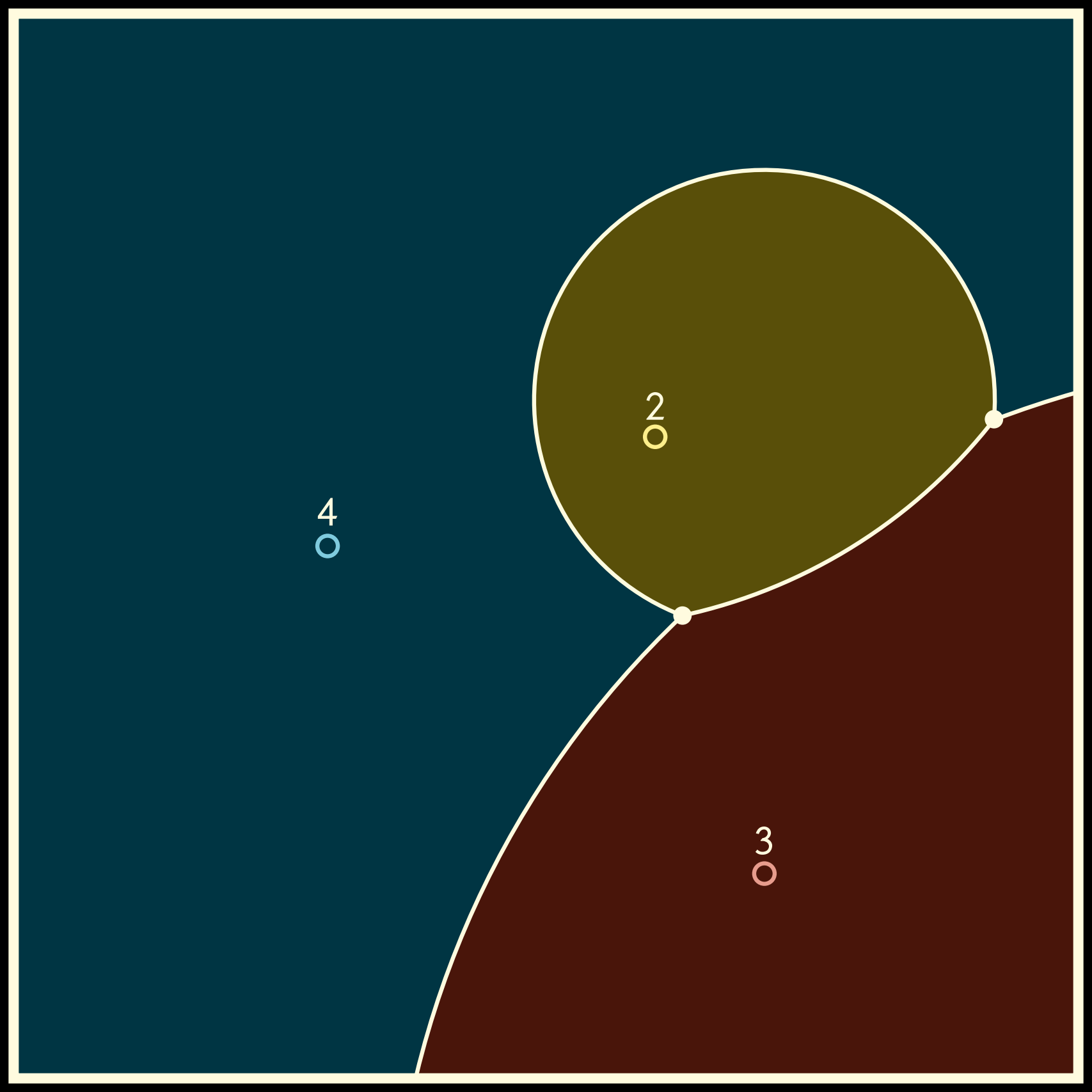
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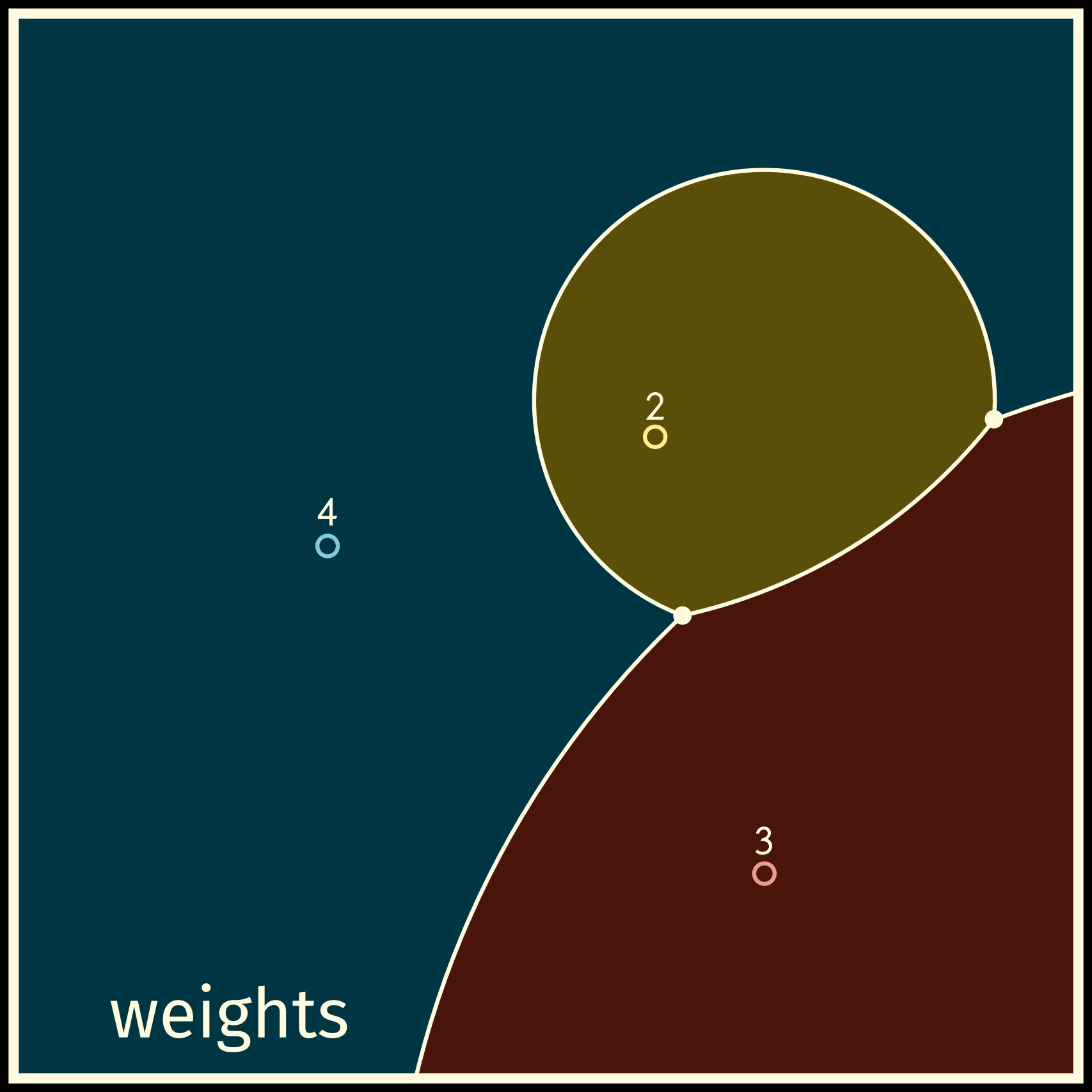
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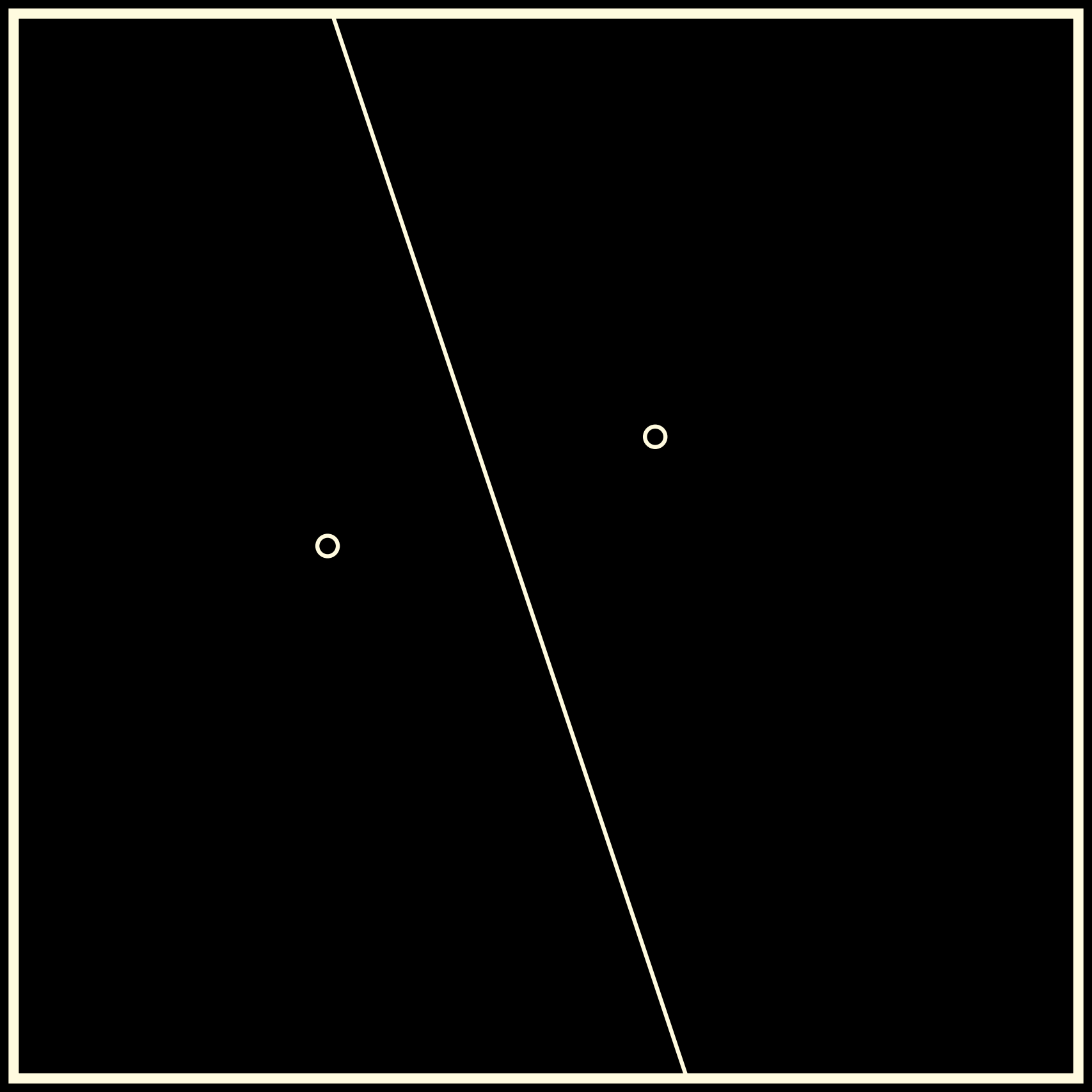

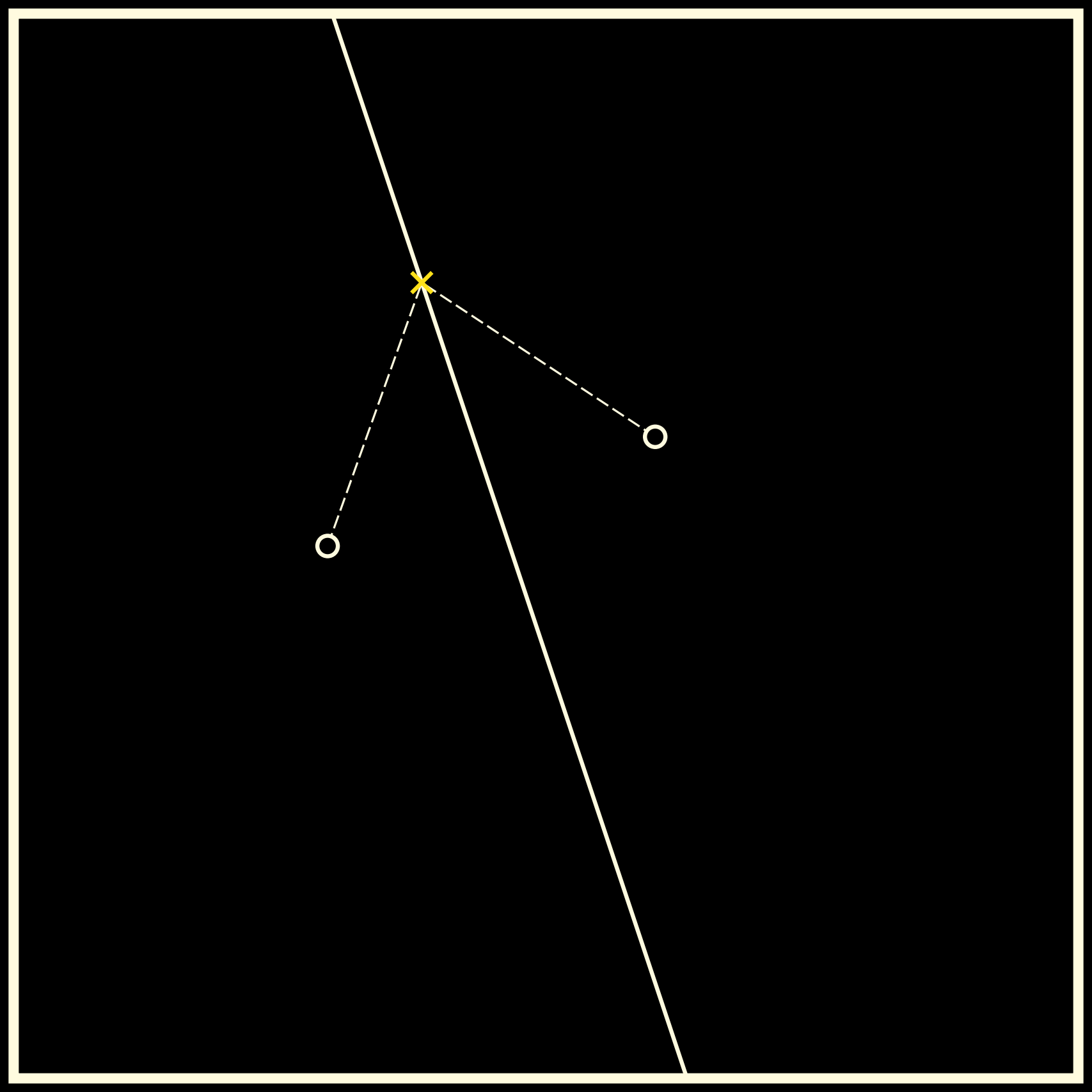

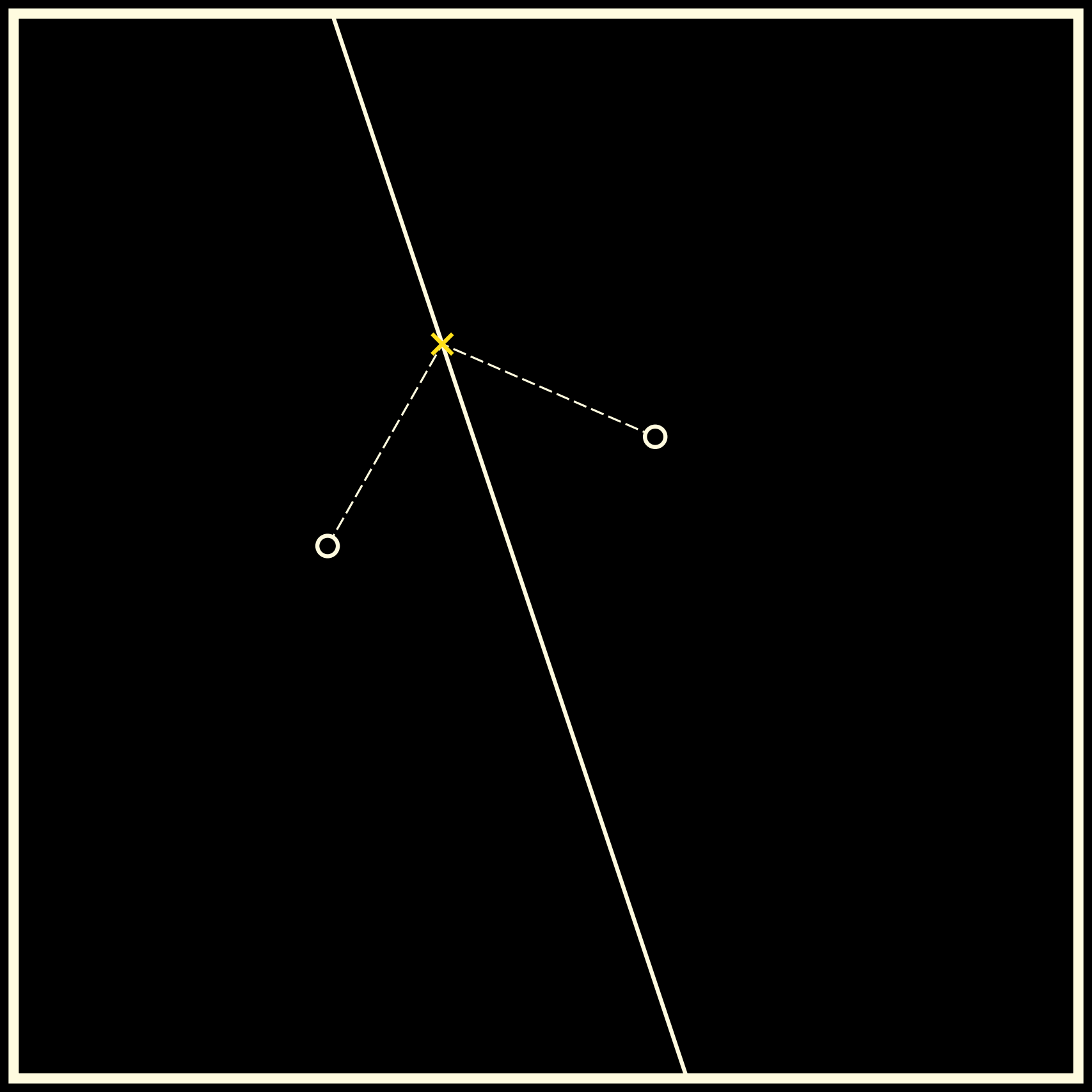

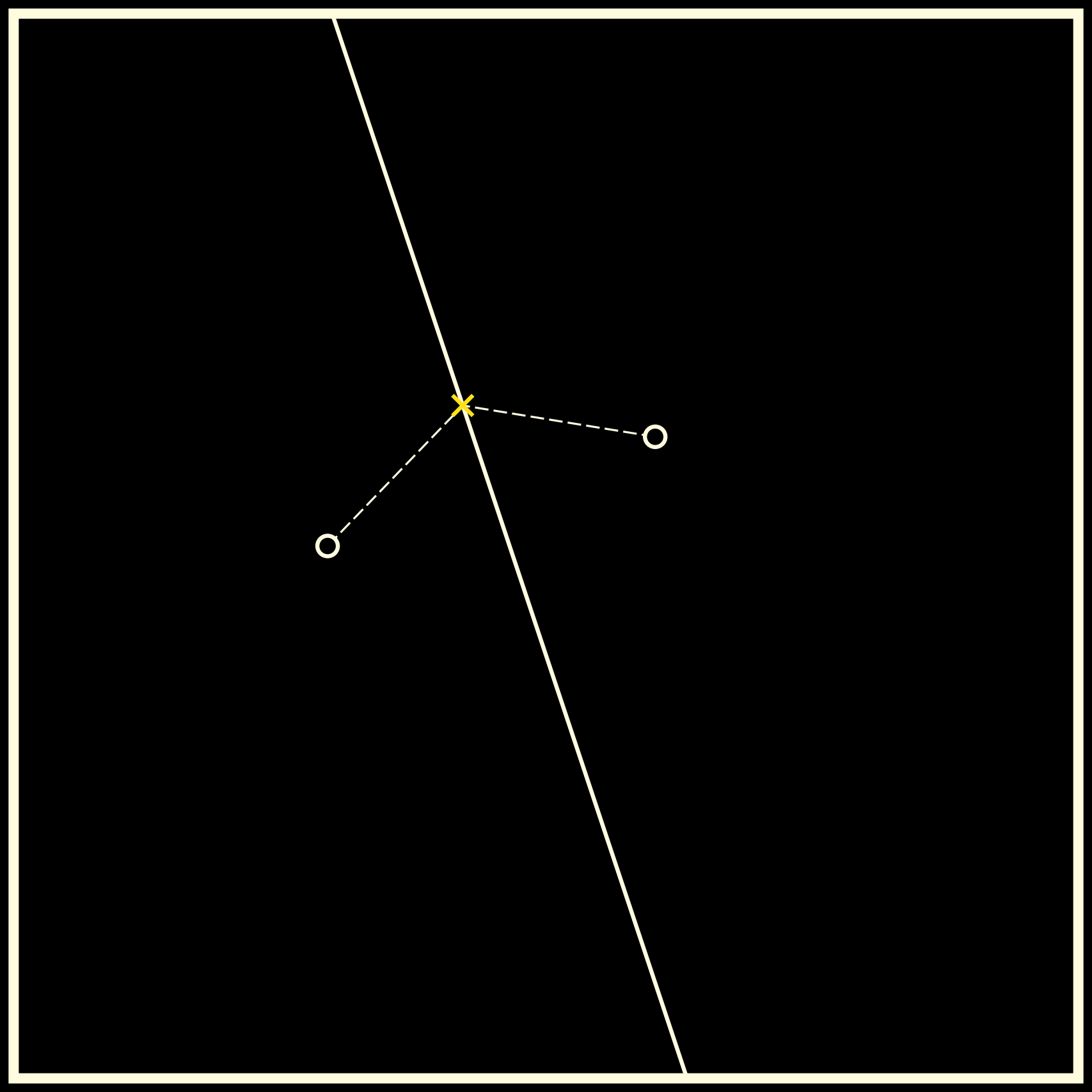

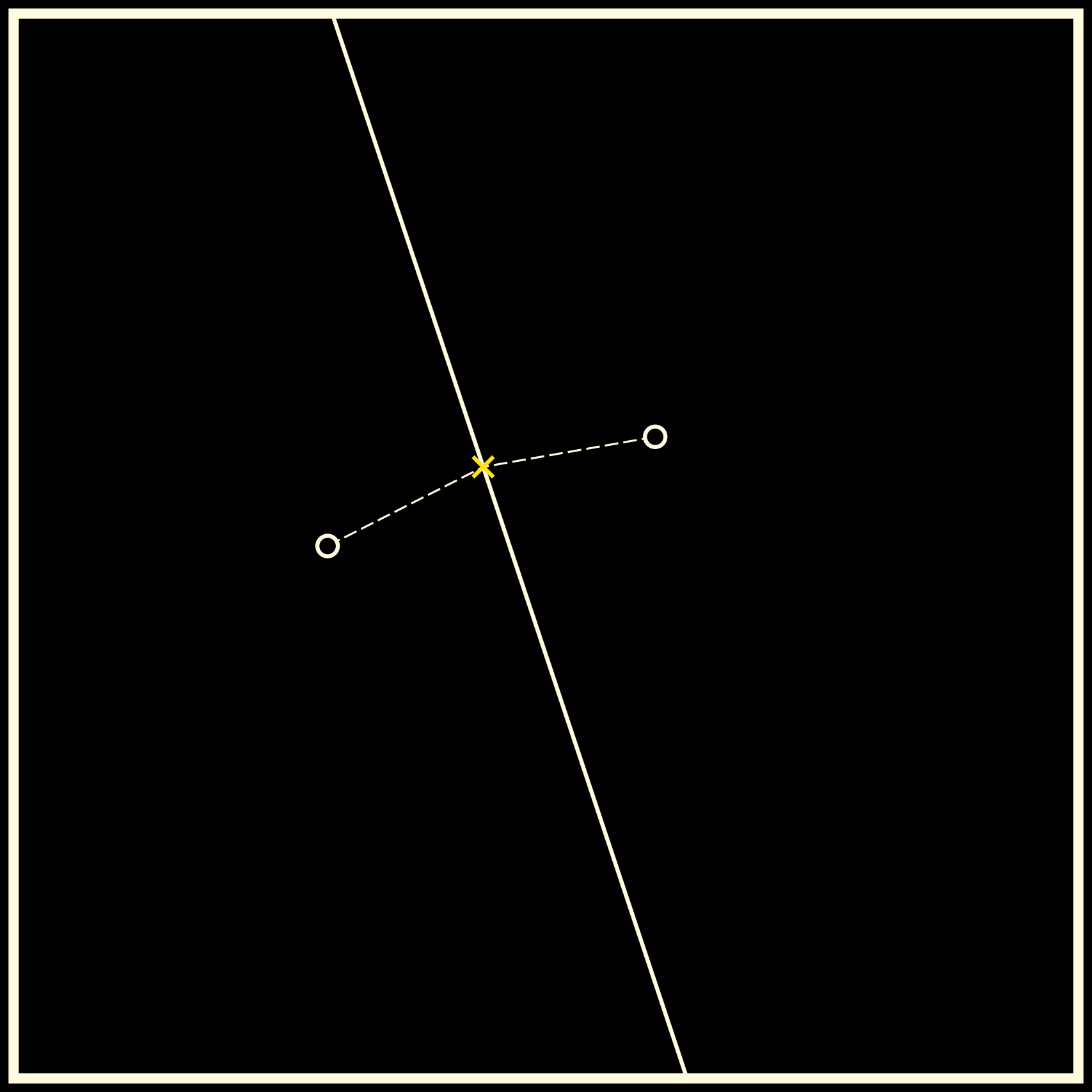

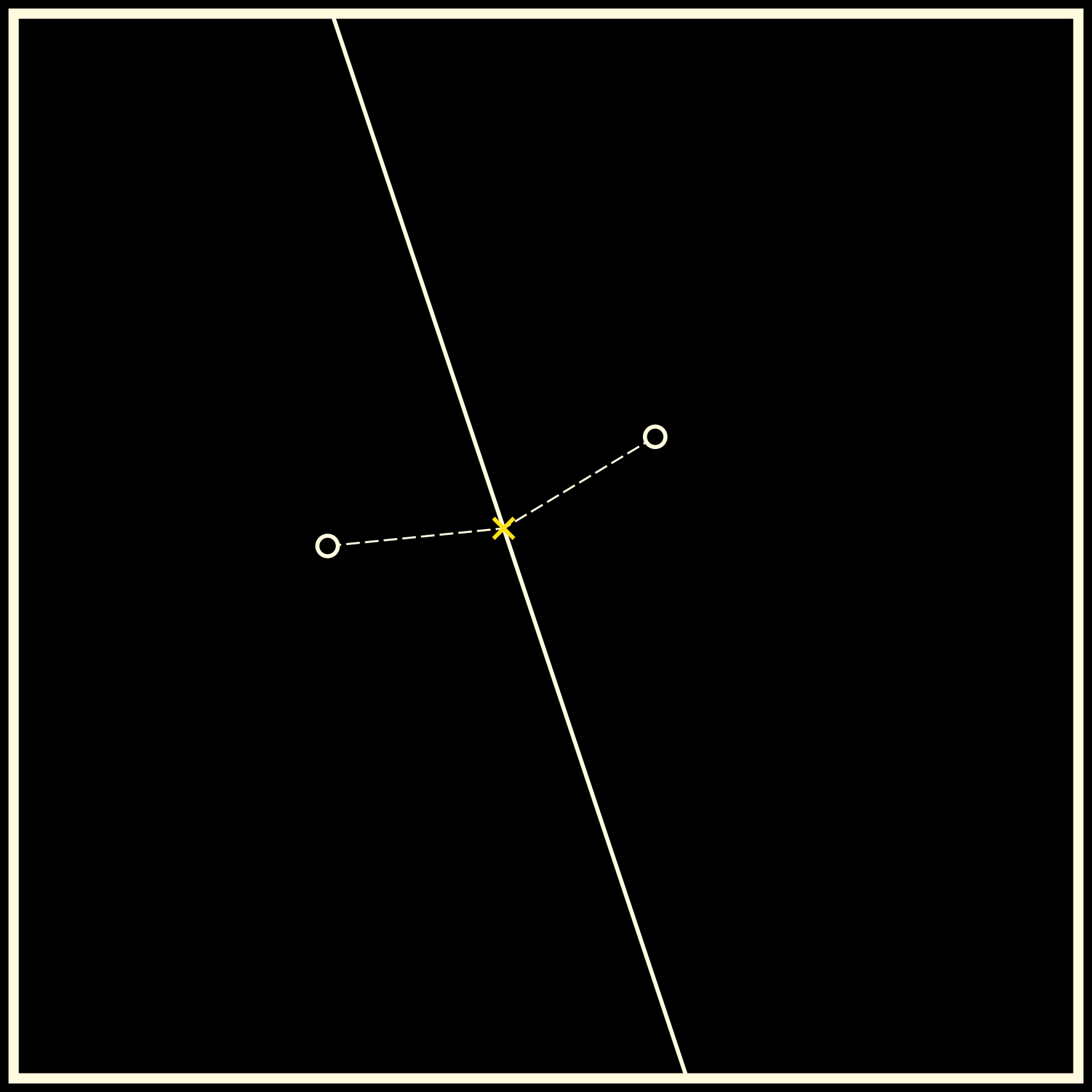

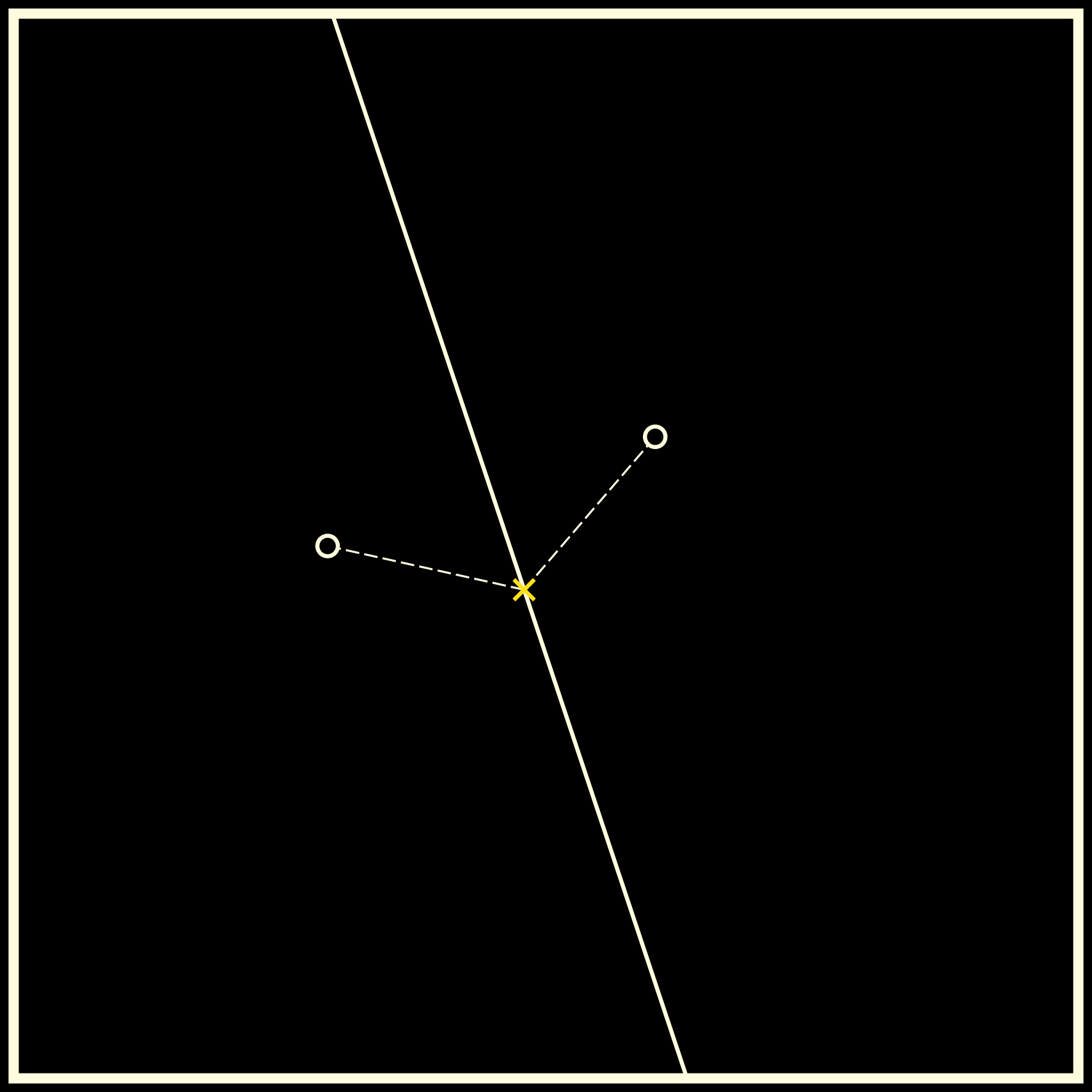

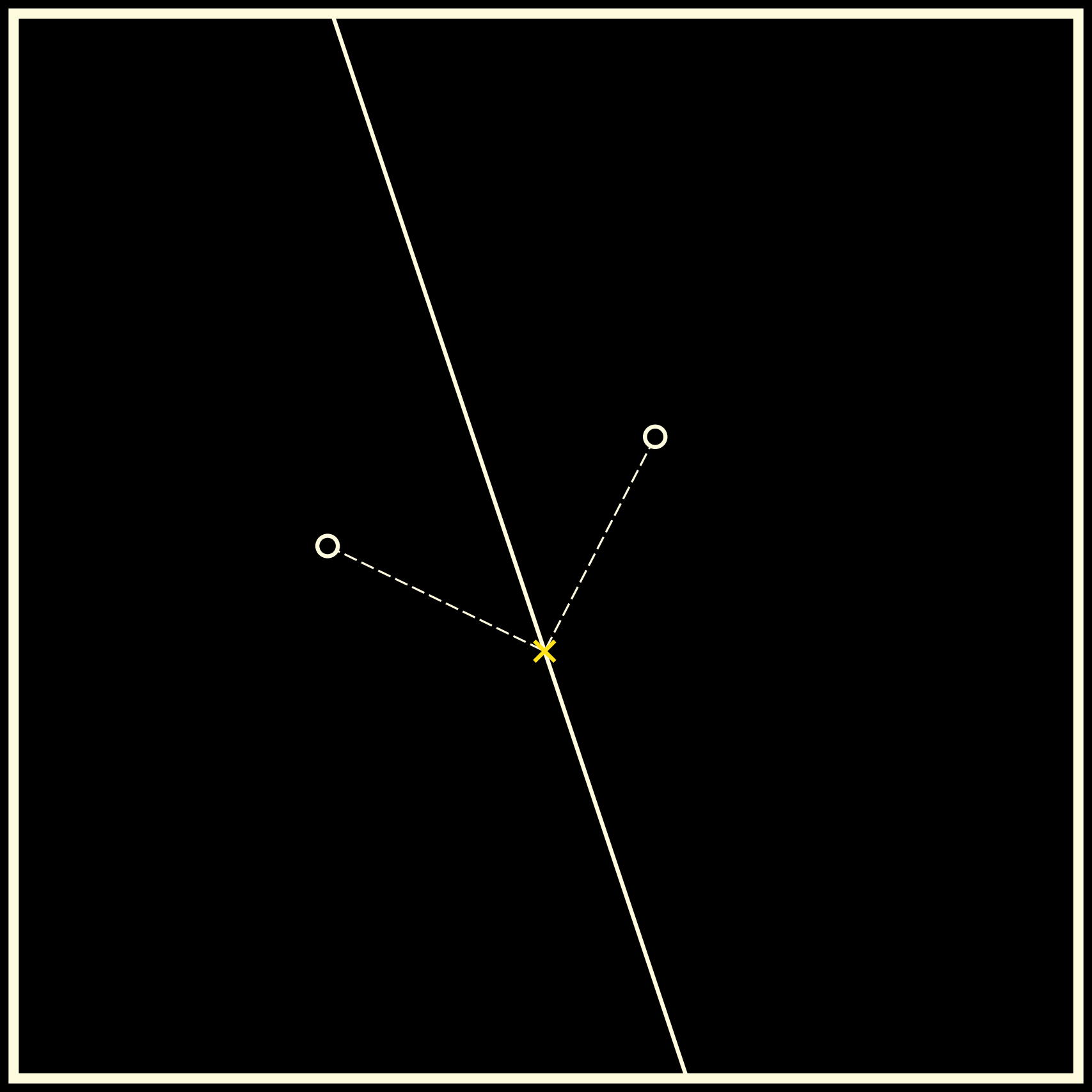

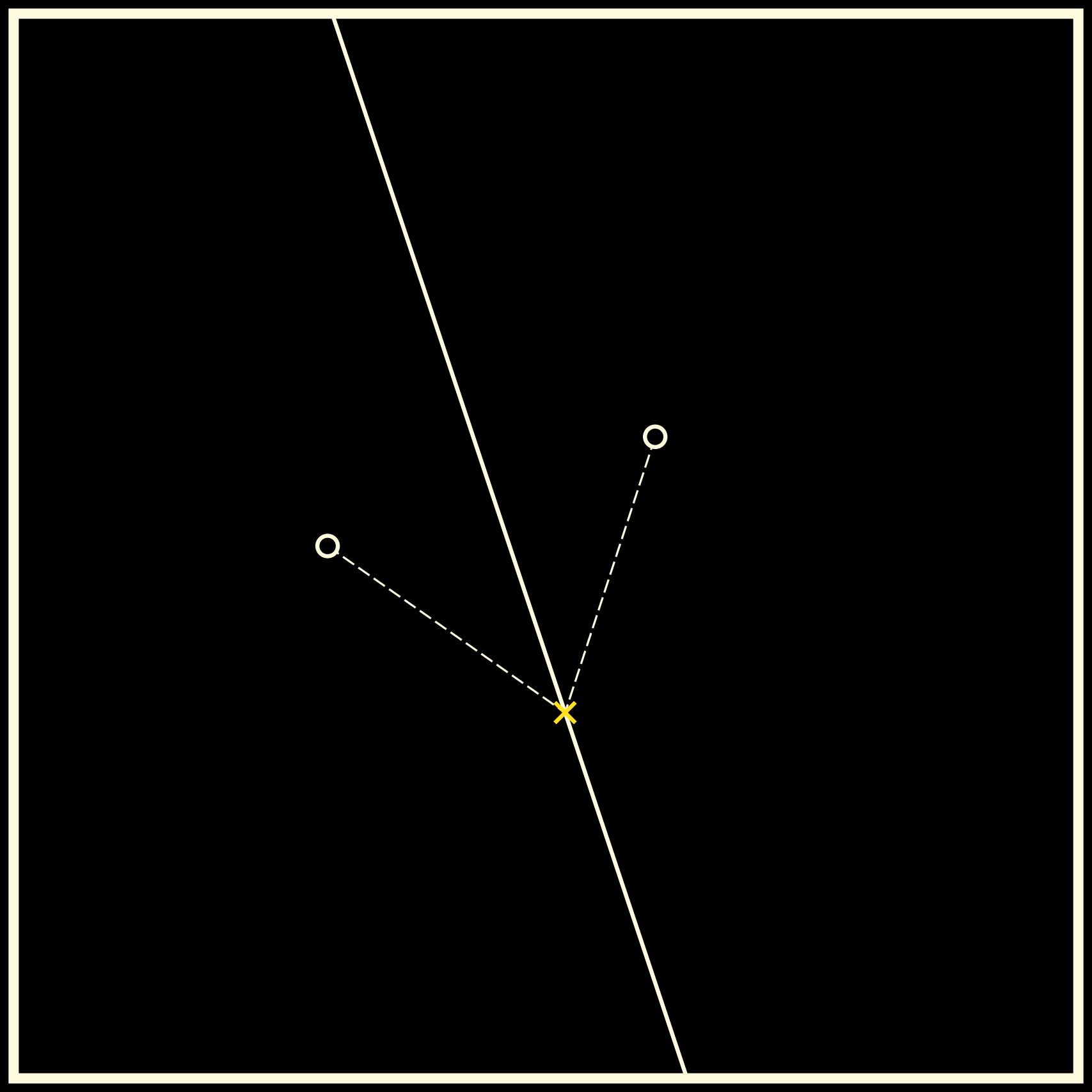

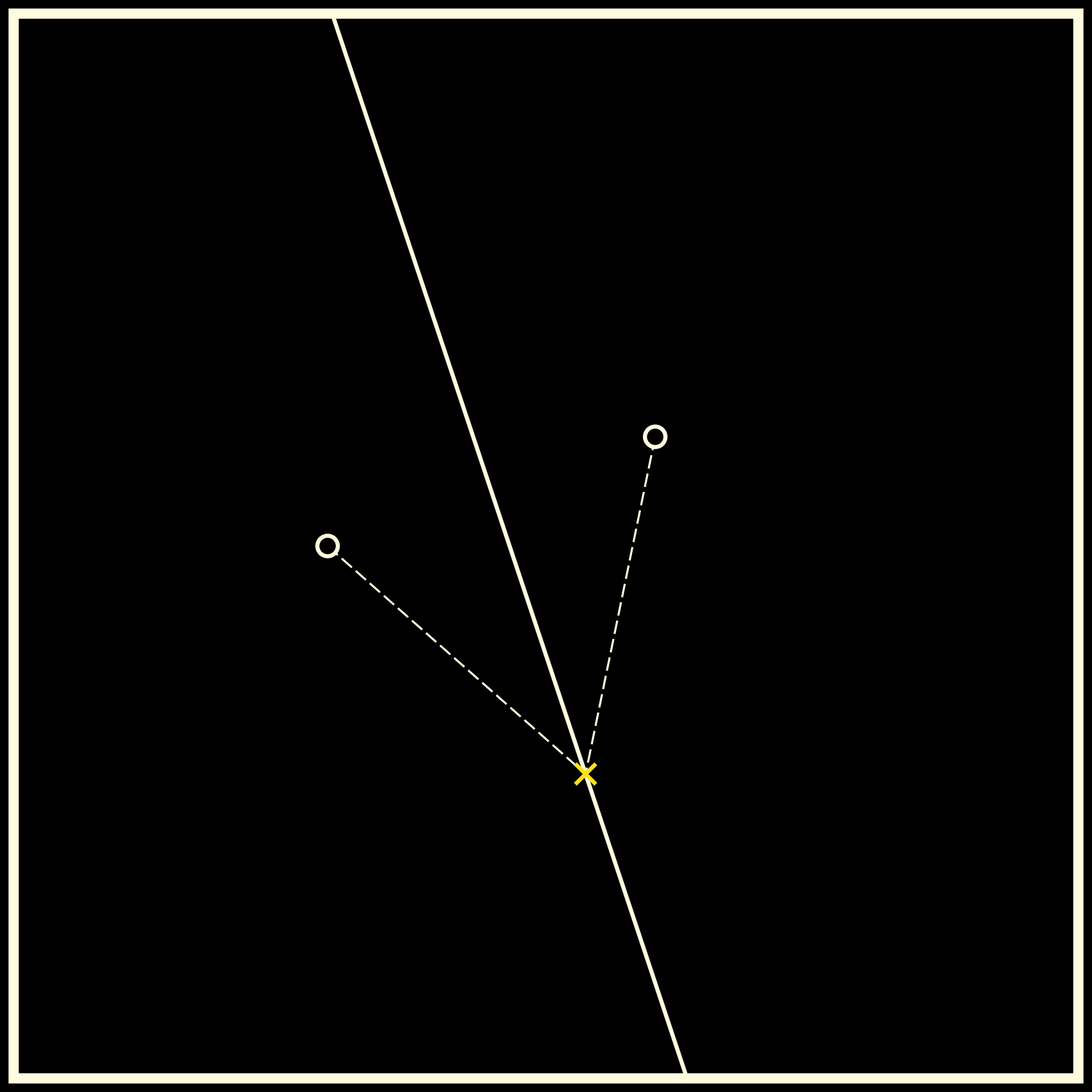

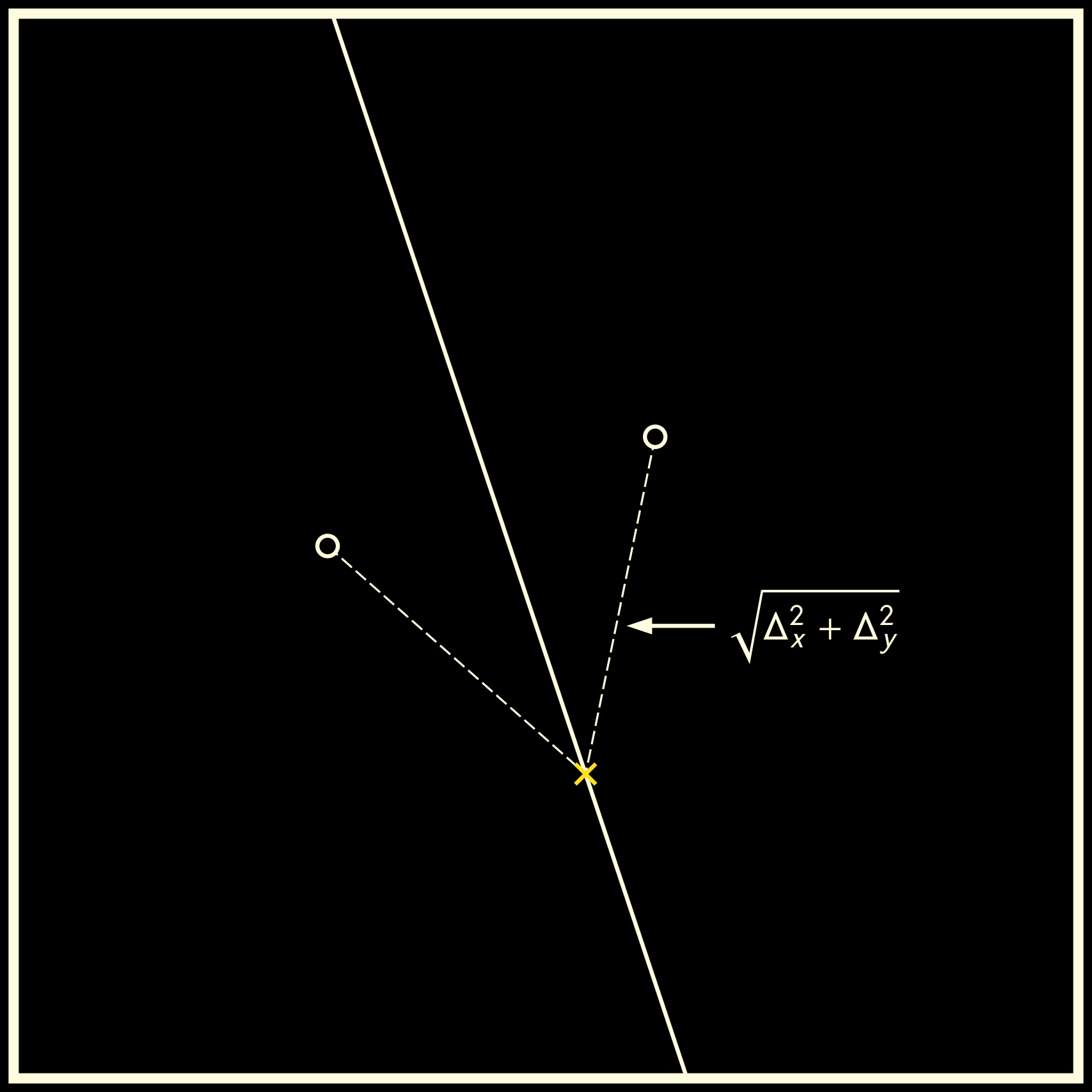
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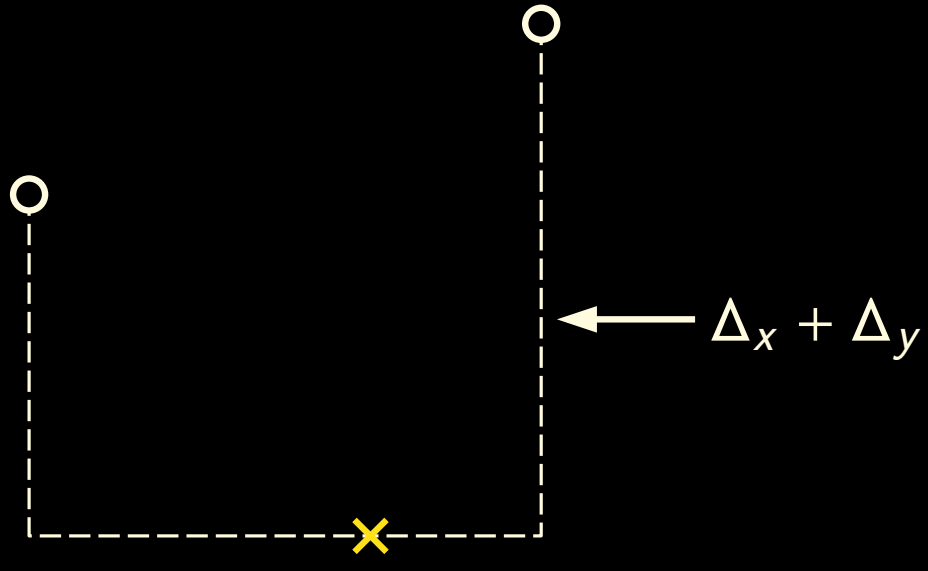
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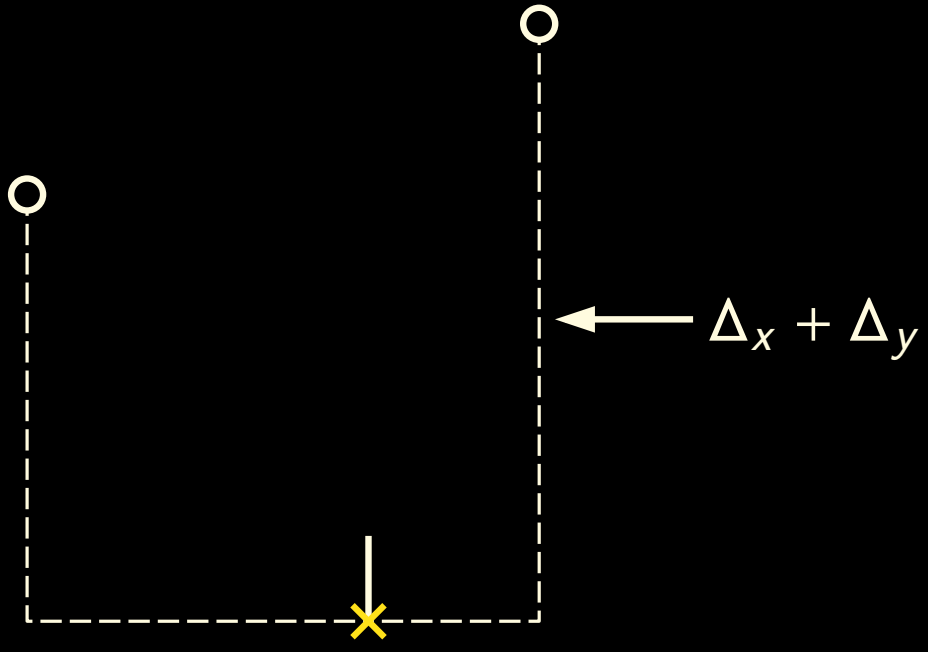
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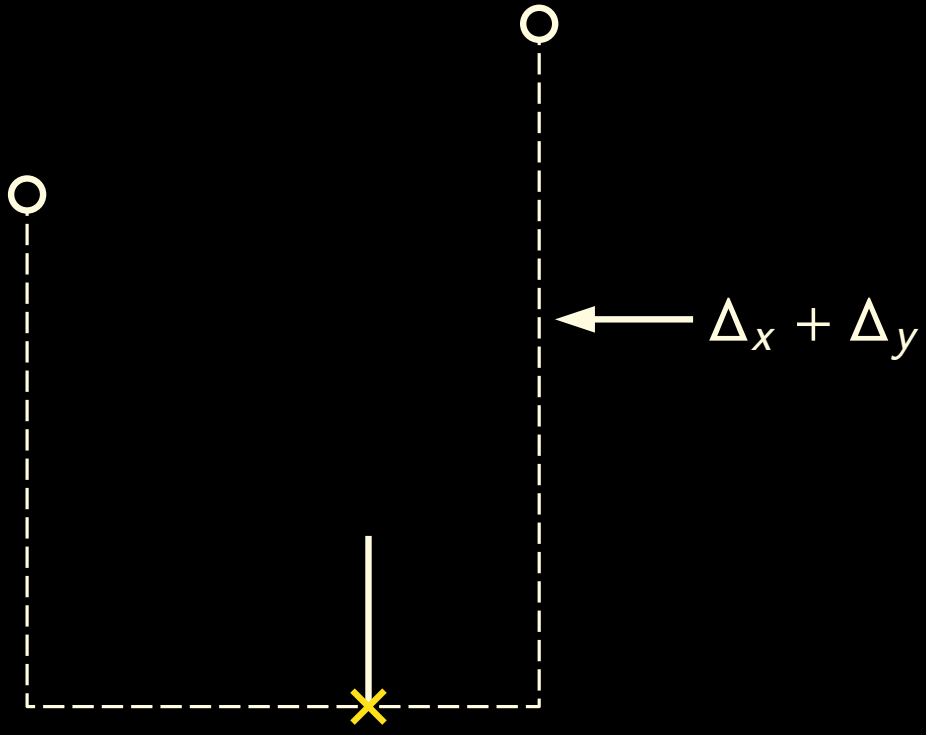
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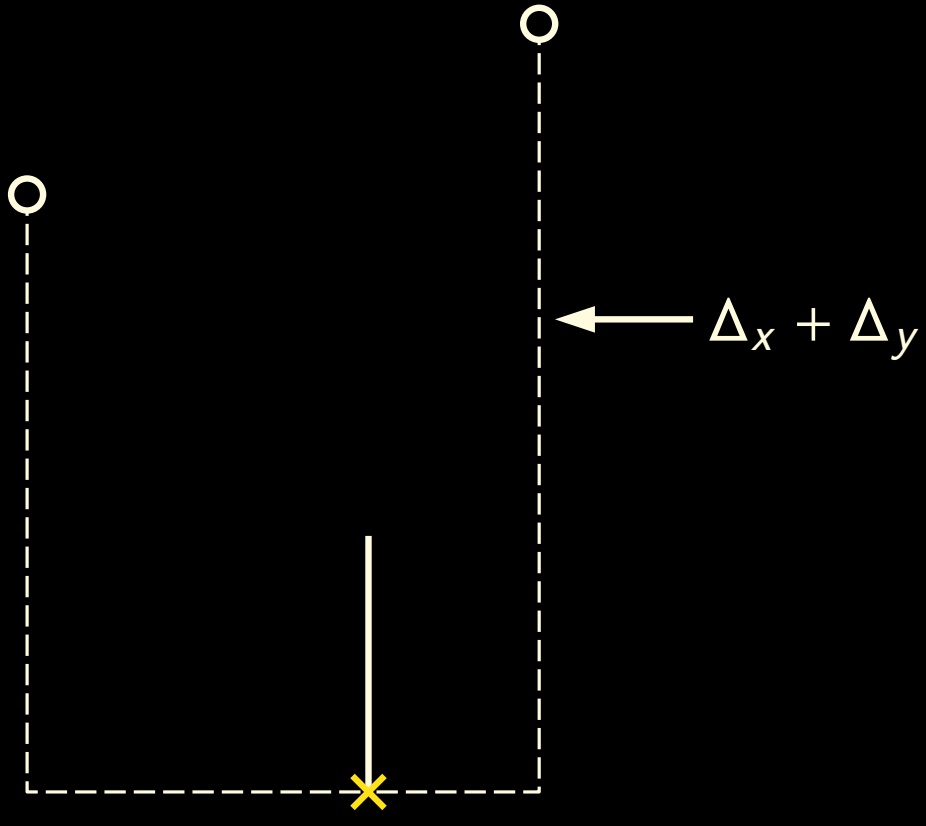
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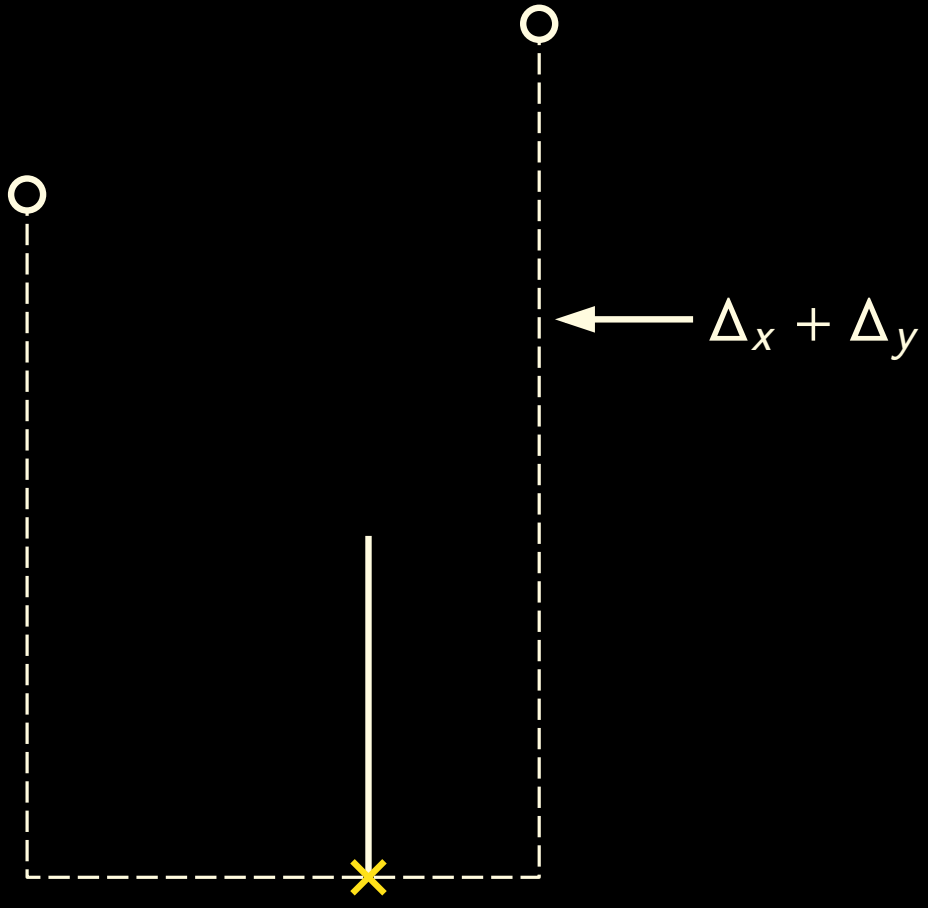
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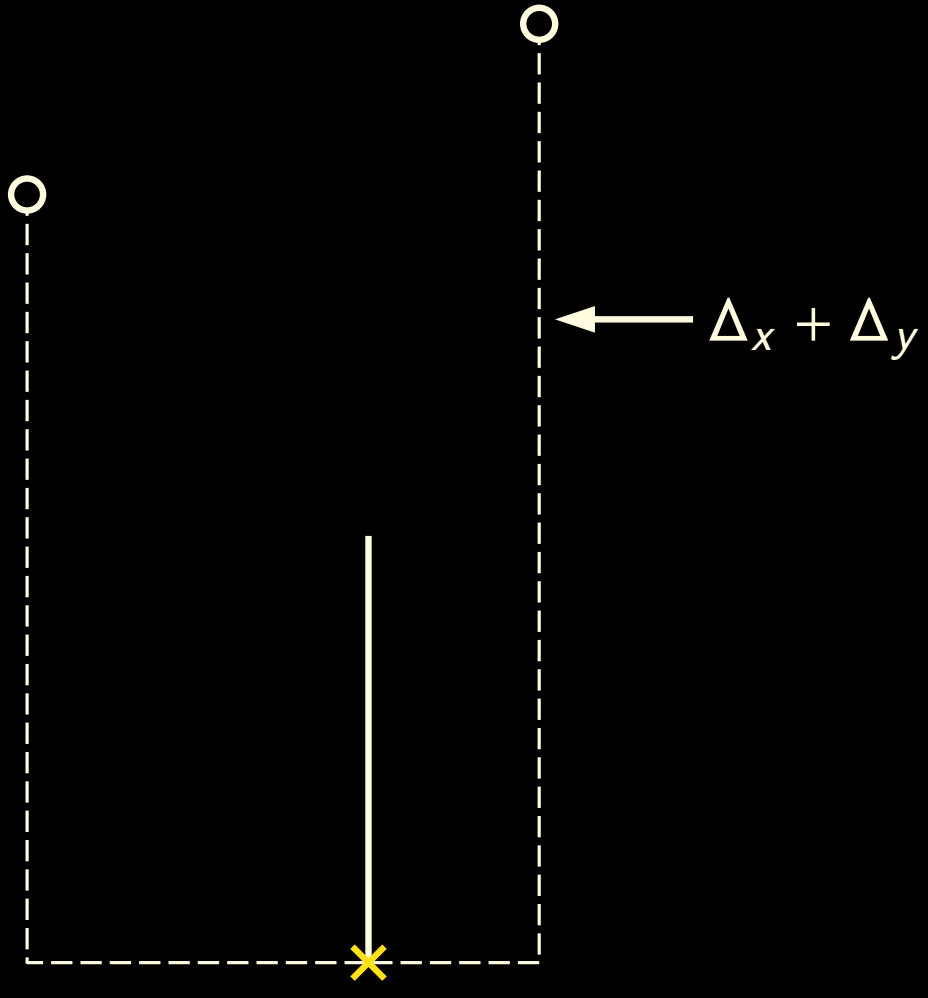
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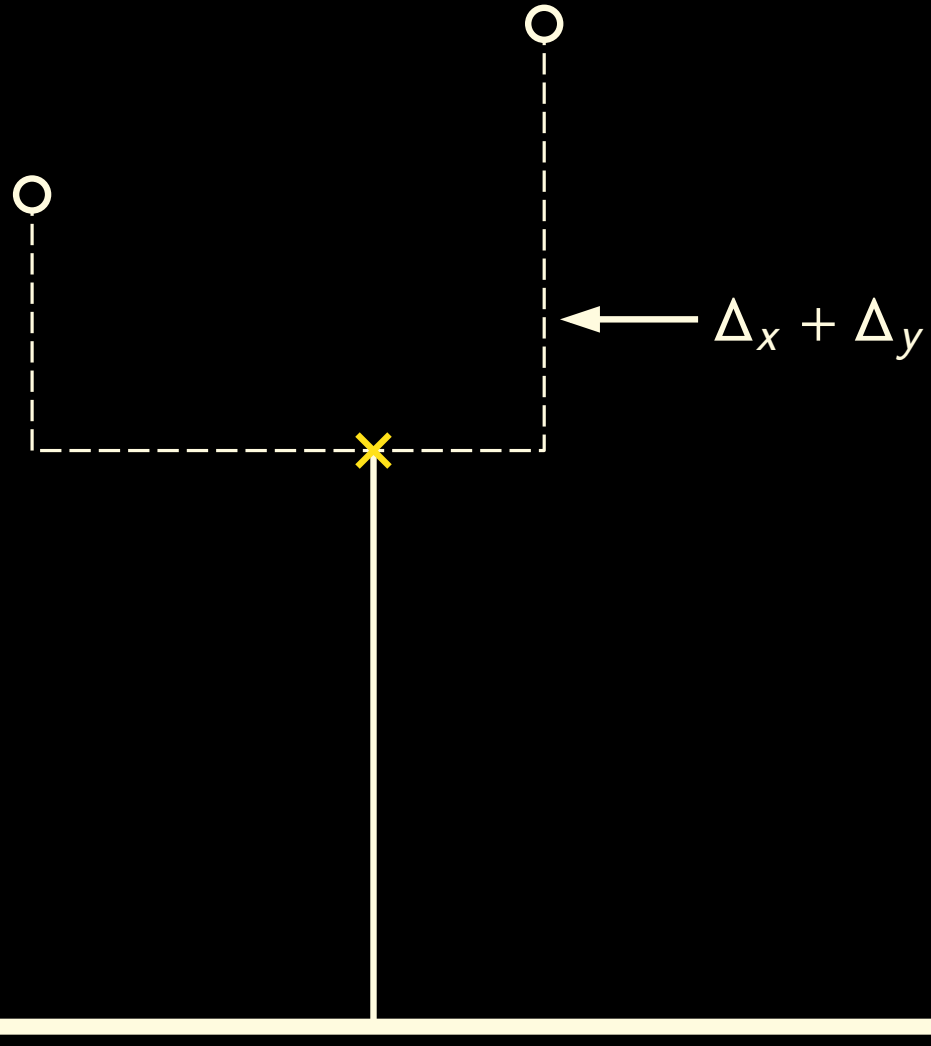
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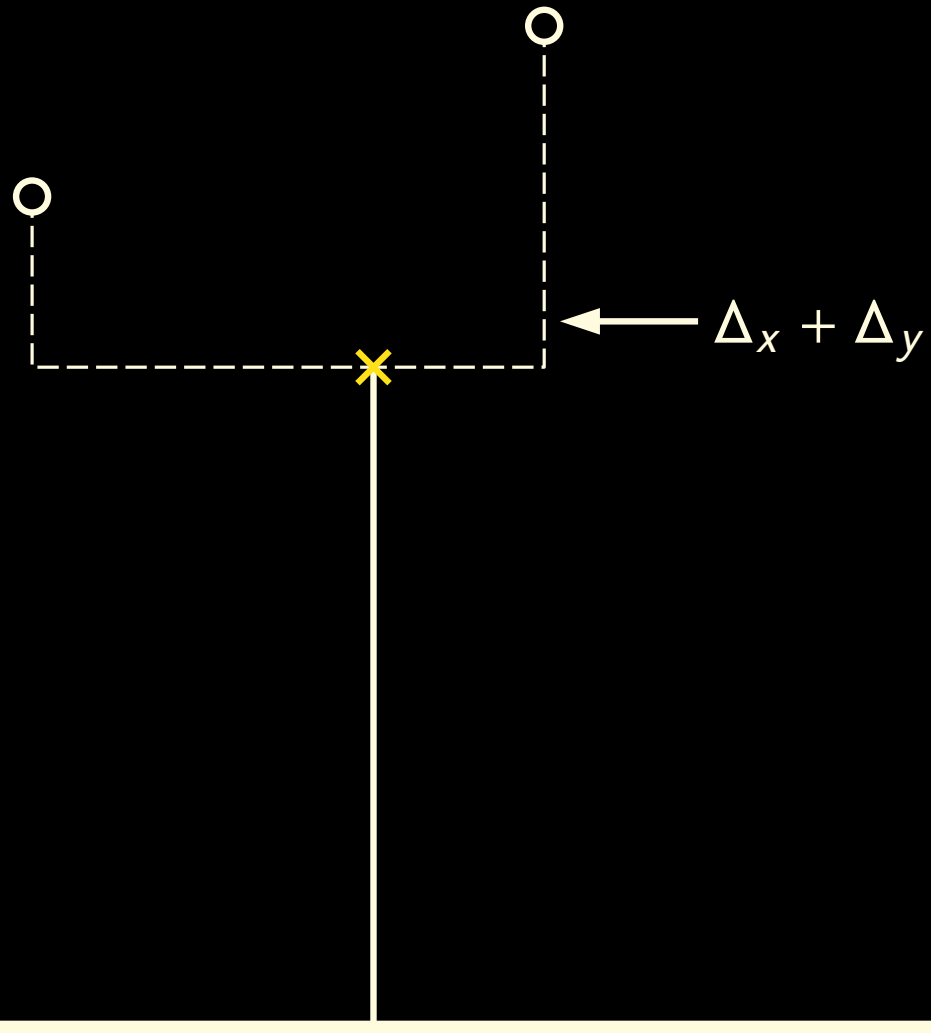
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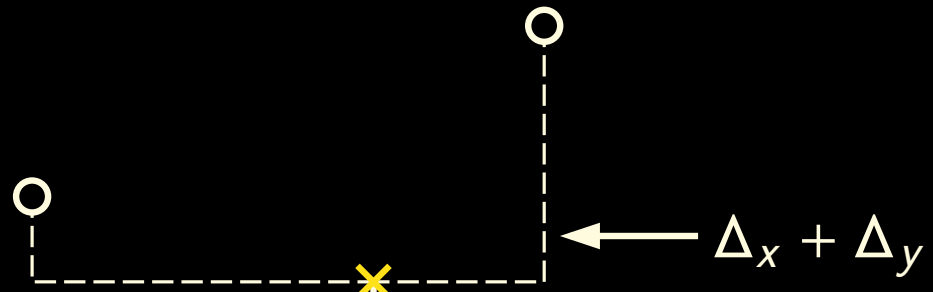
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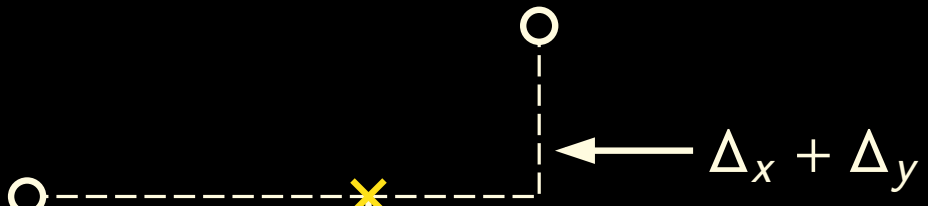
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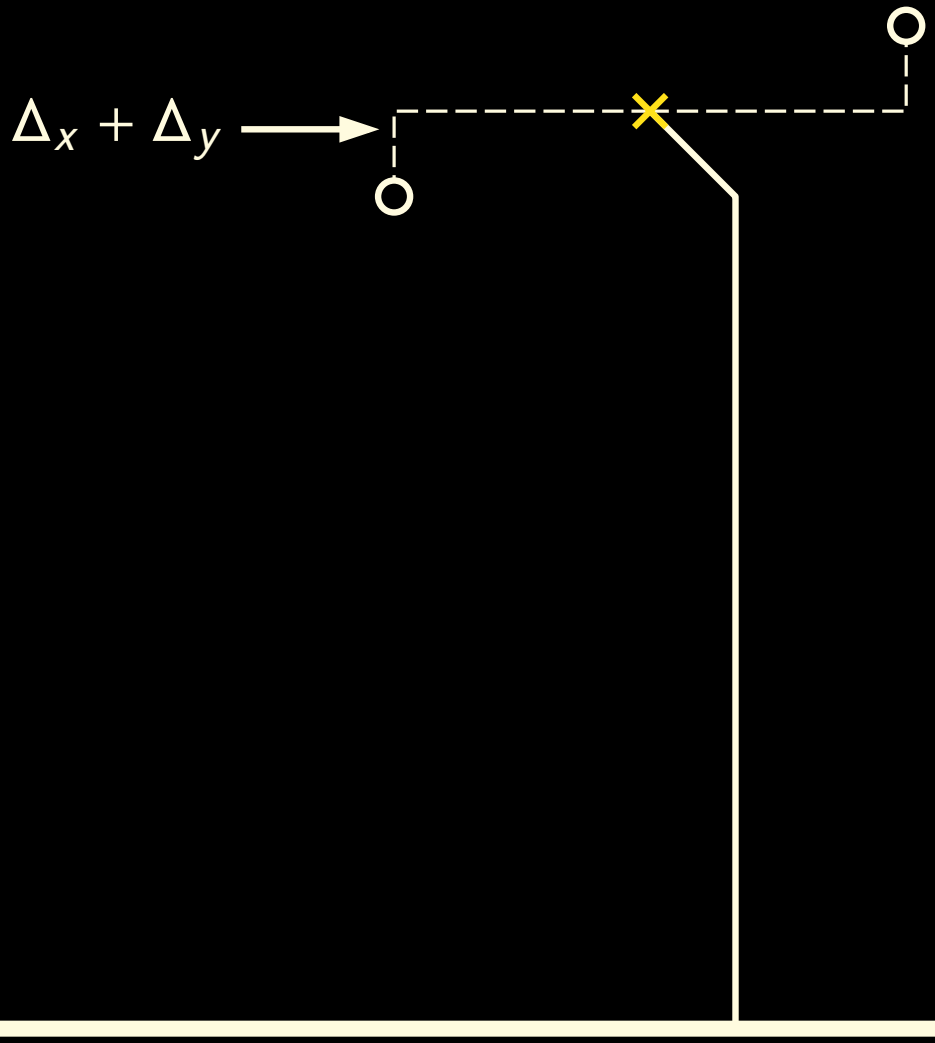
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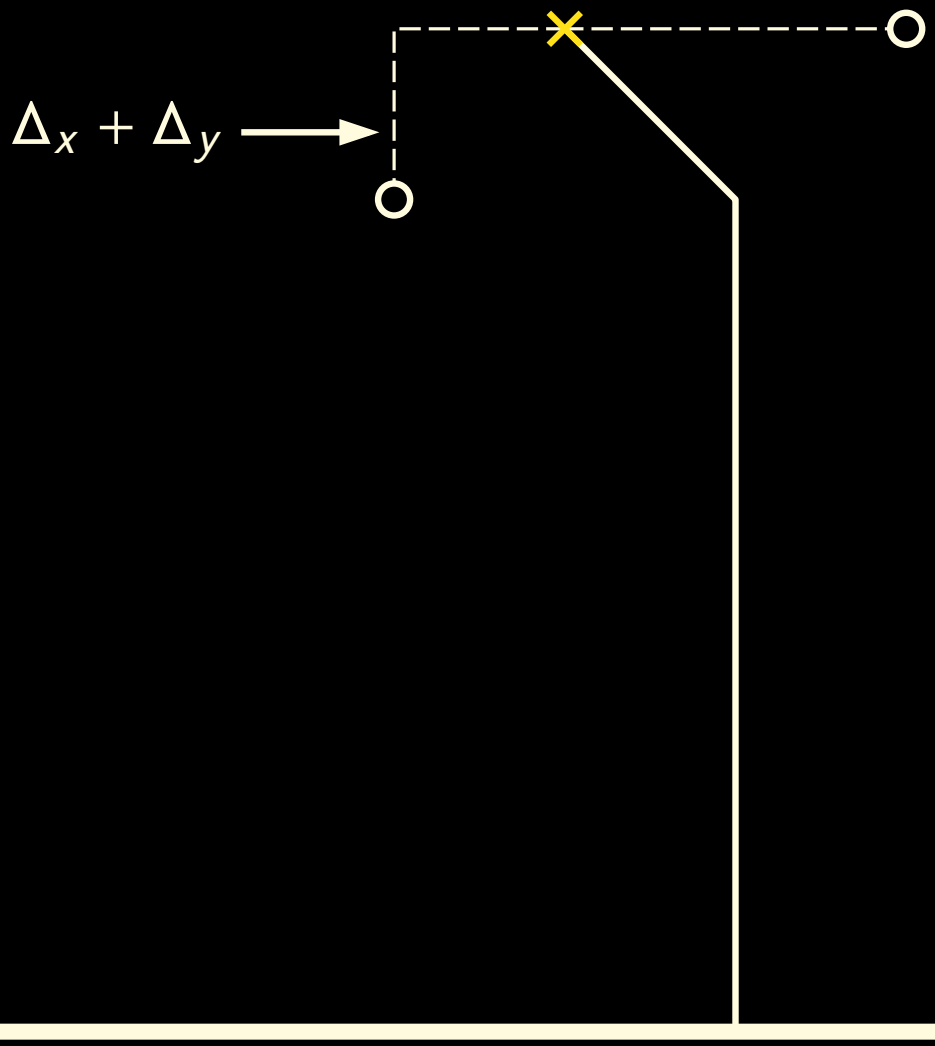
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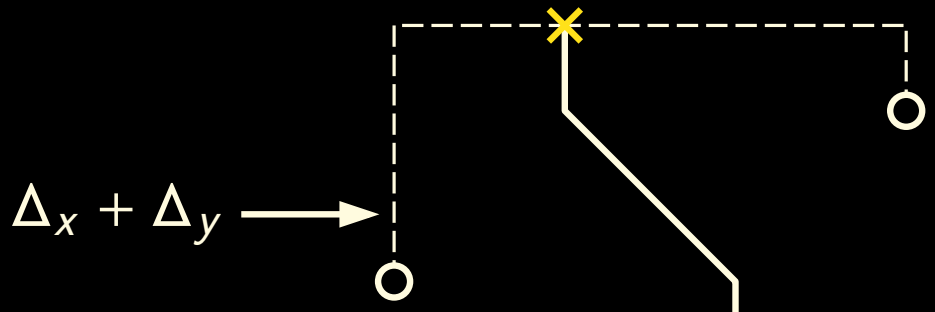
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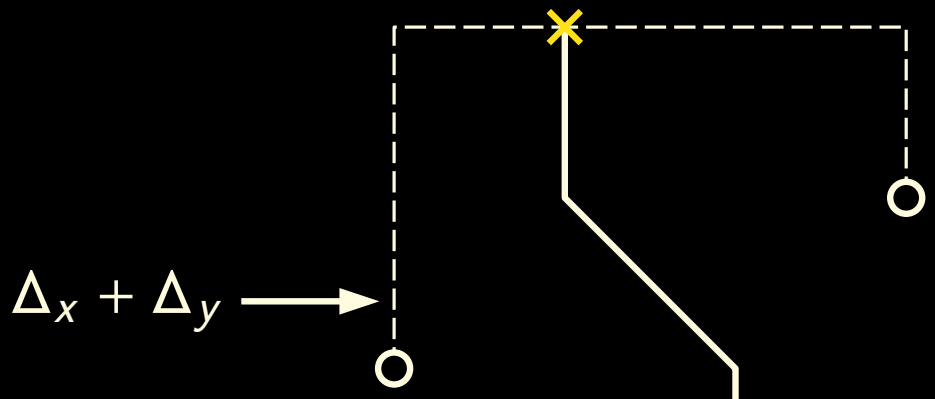
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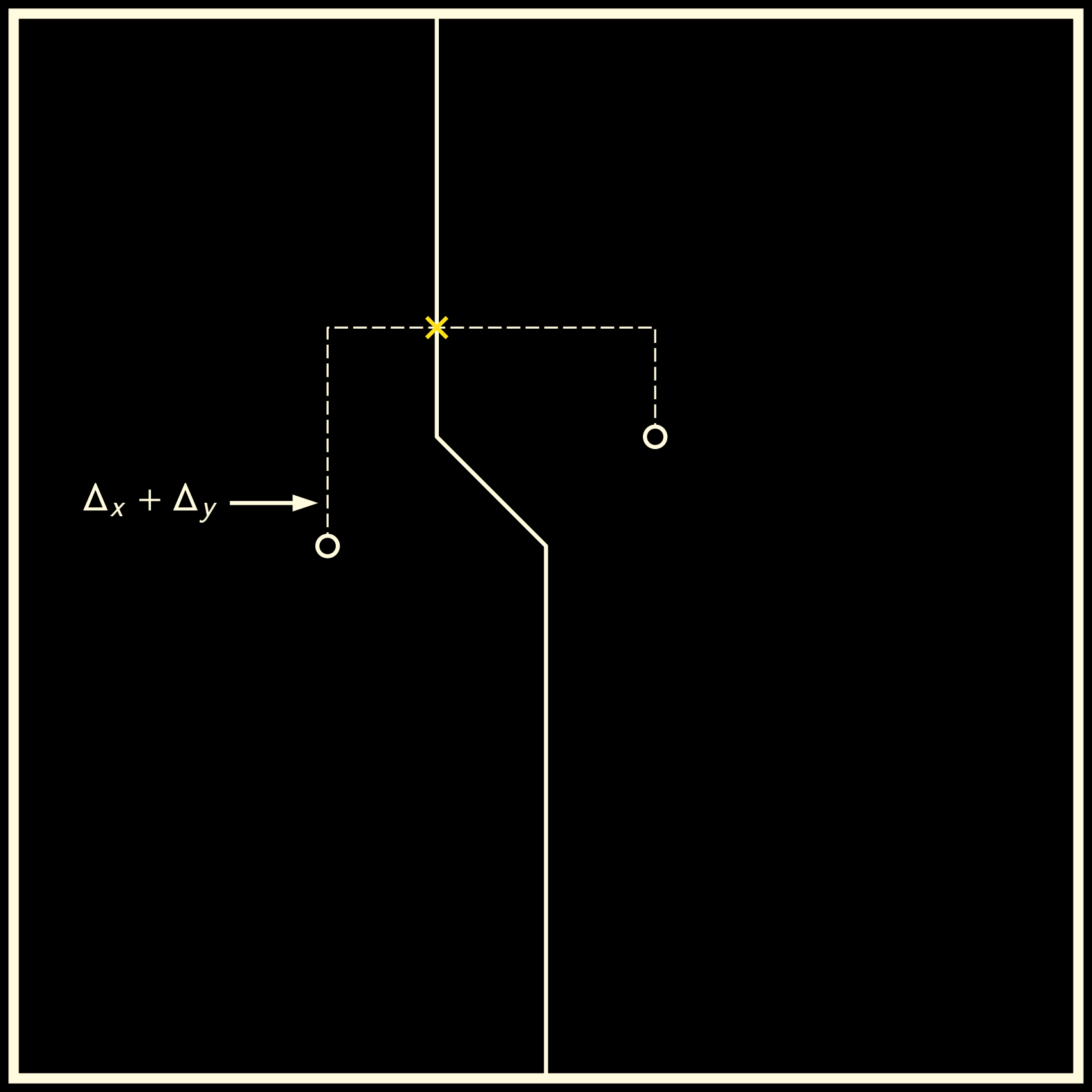
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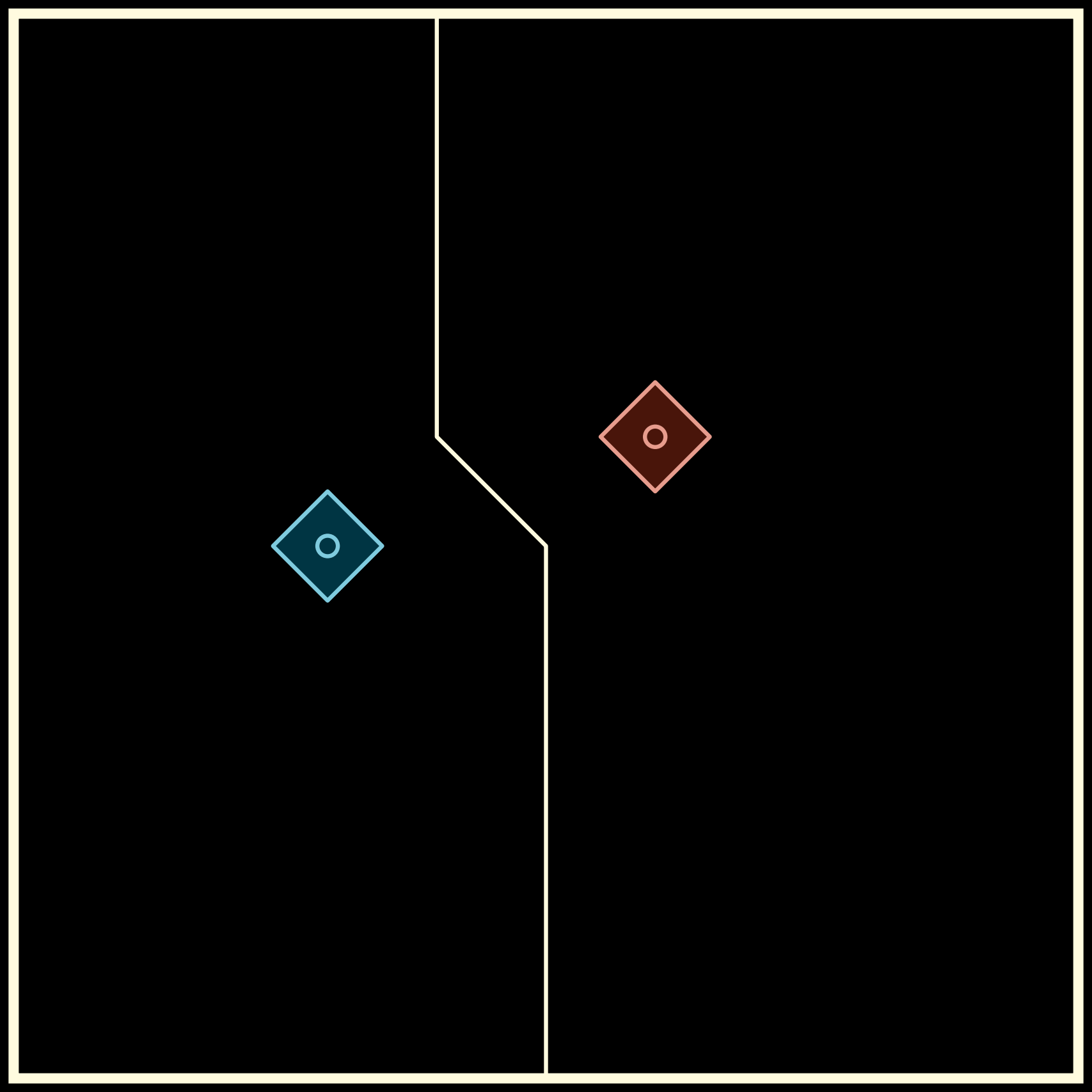
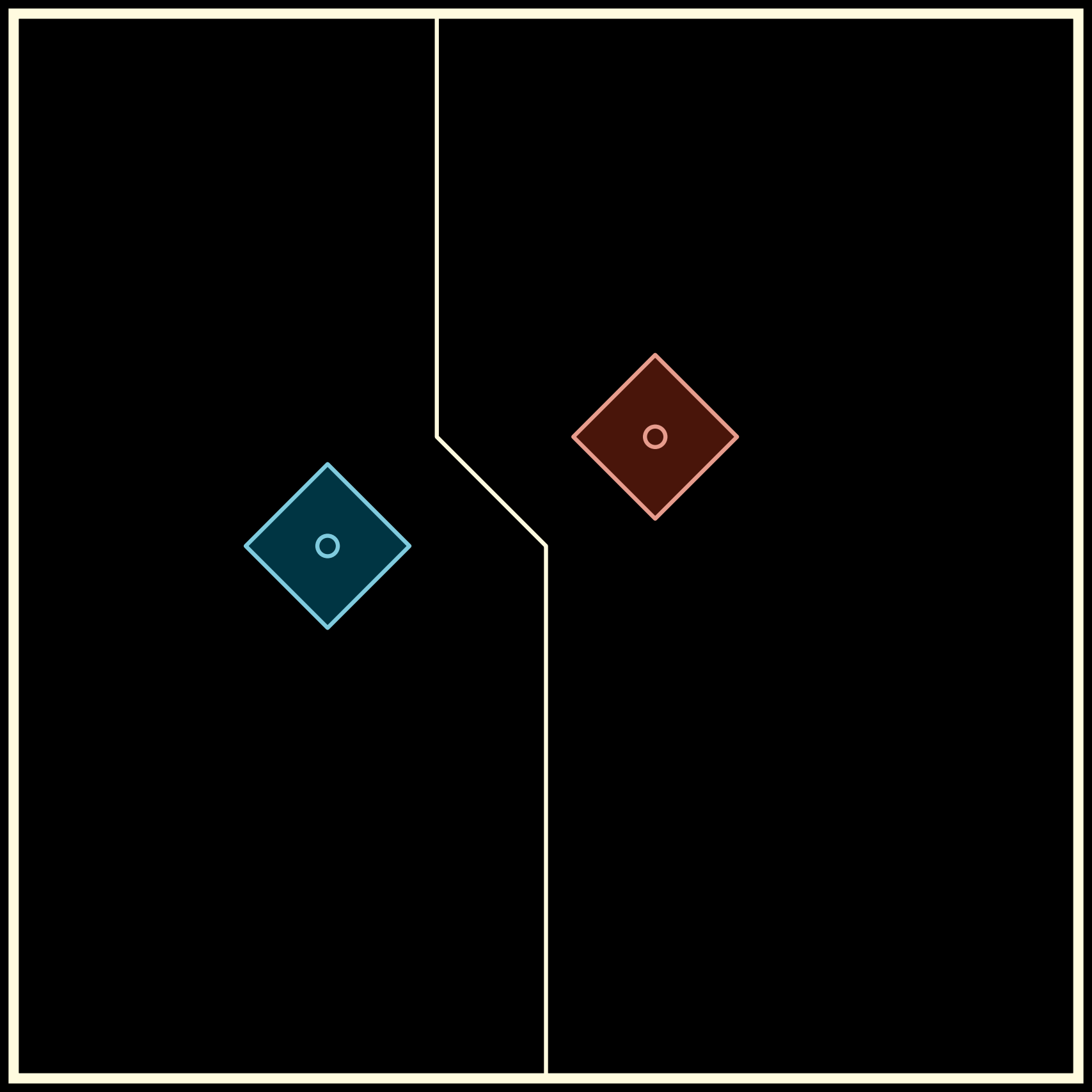
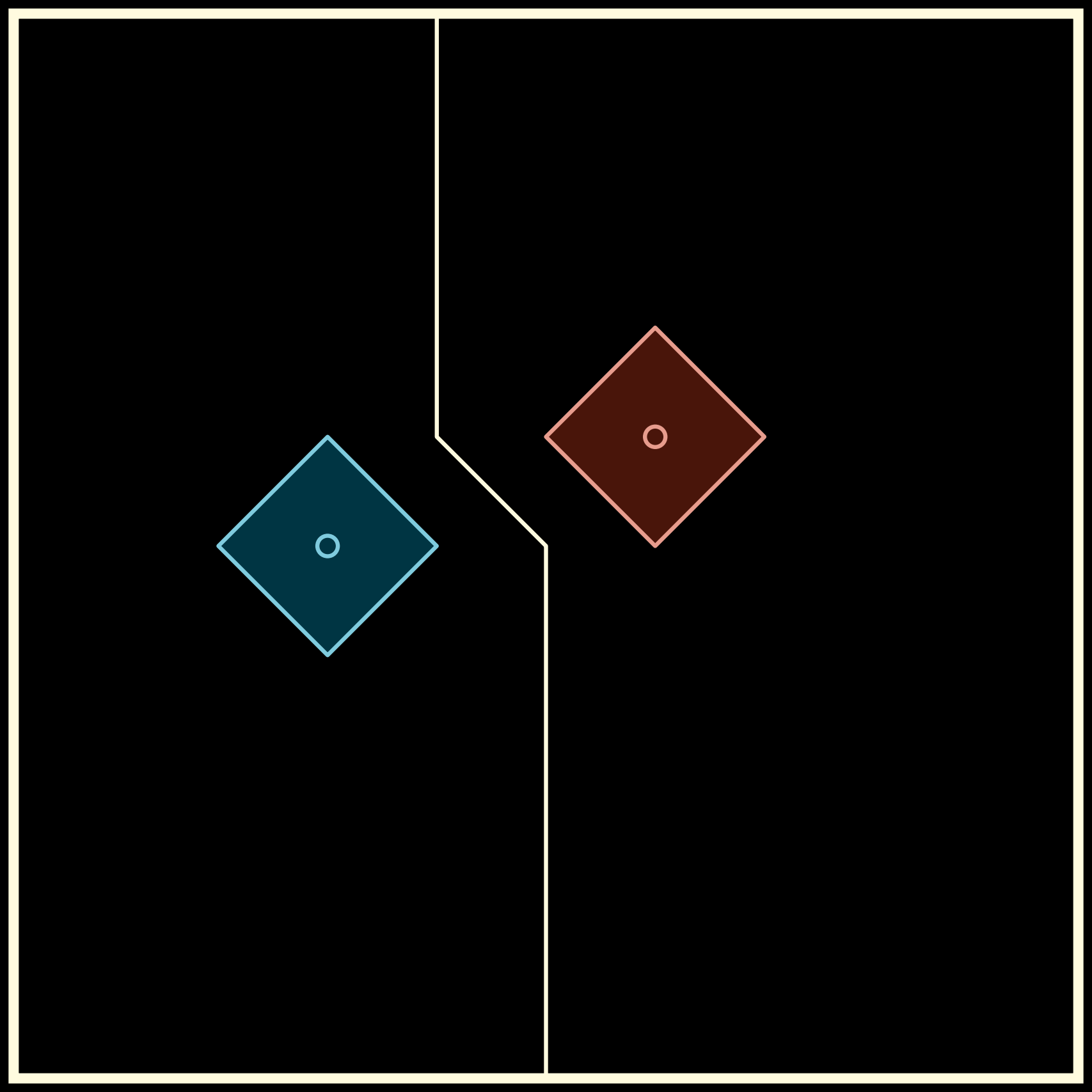
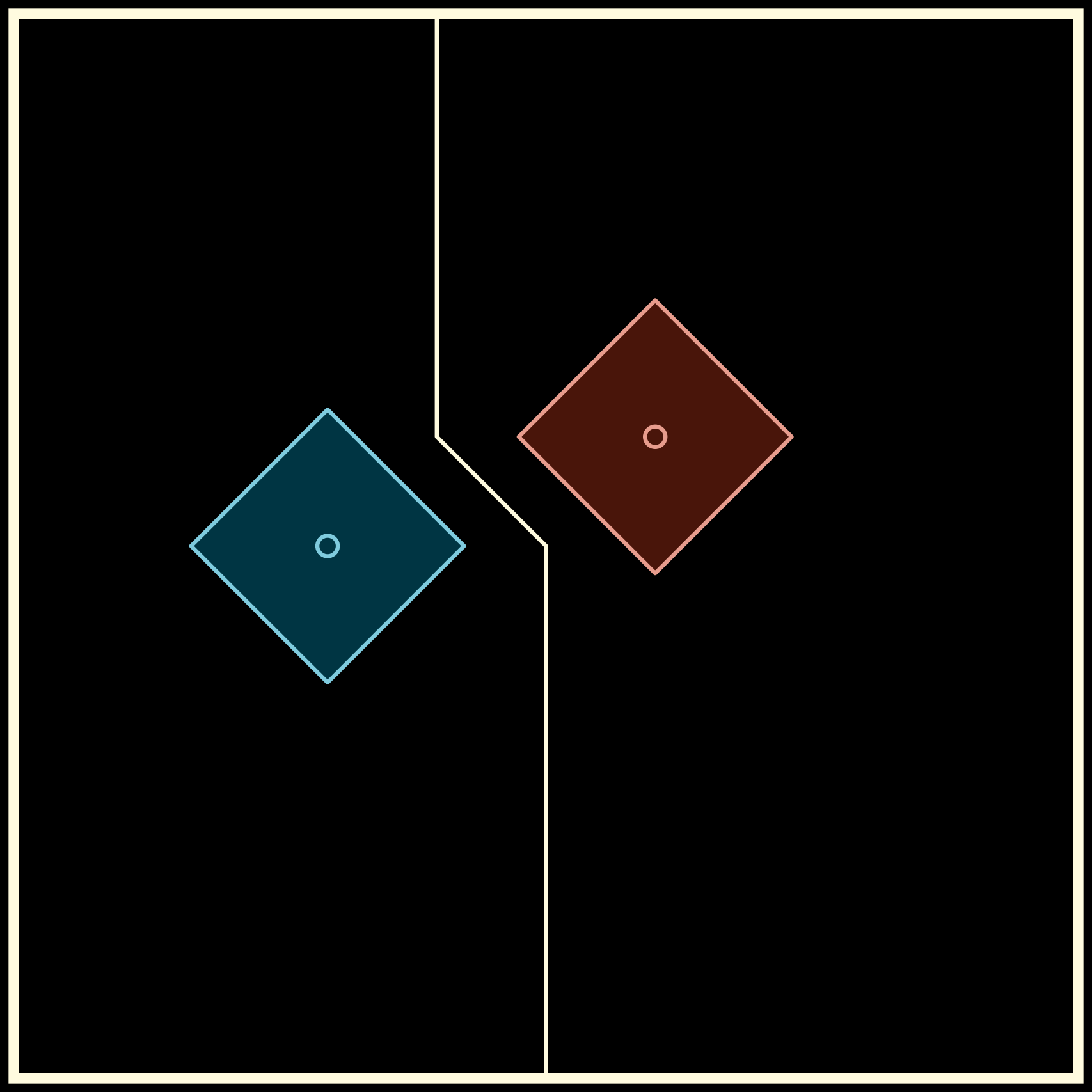
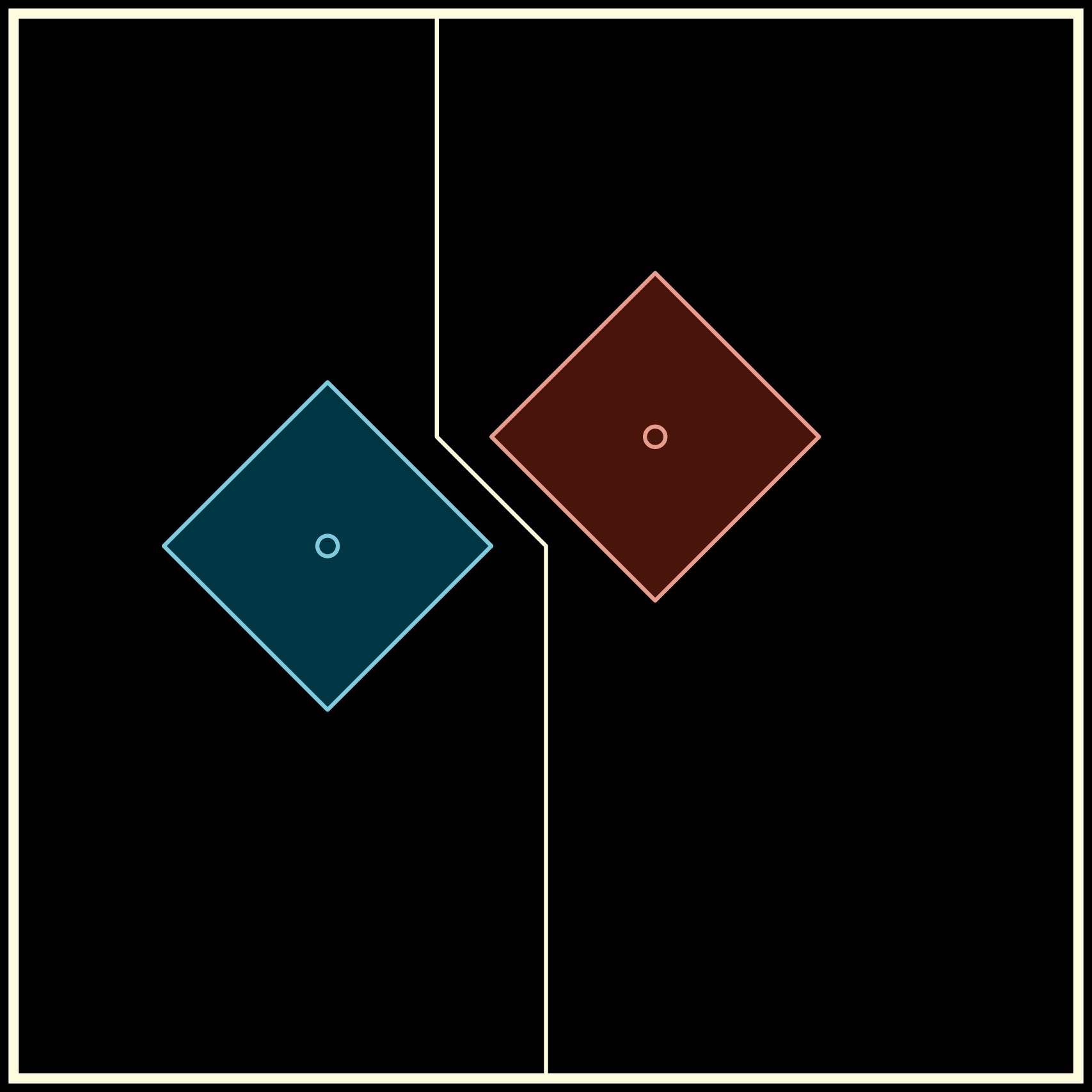
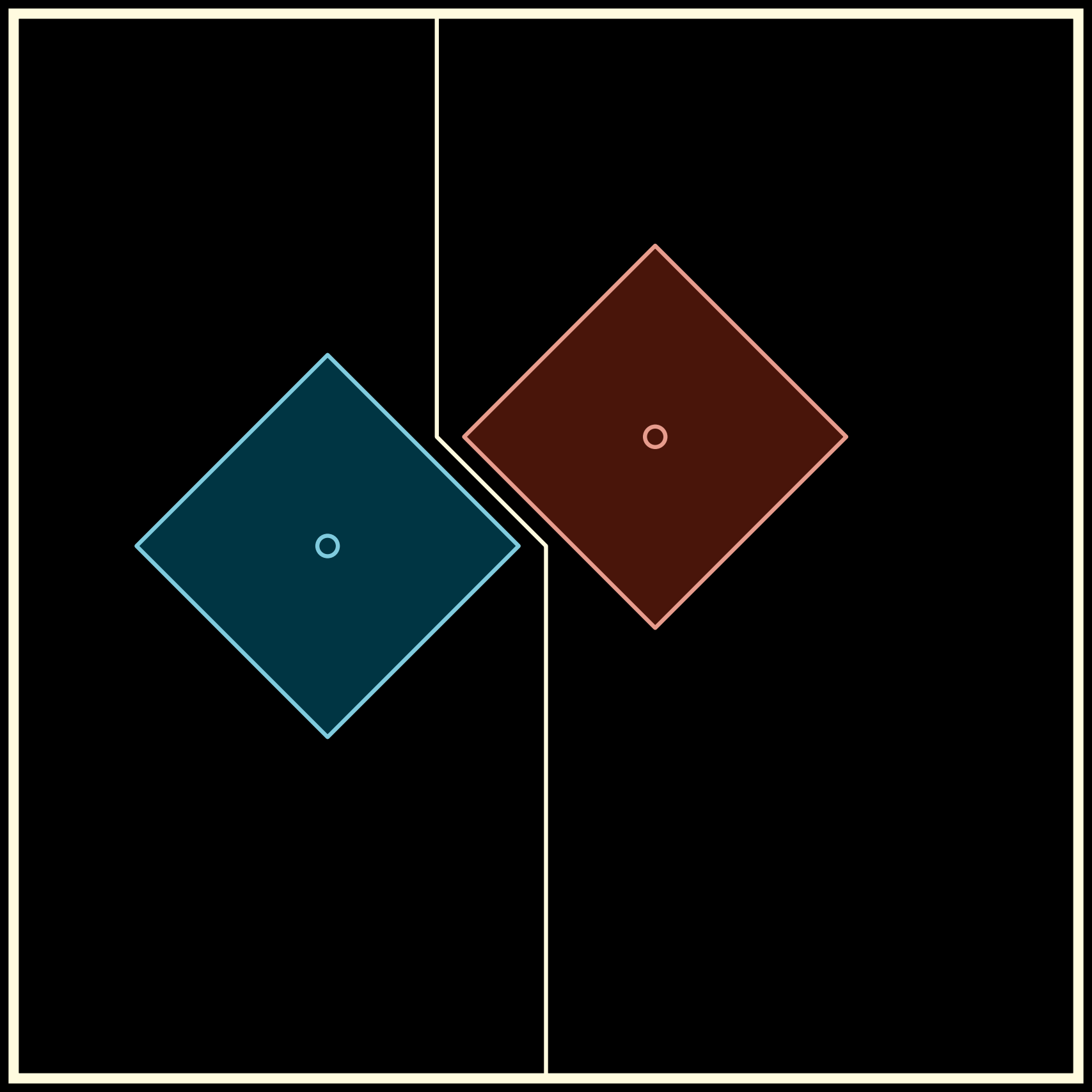


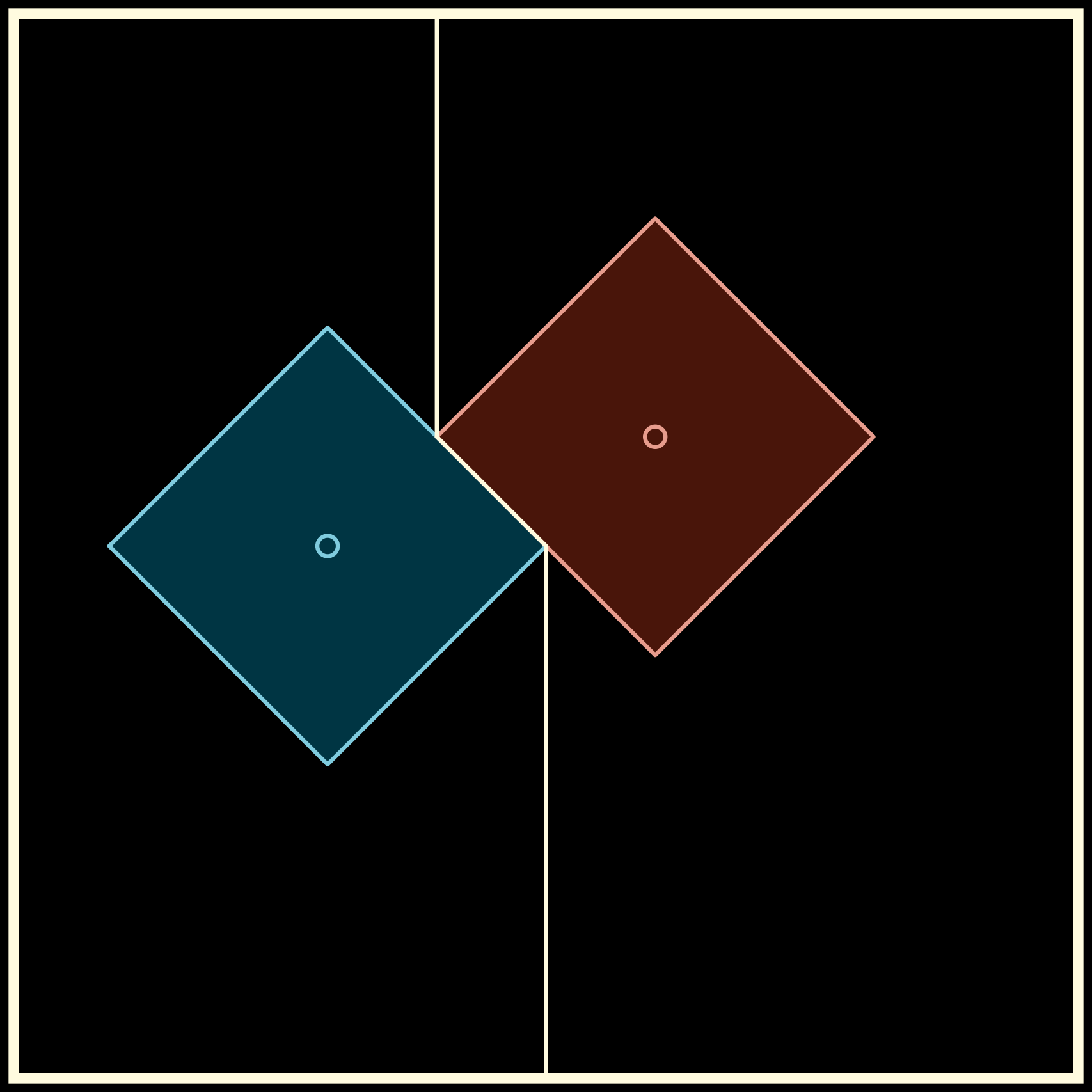


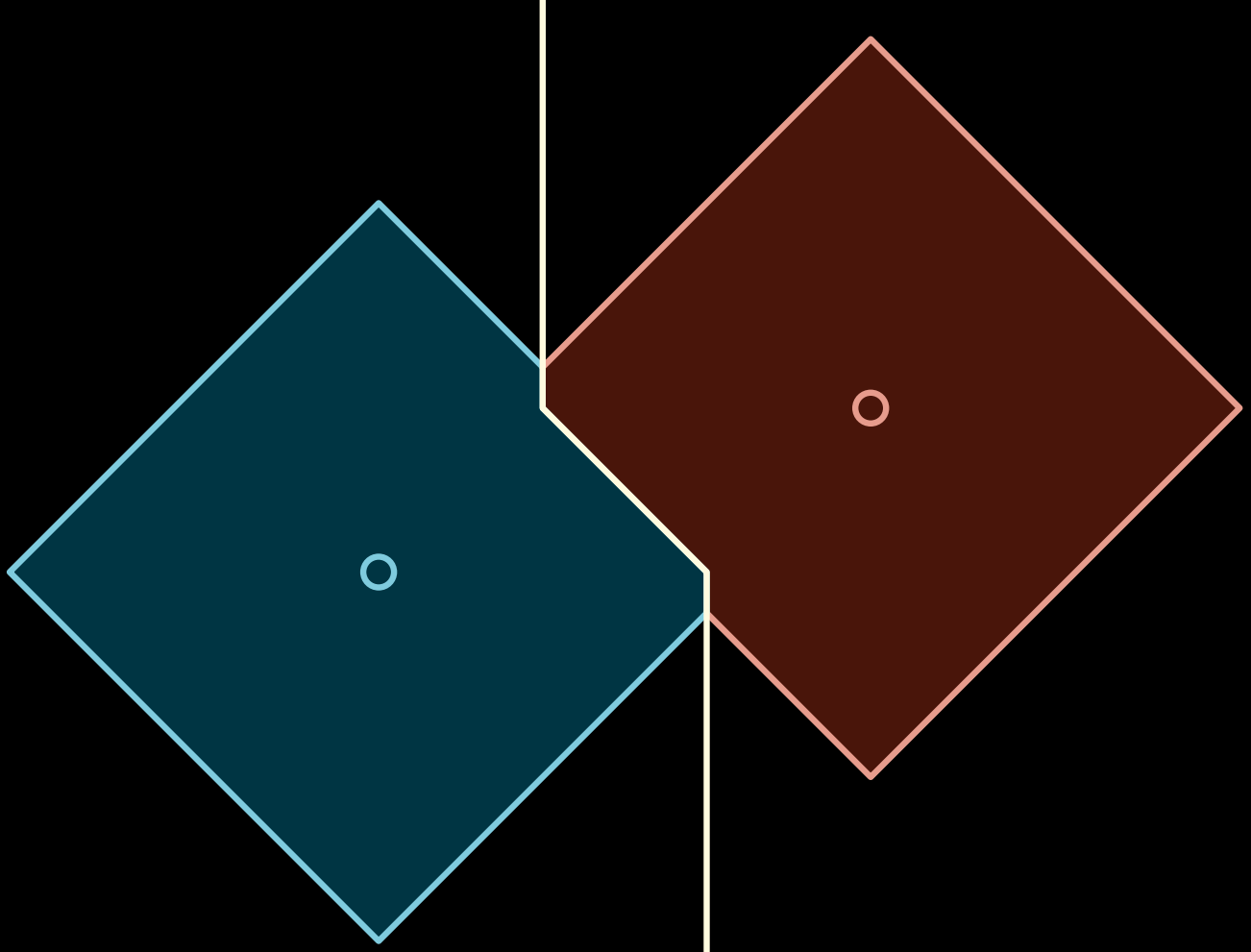


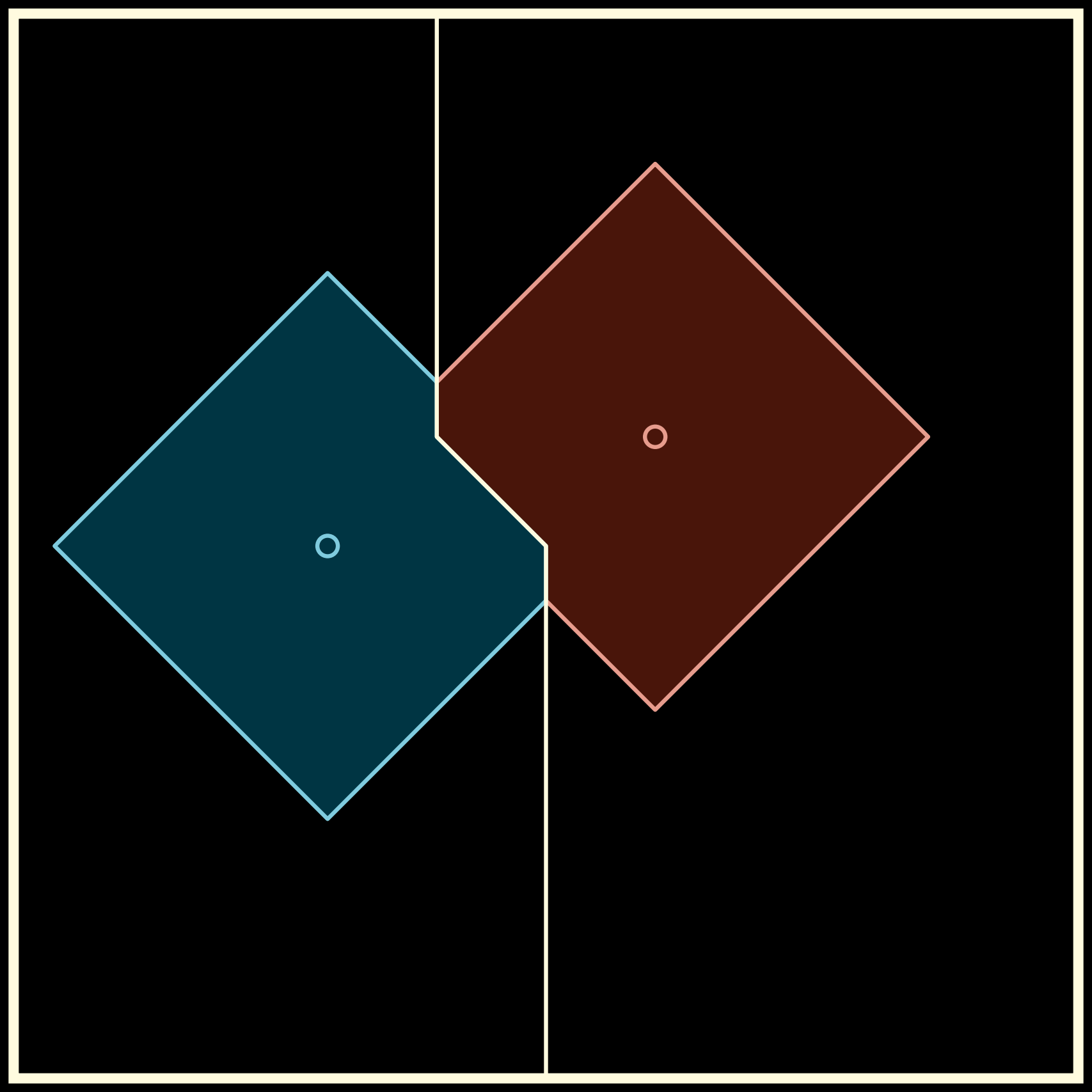


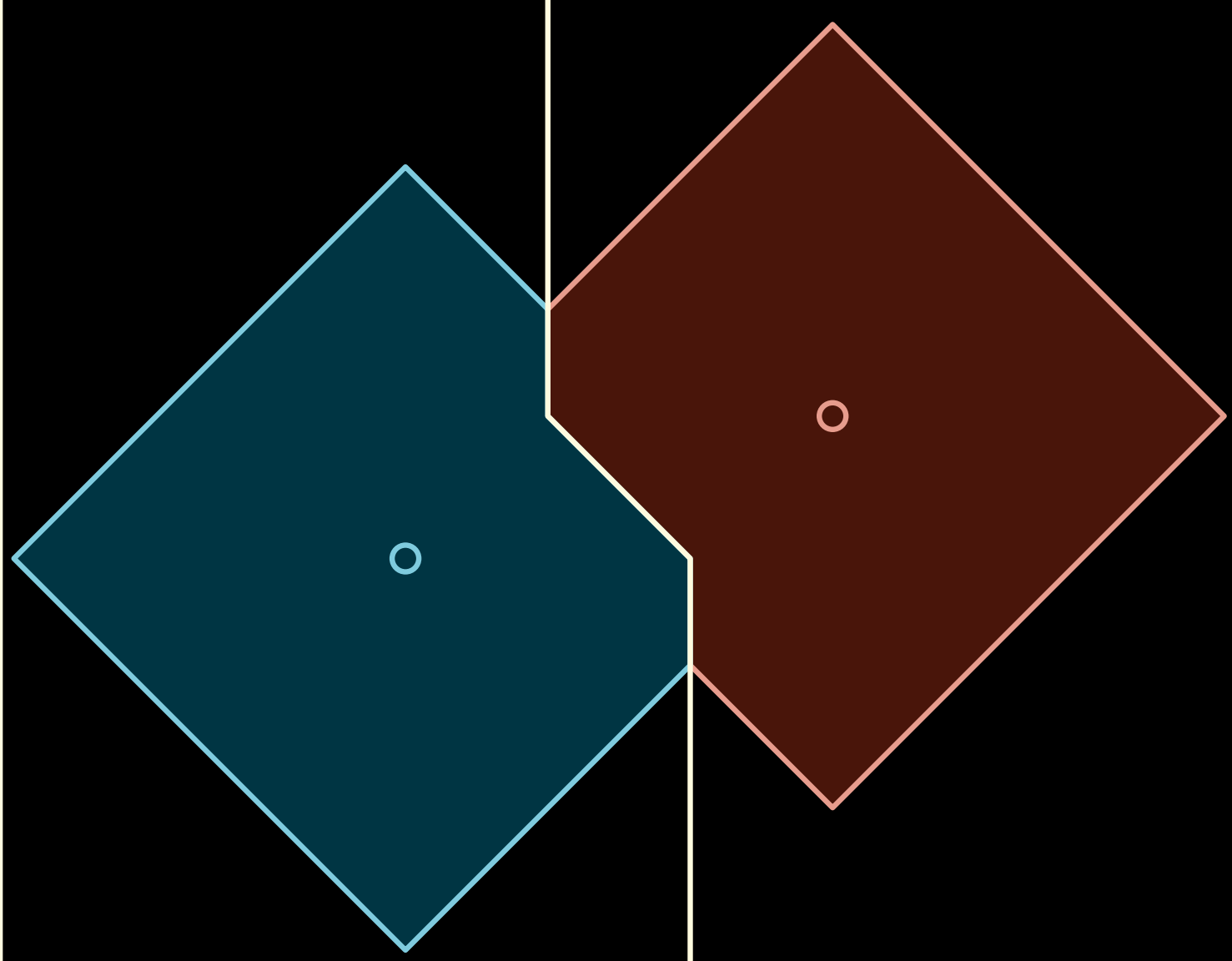


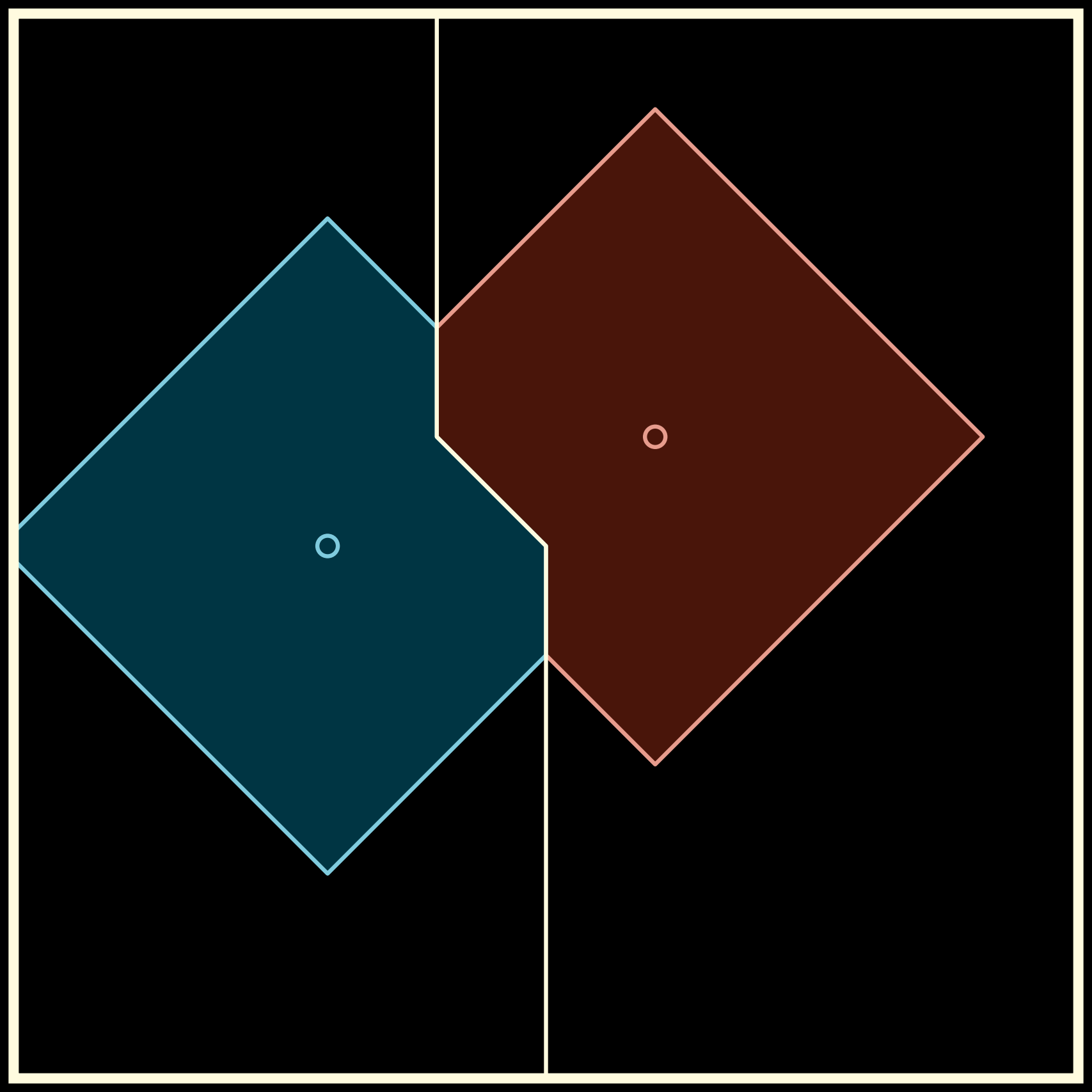


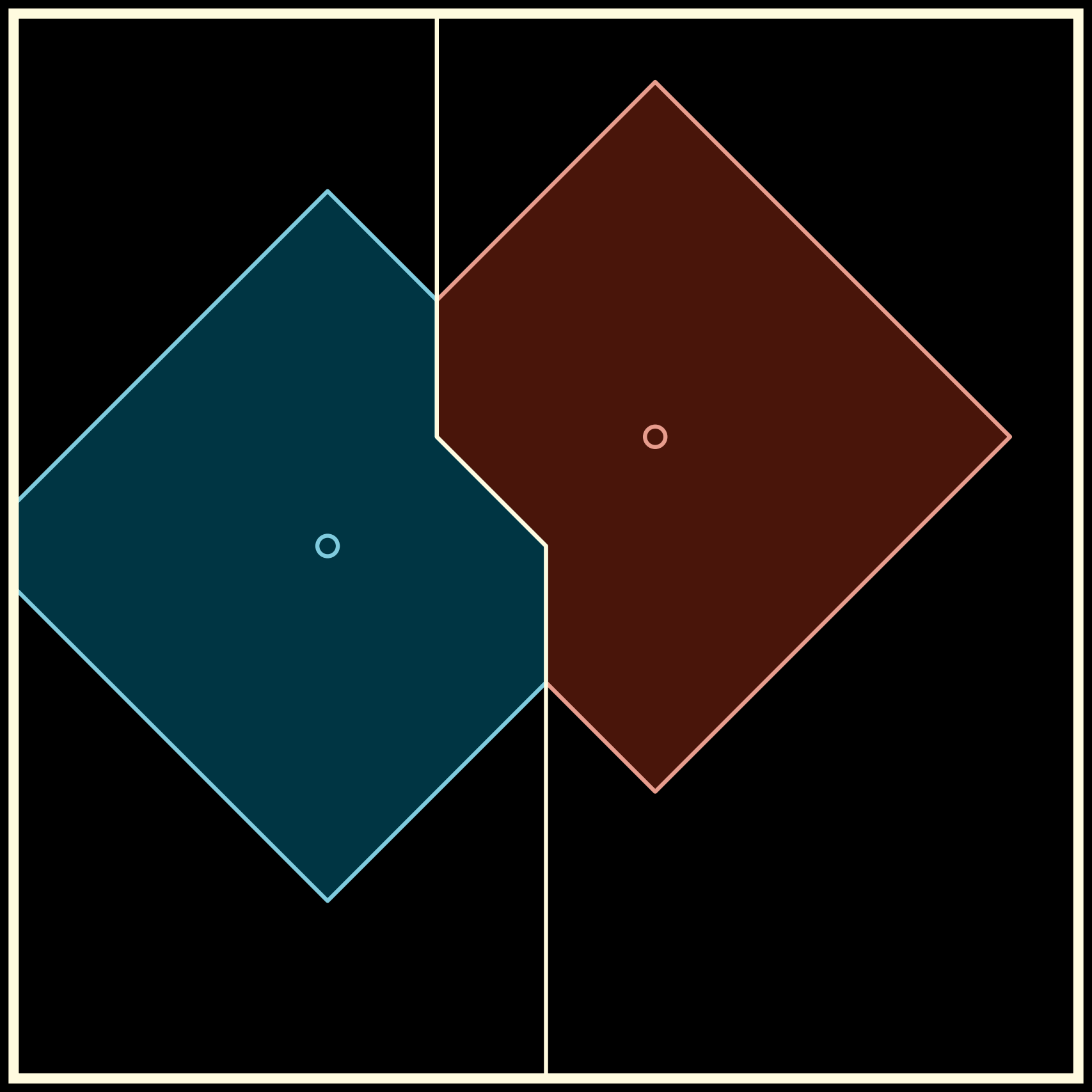


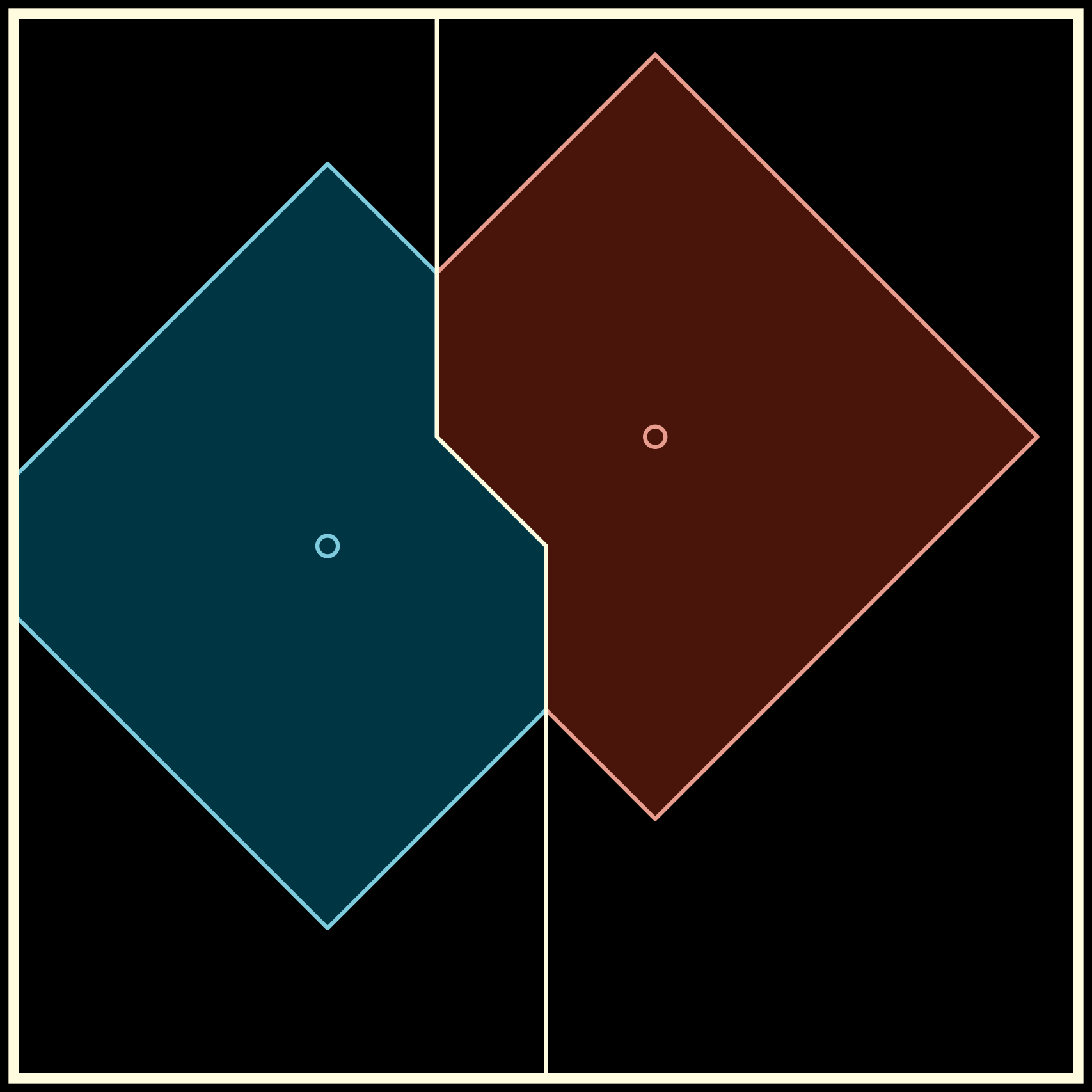


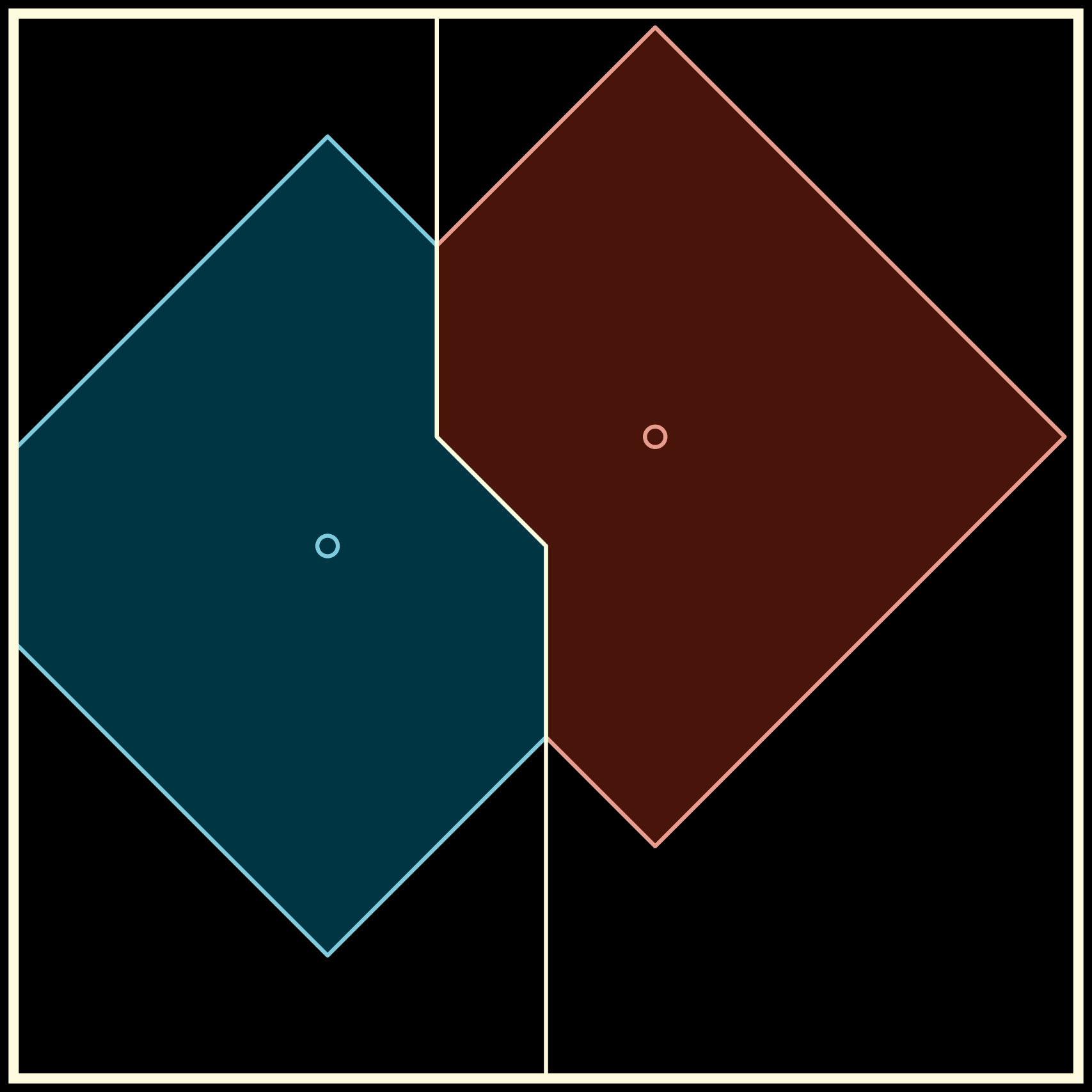


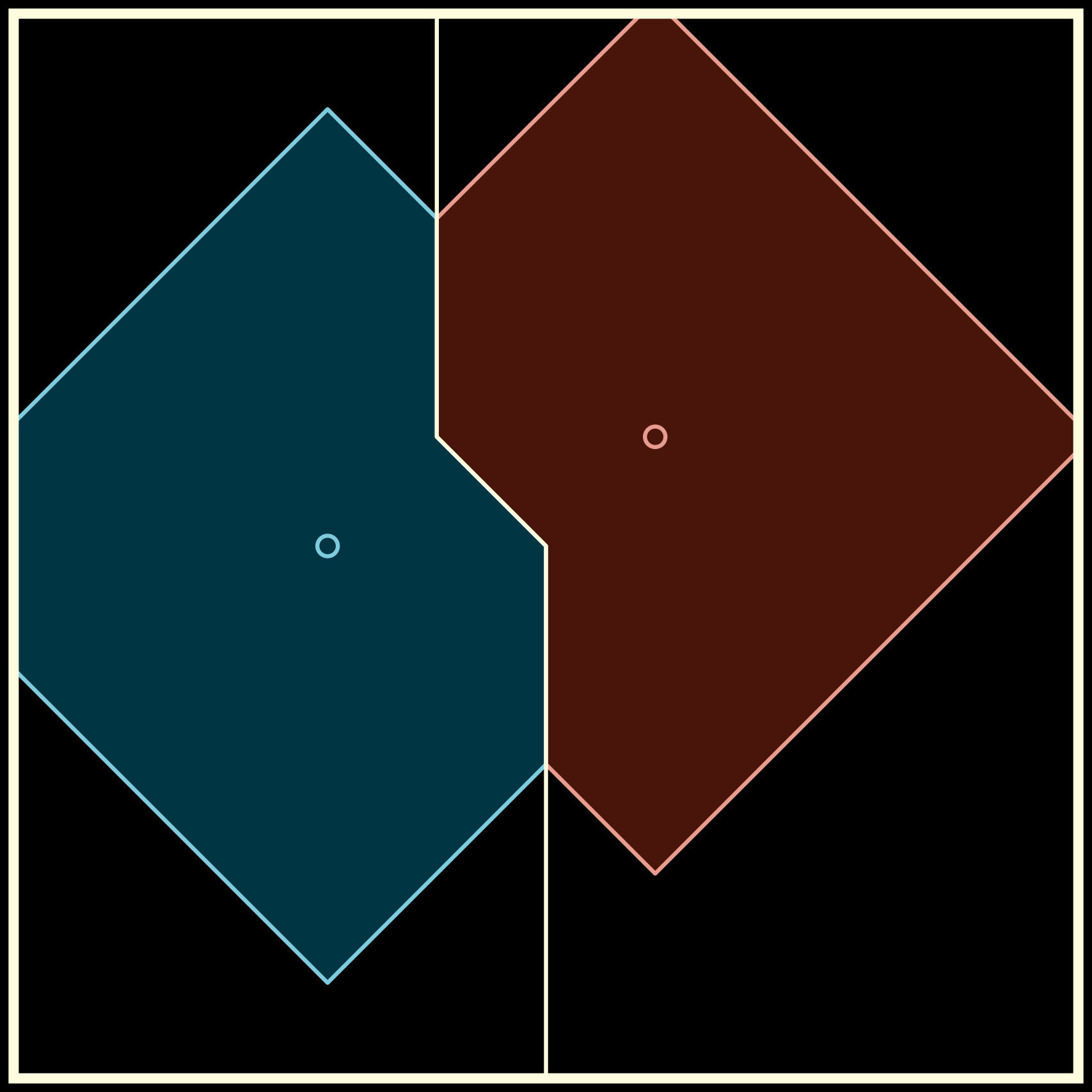


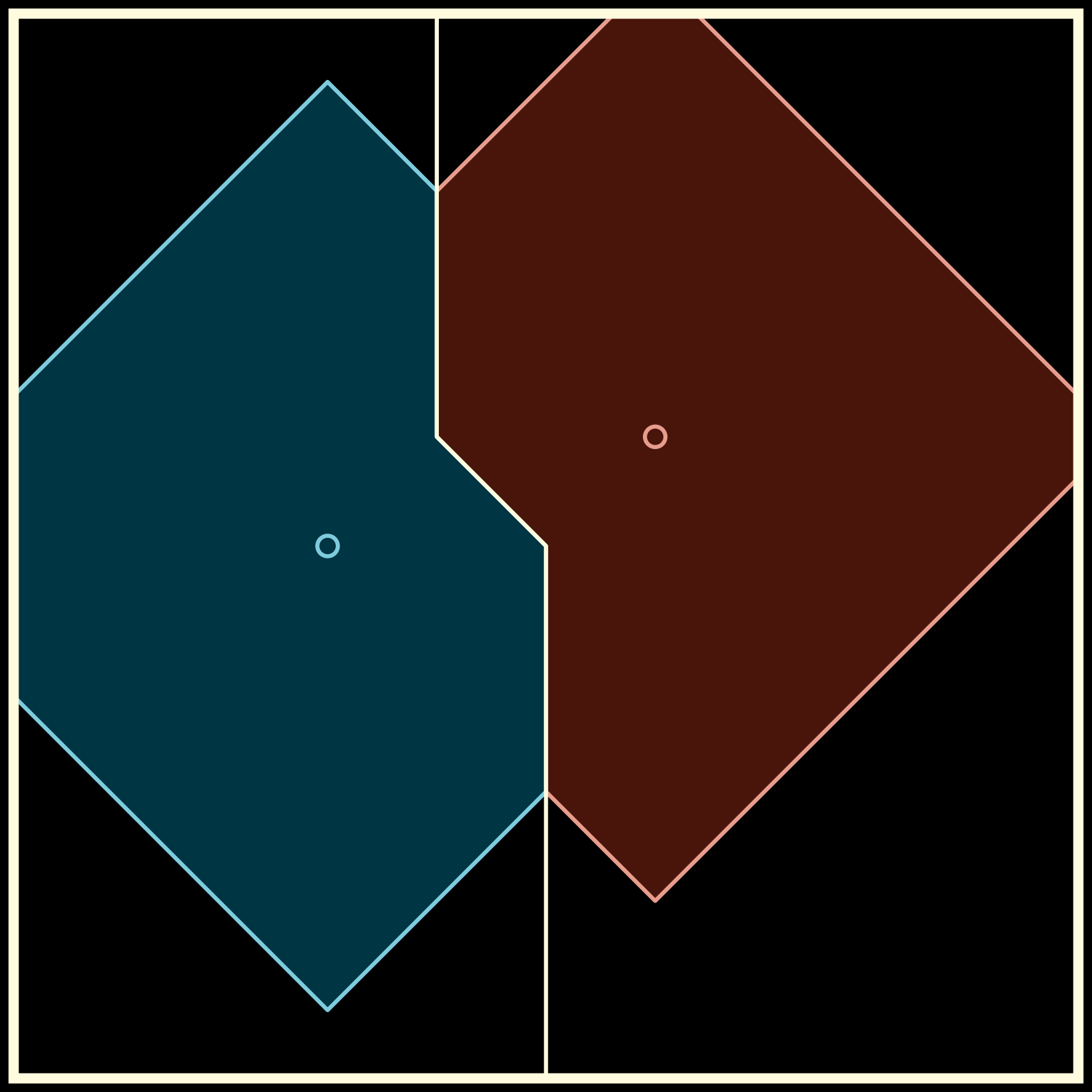


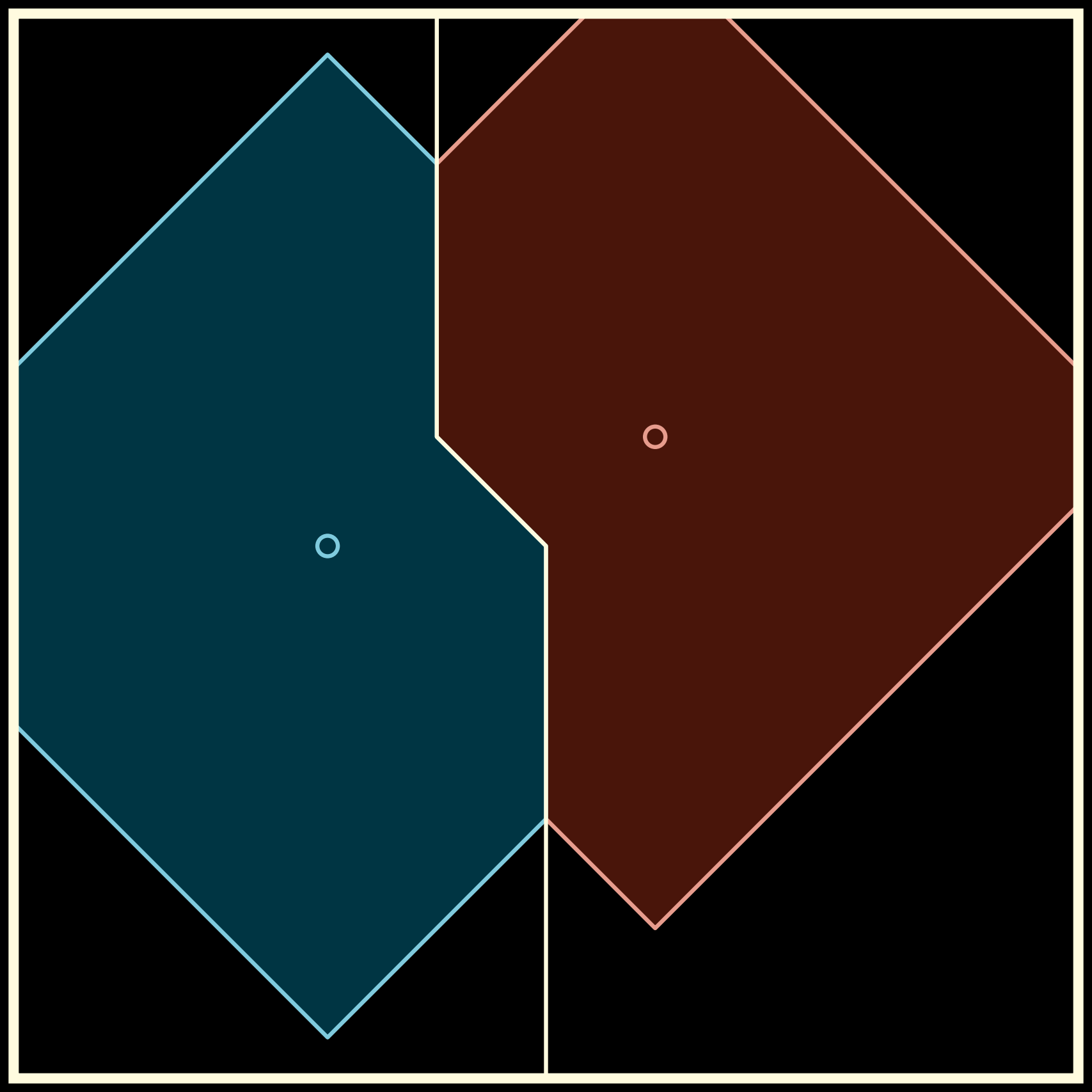


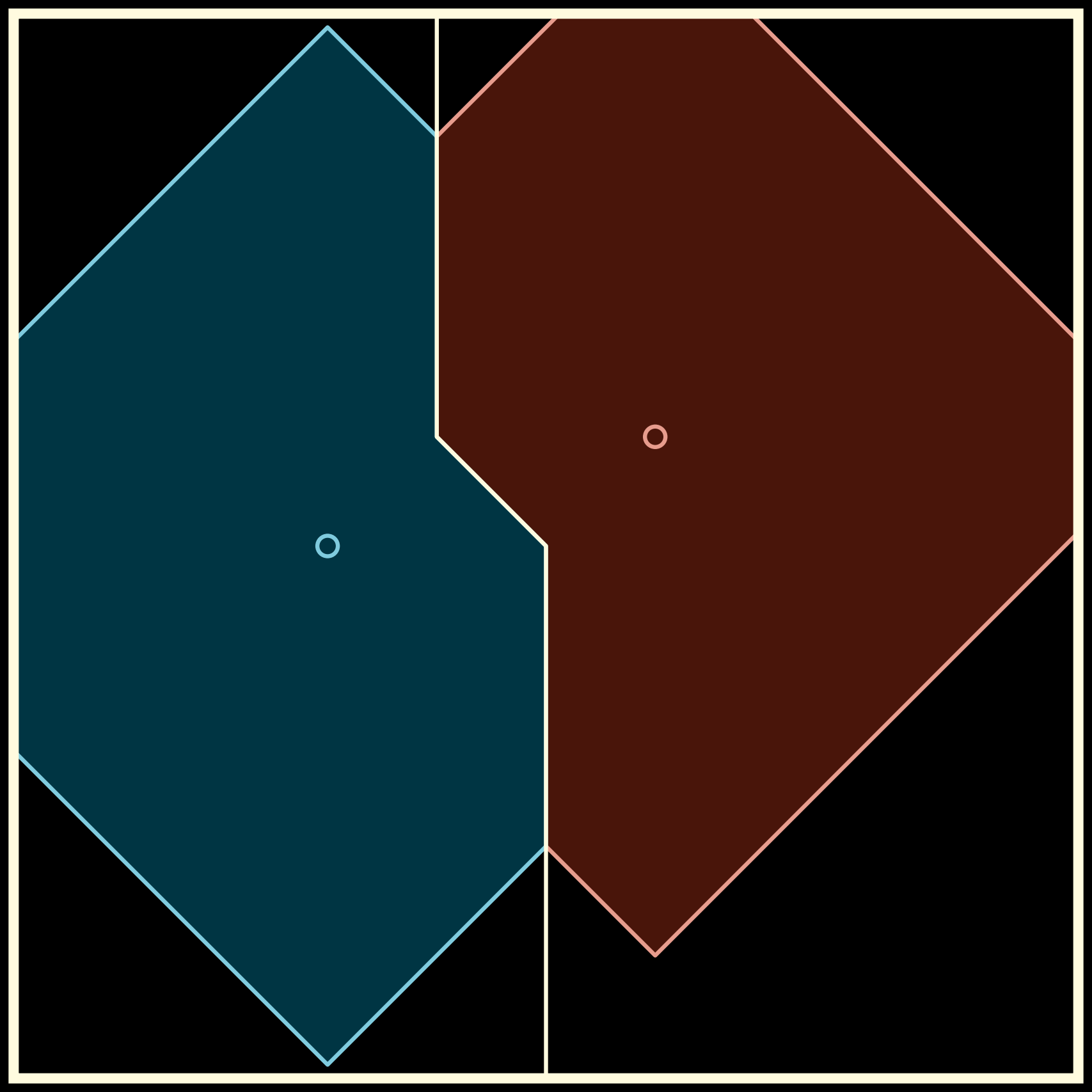


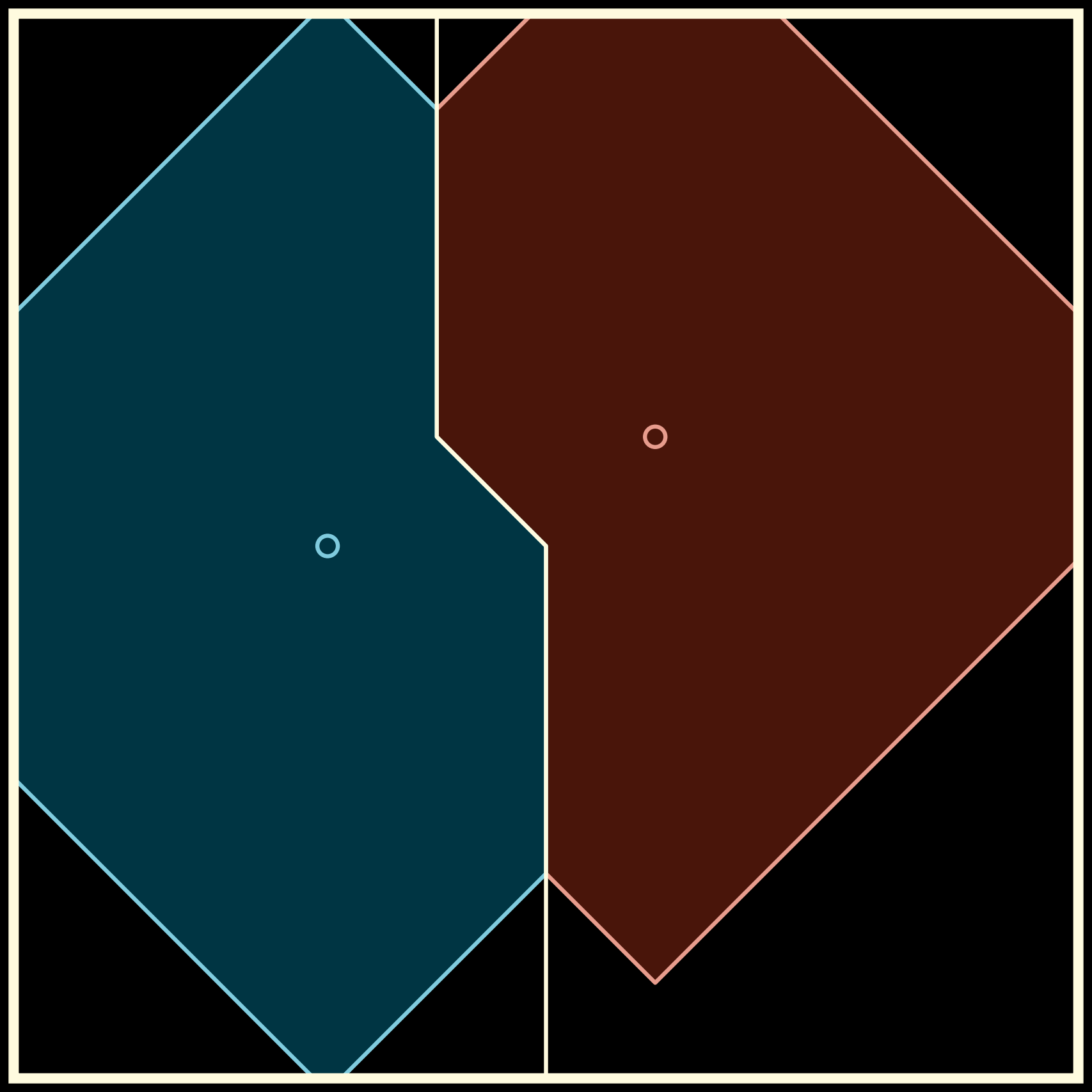


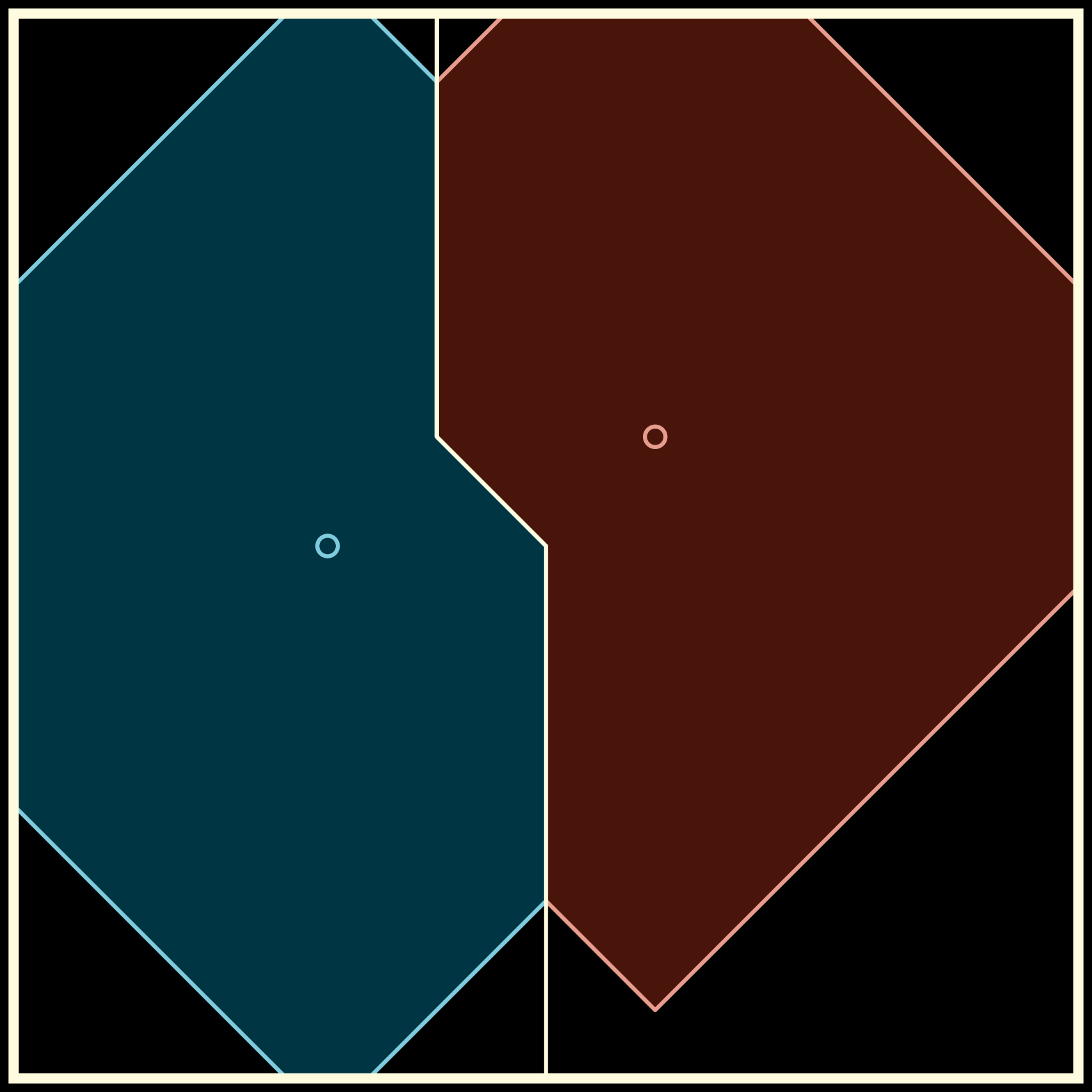


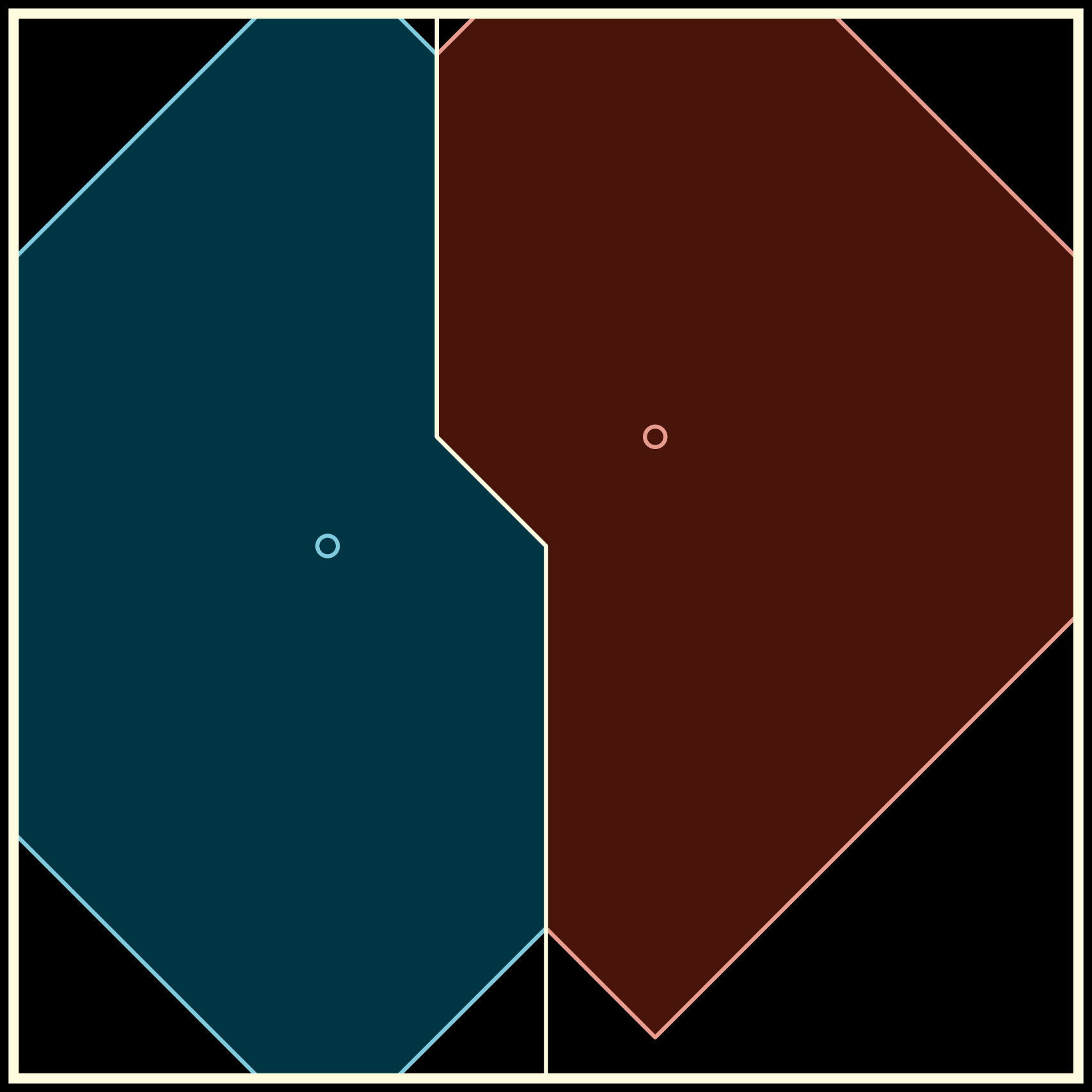


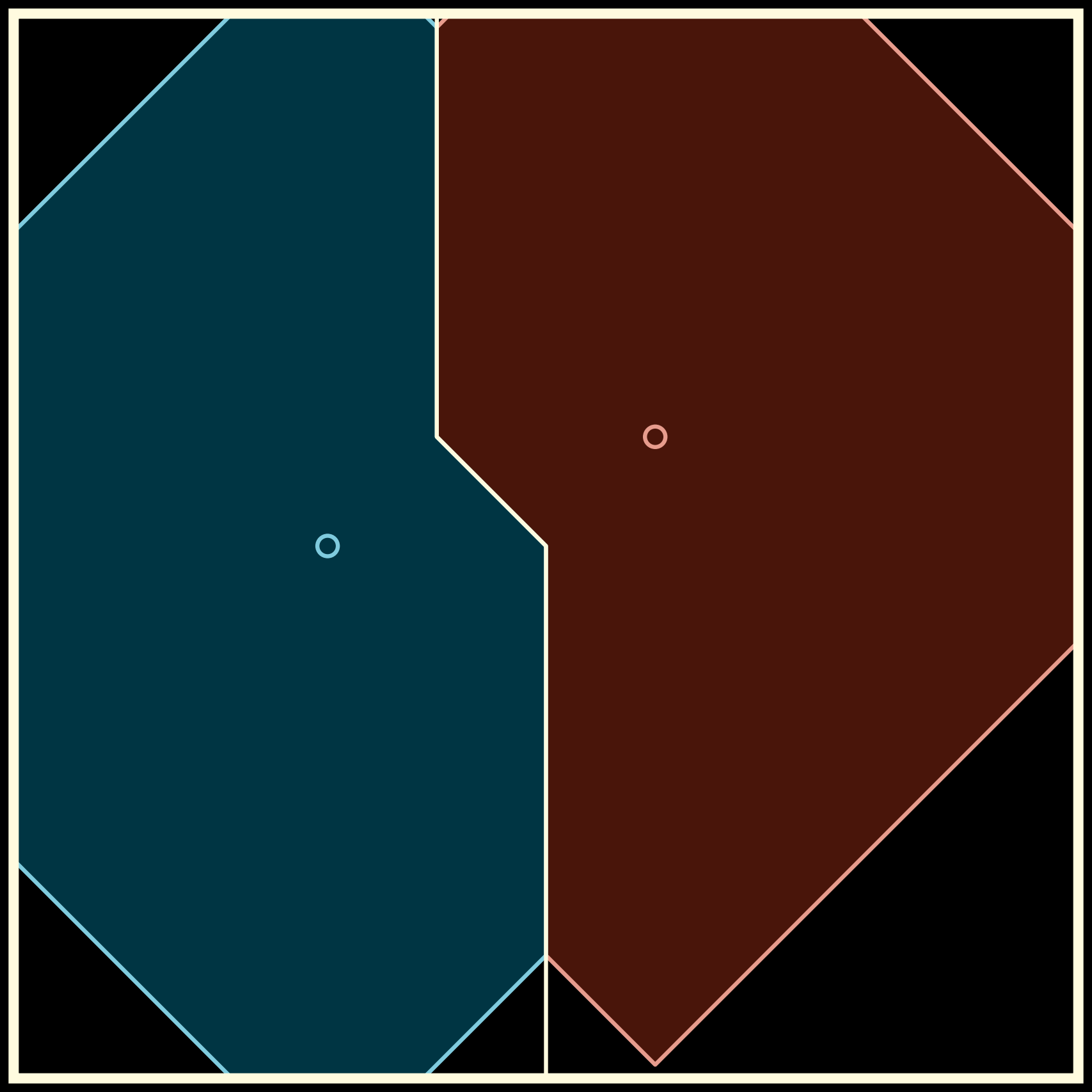


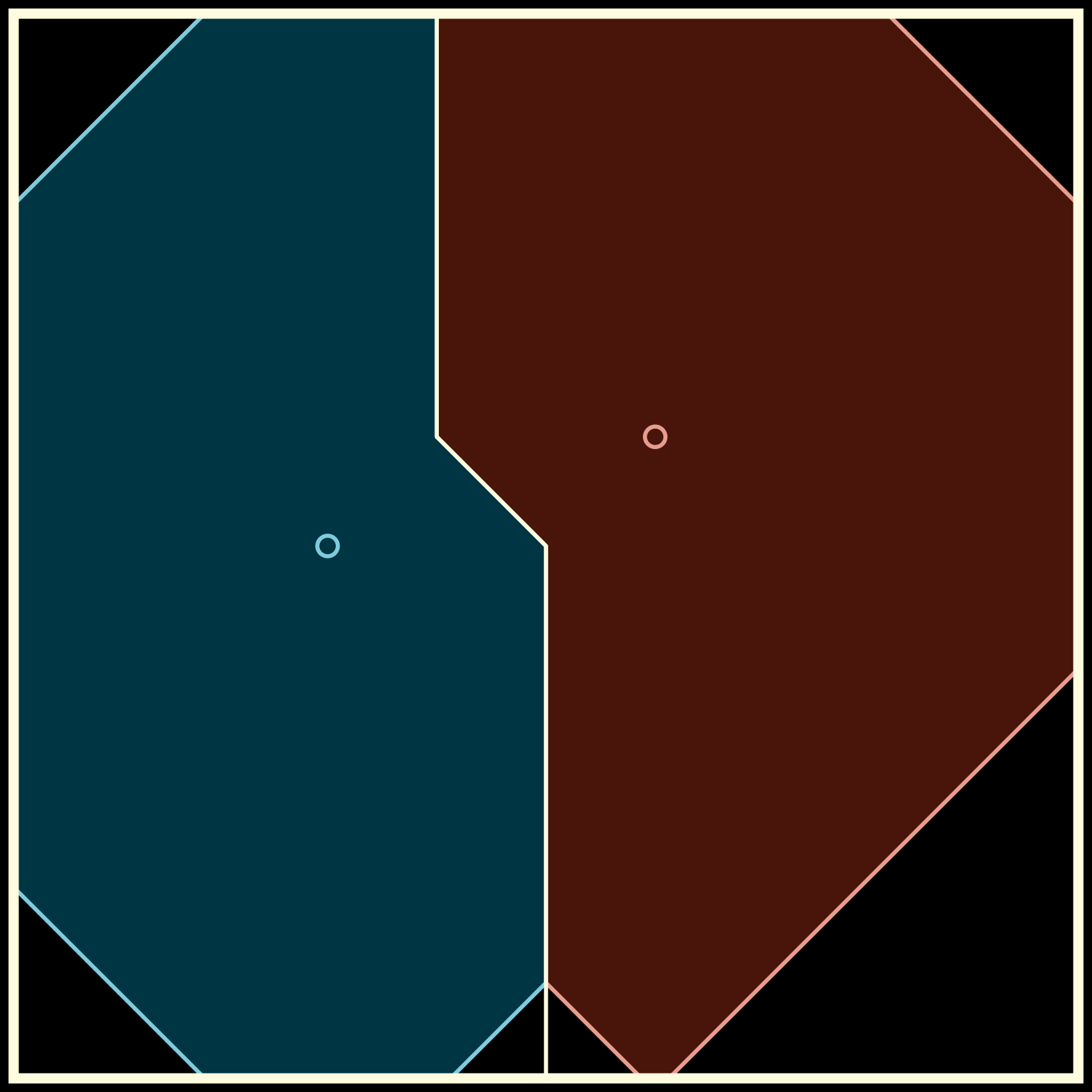


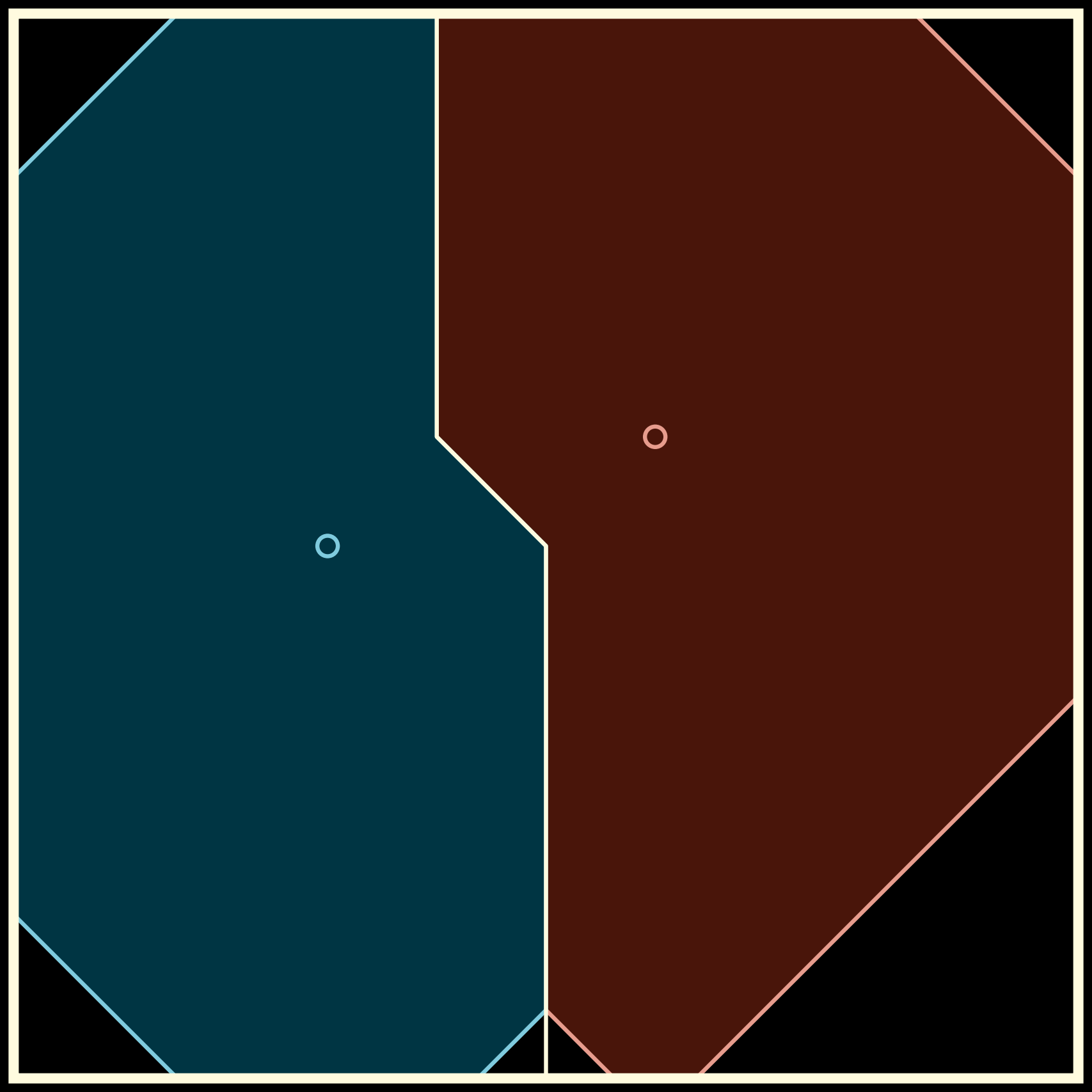


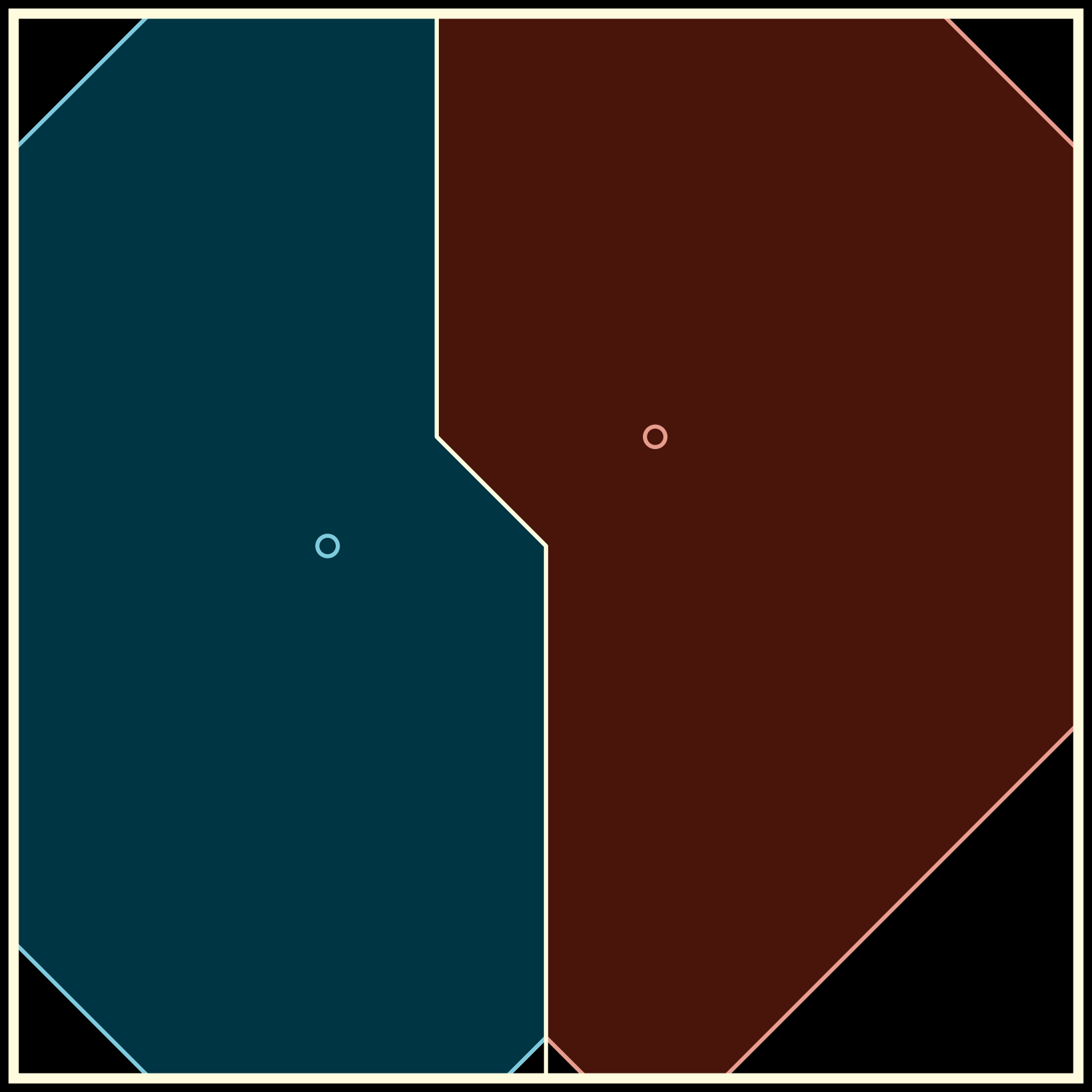


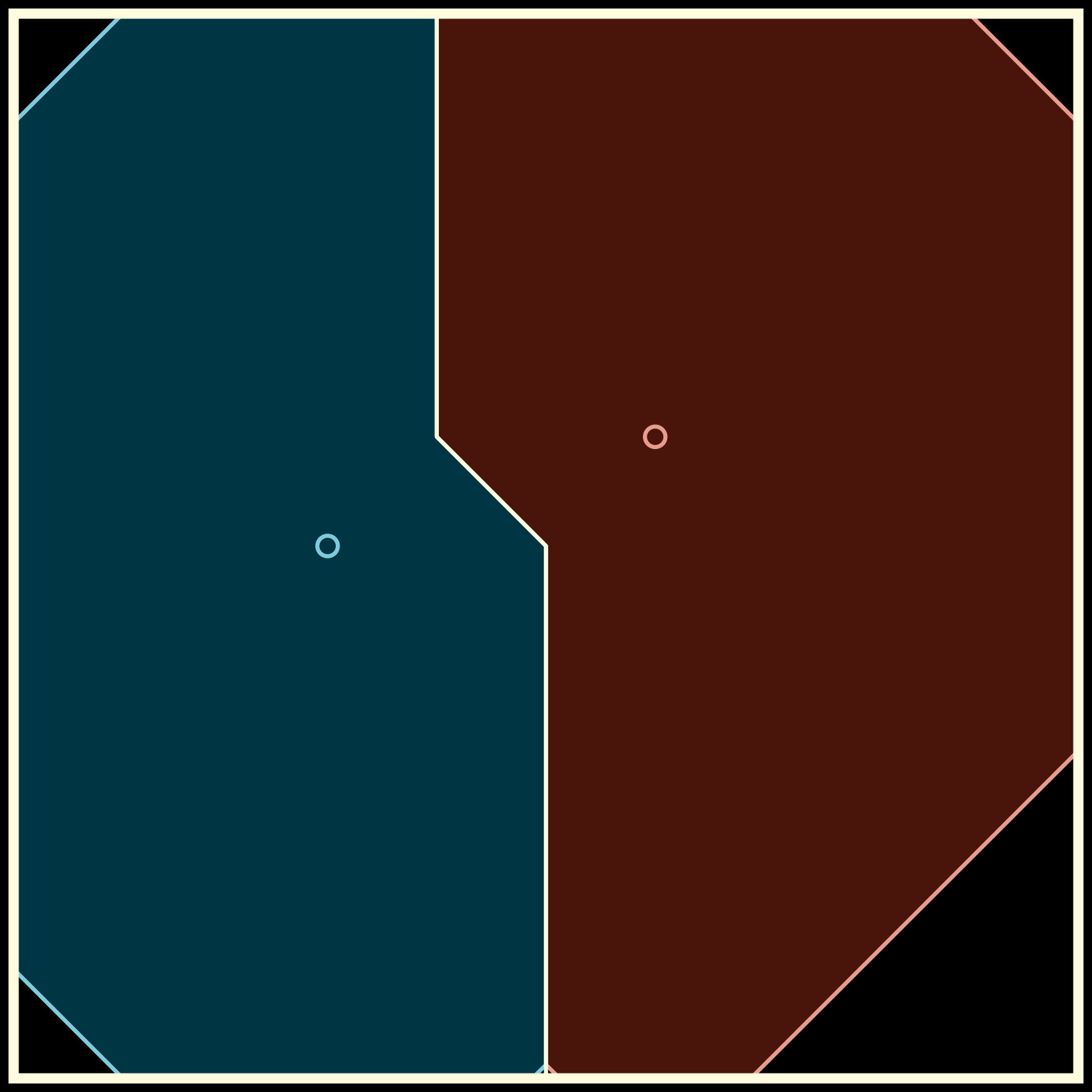


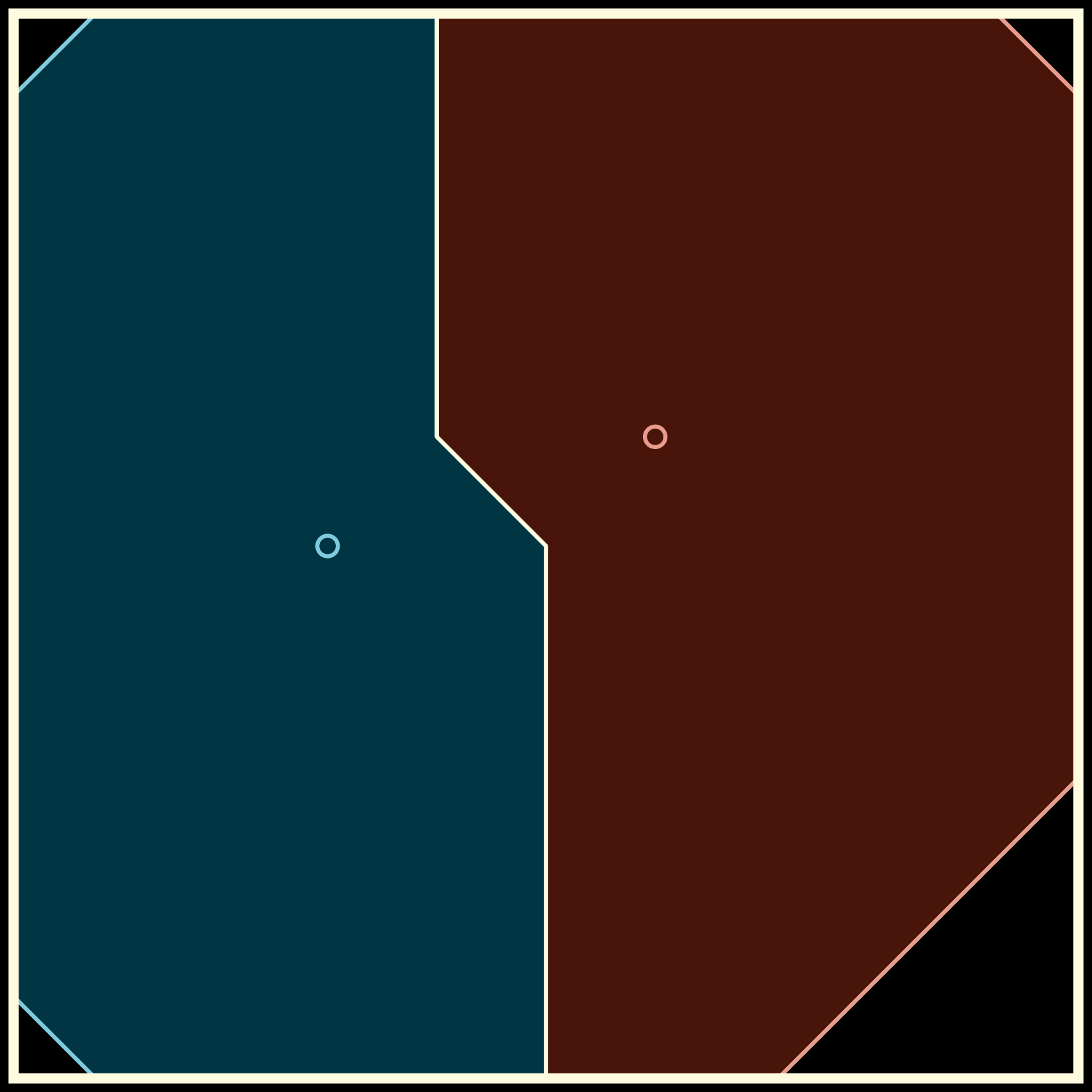


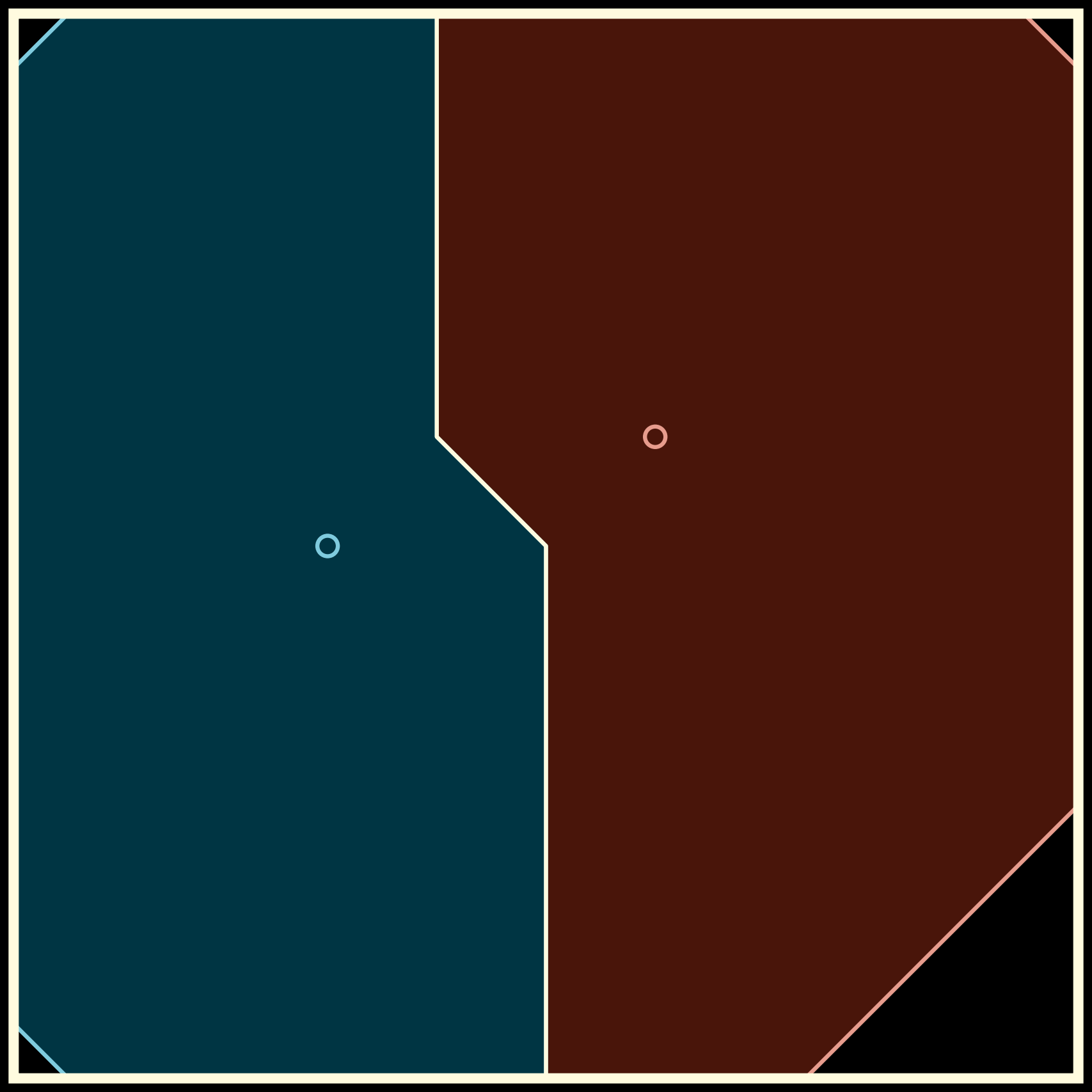


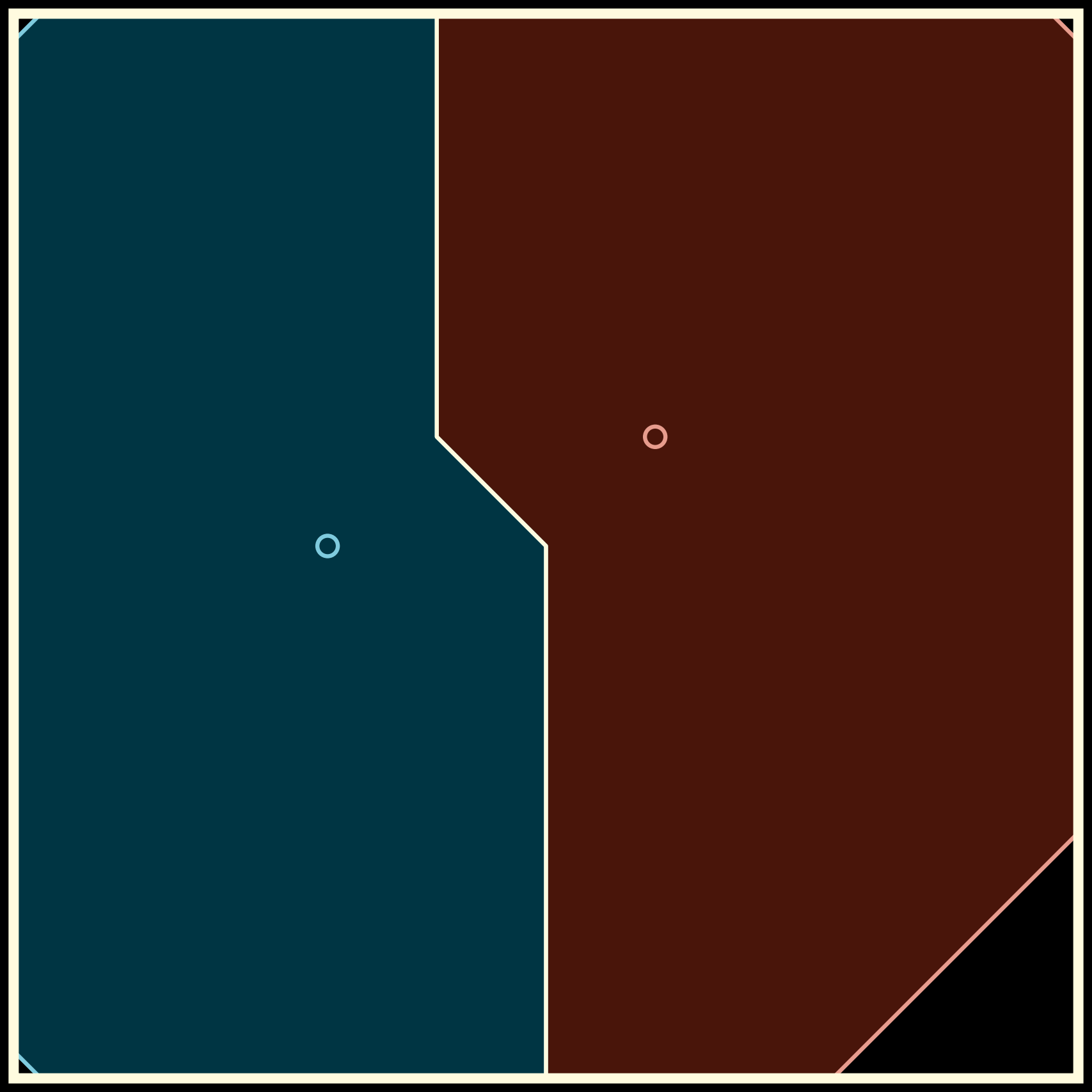


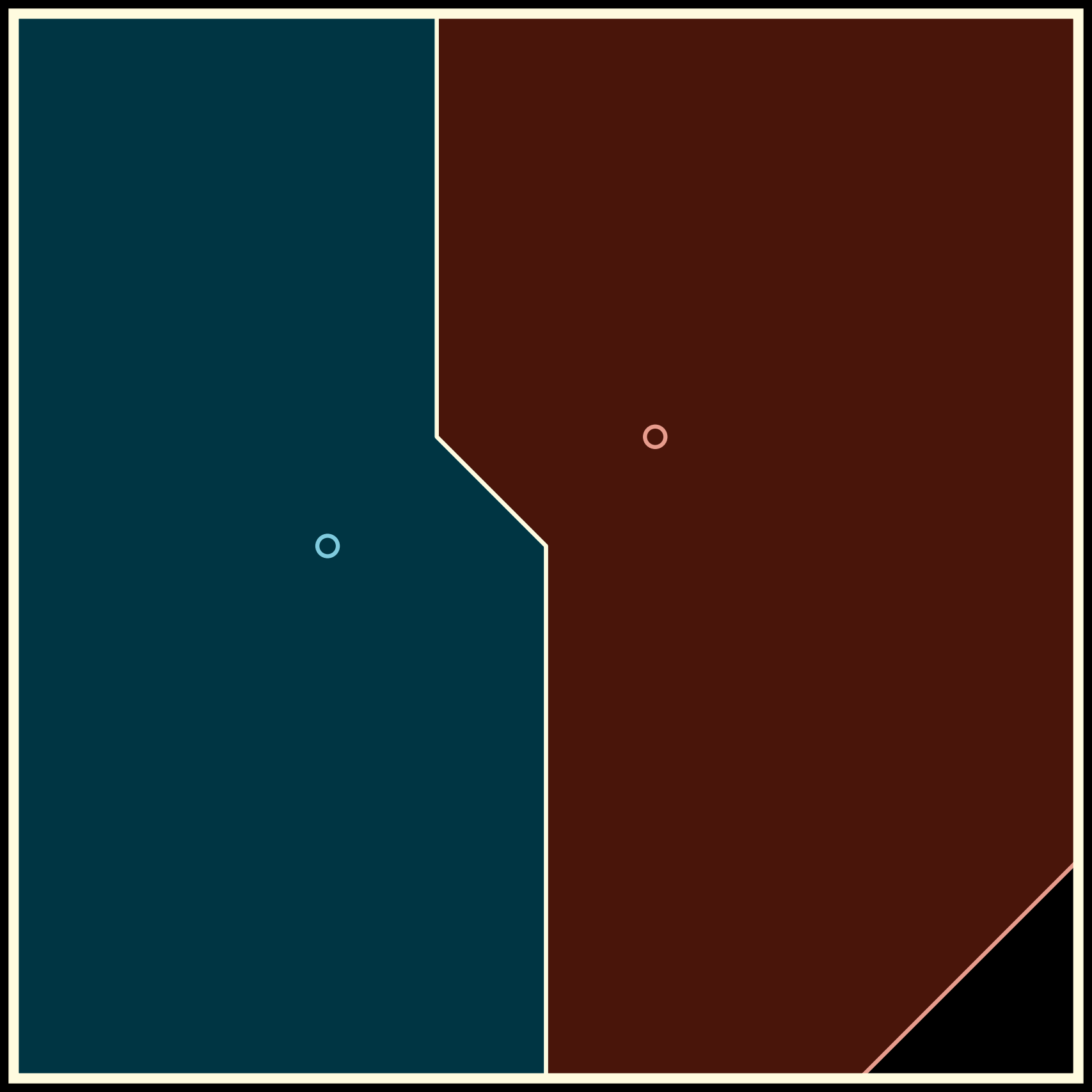


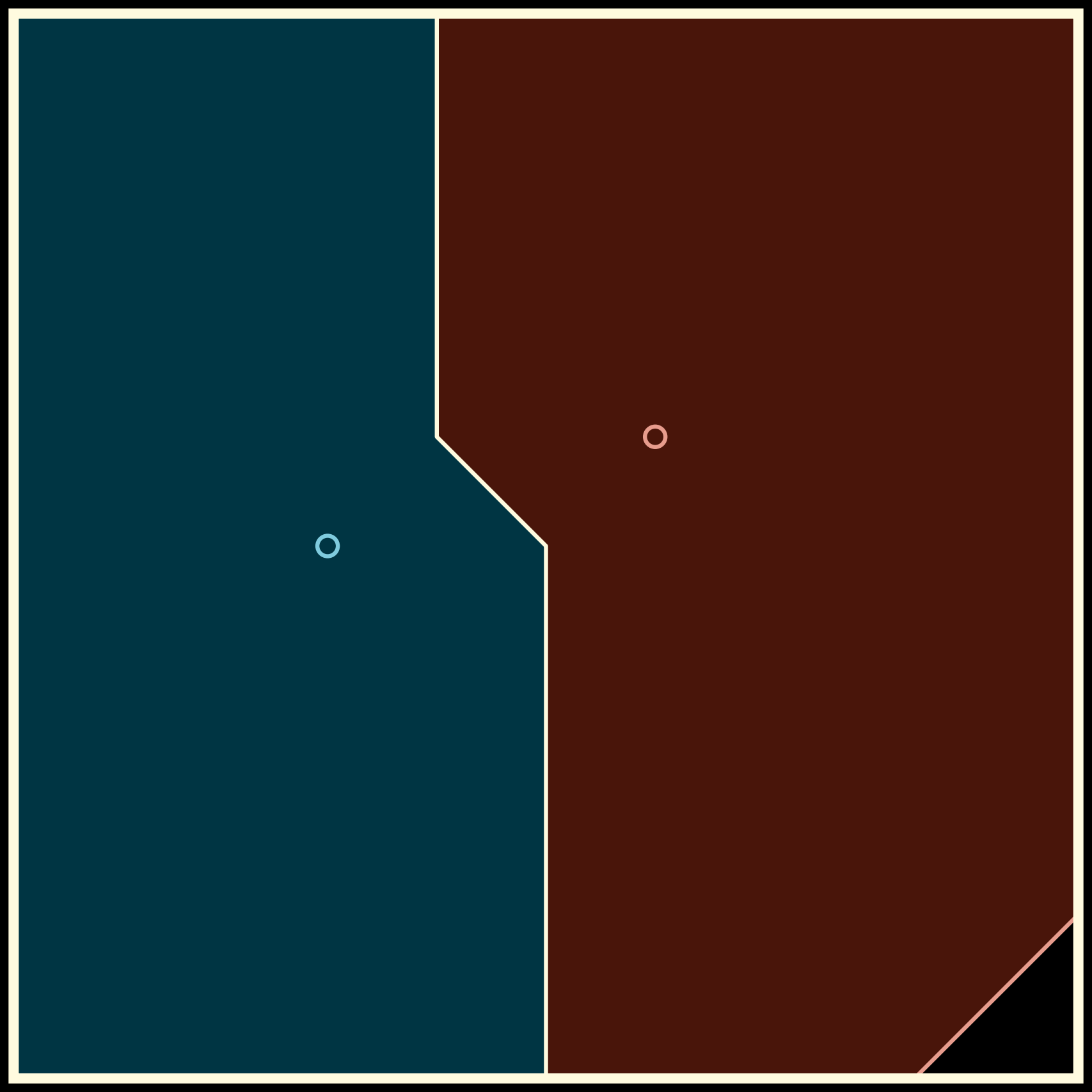


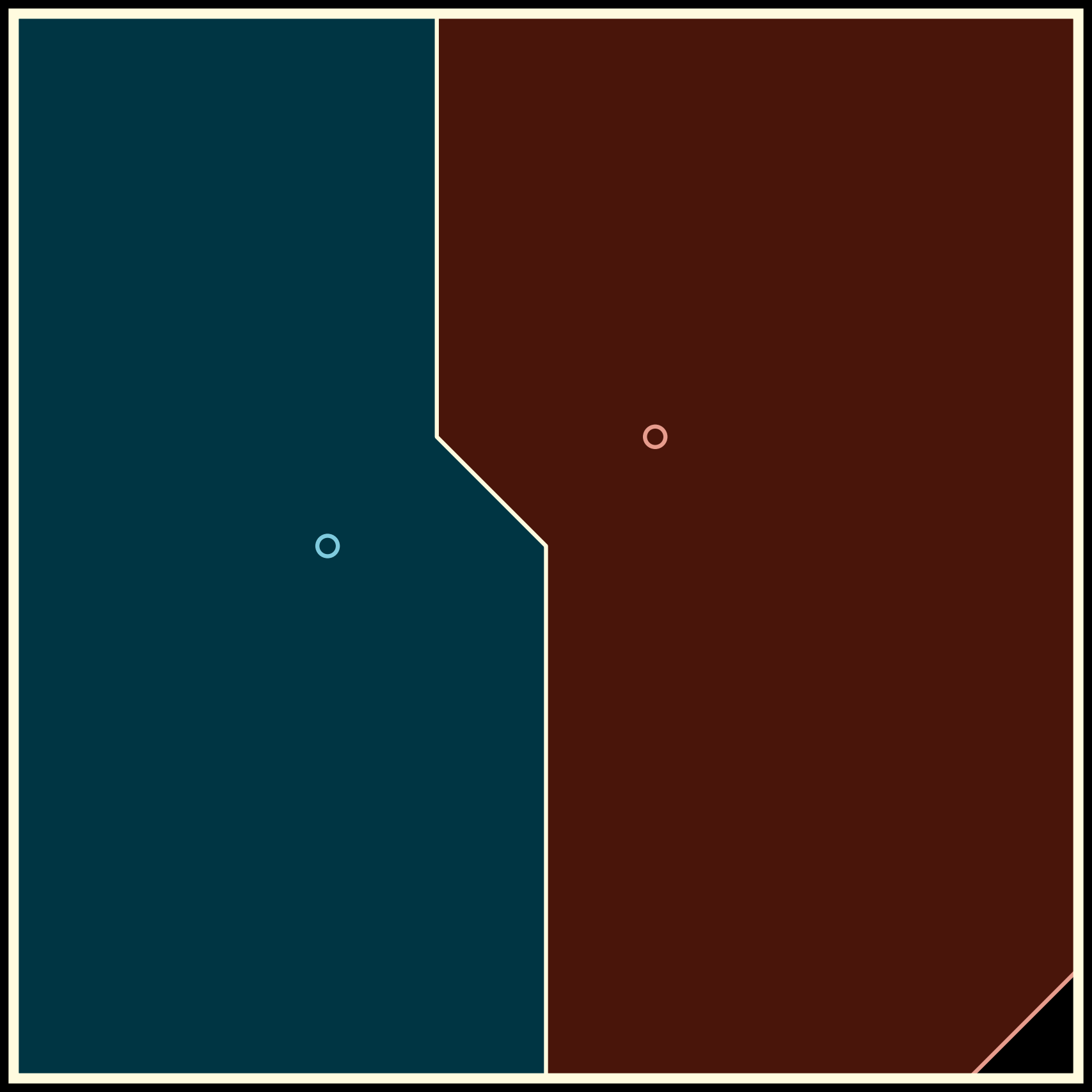


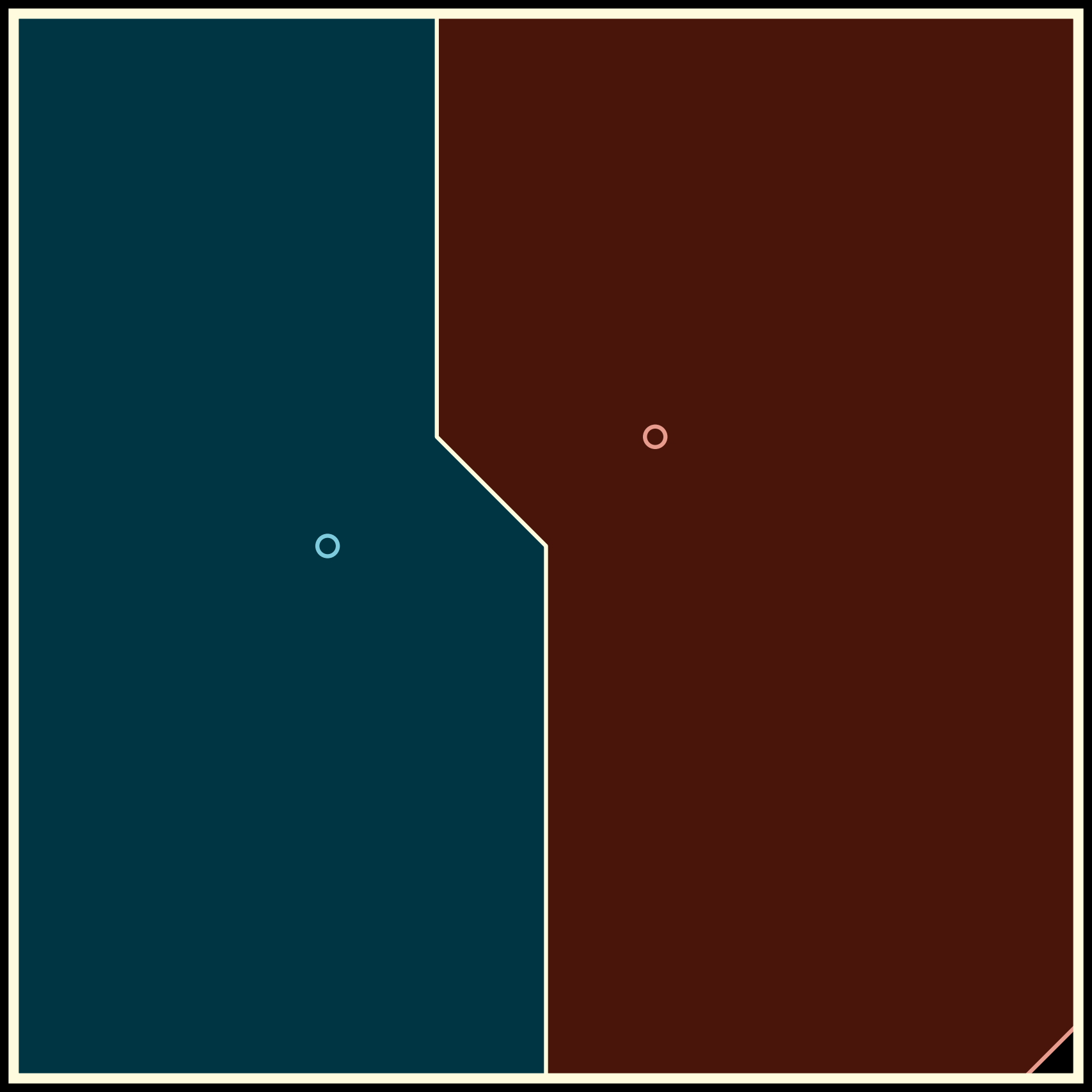


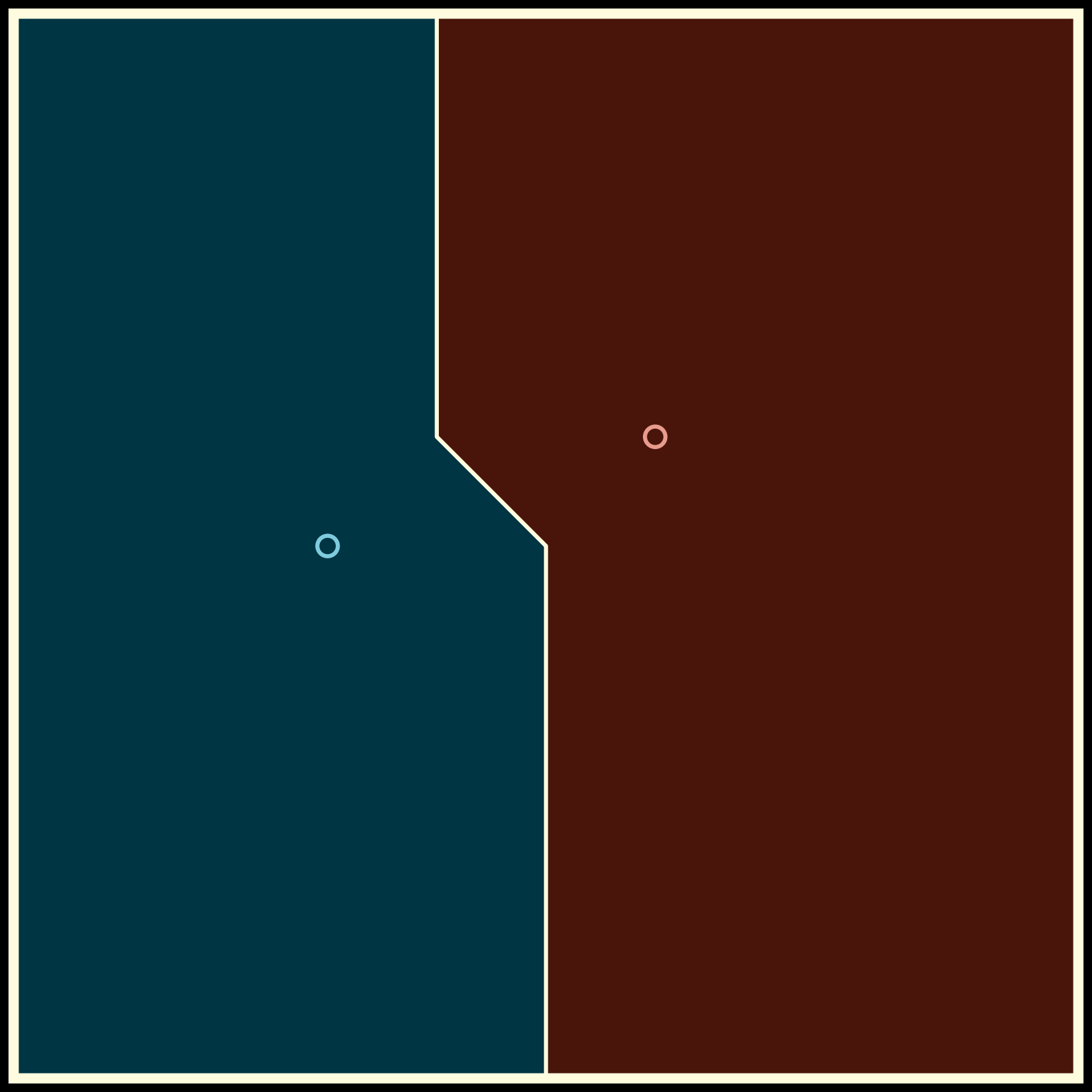


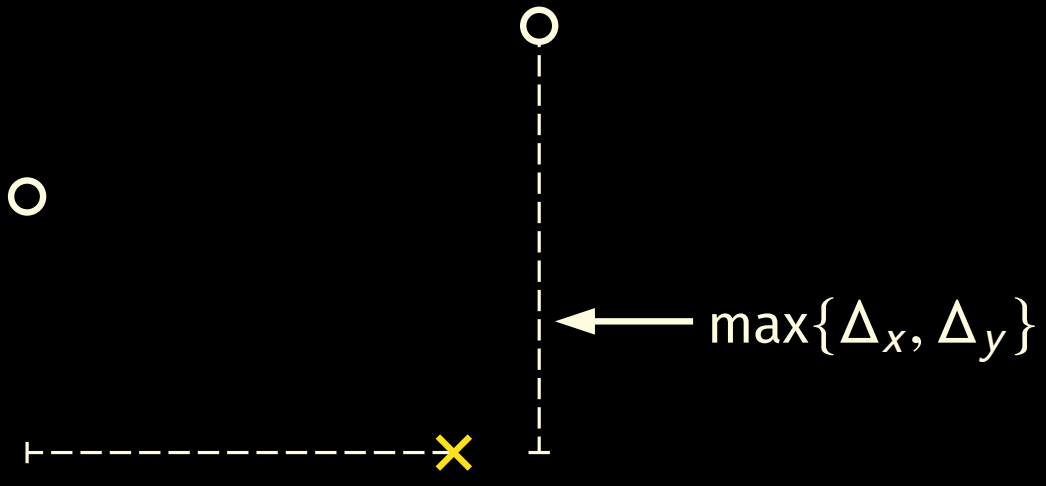


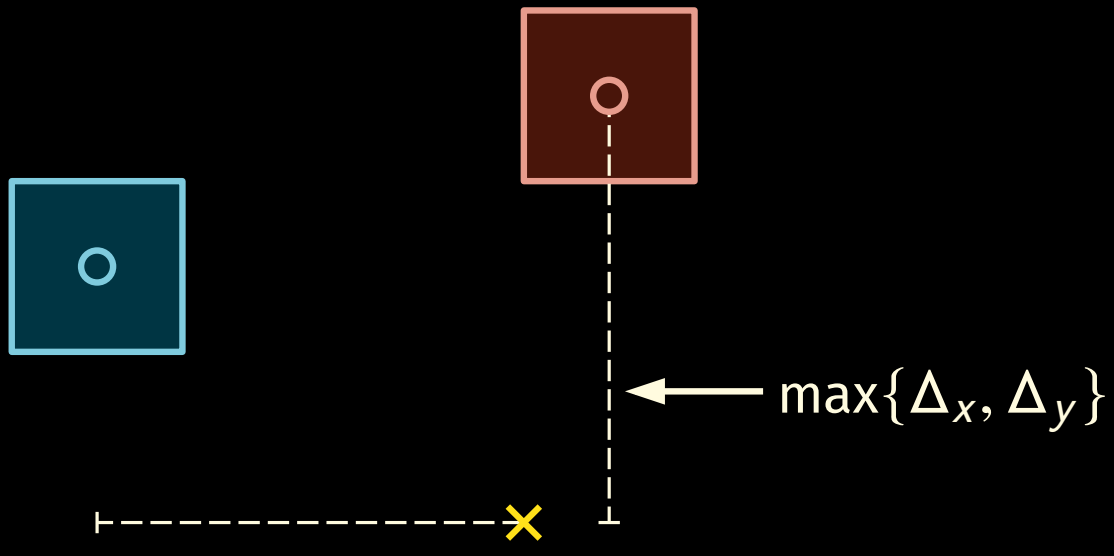


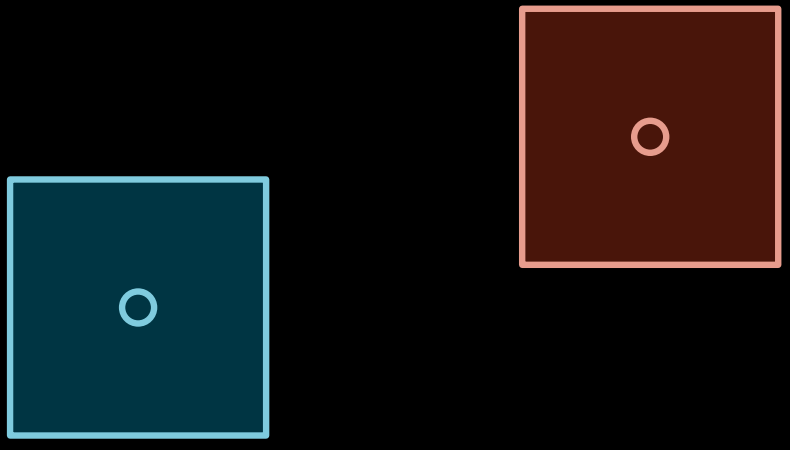


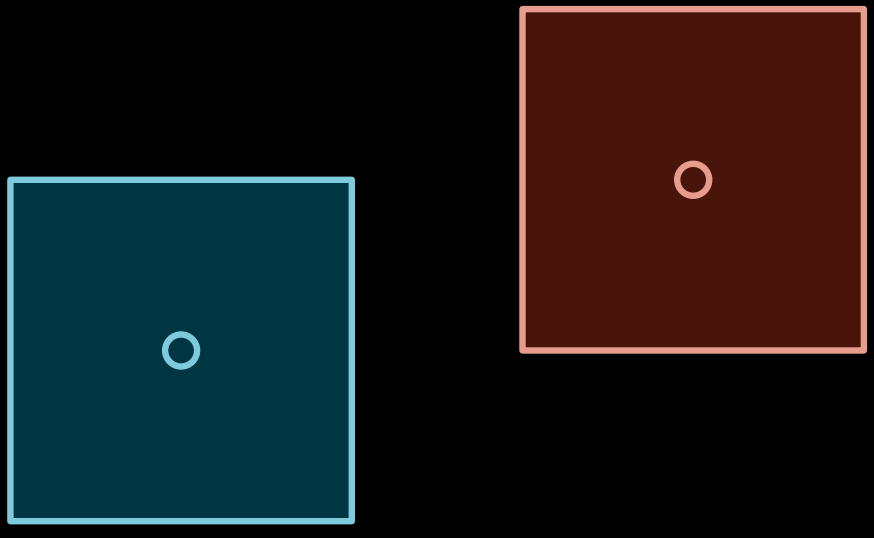


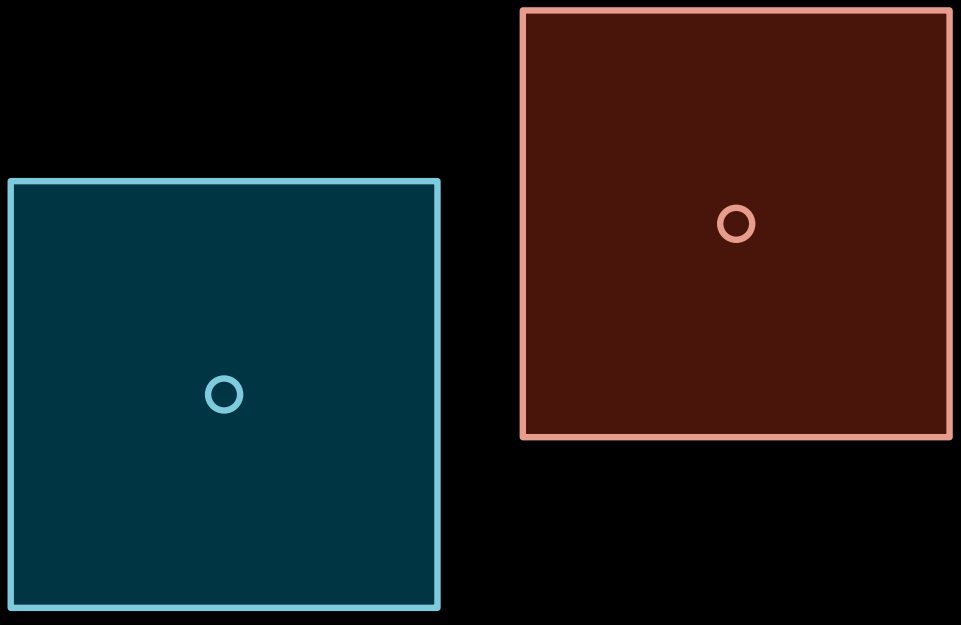


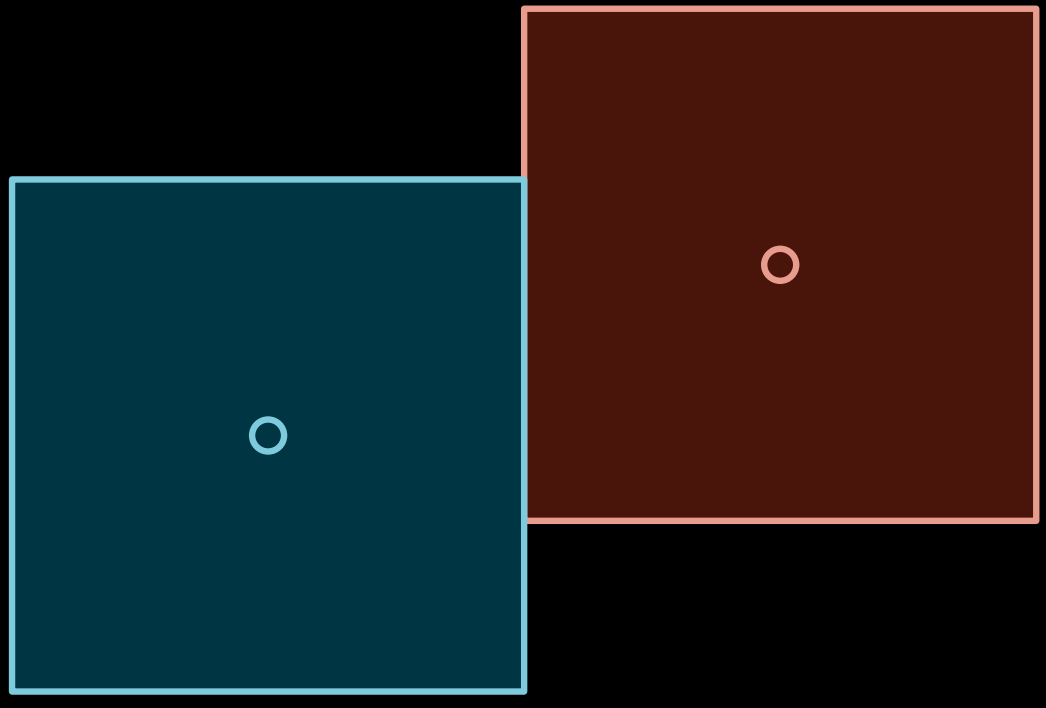

$\max\{\Delta_x, \Delta_y\}$

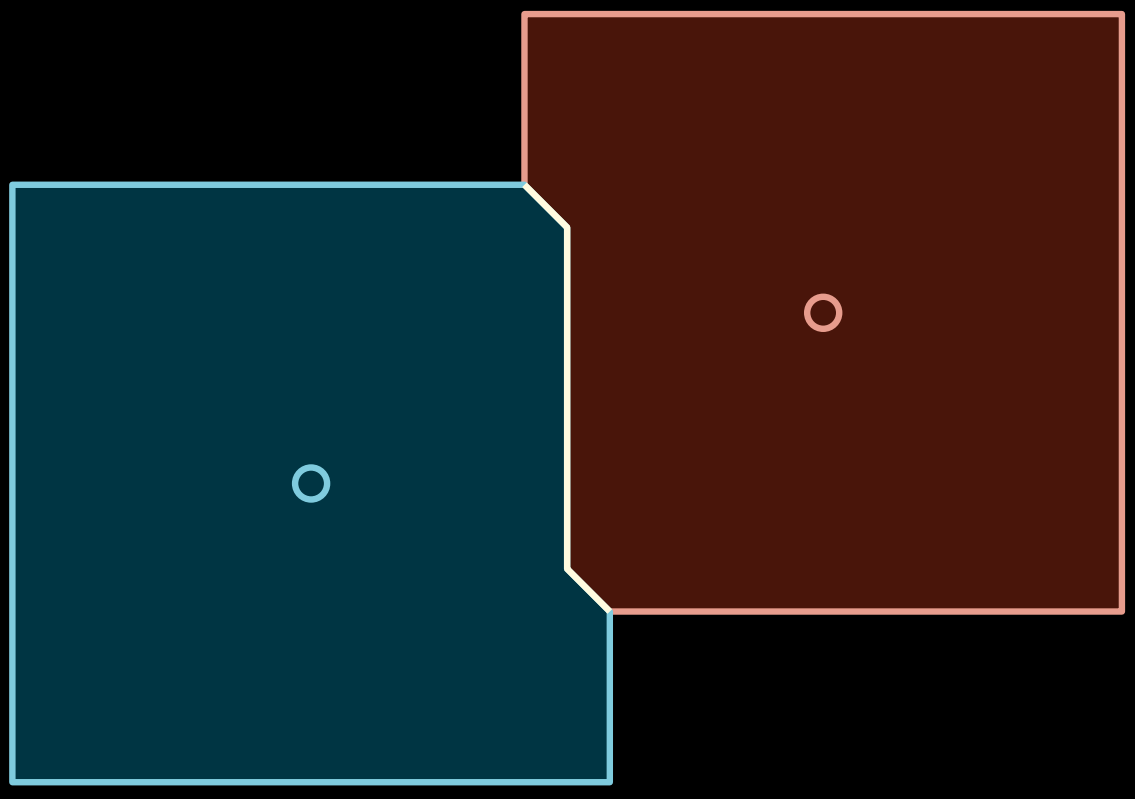

$\max\{\Delta_x, \Delta_y\}$

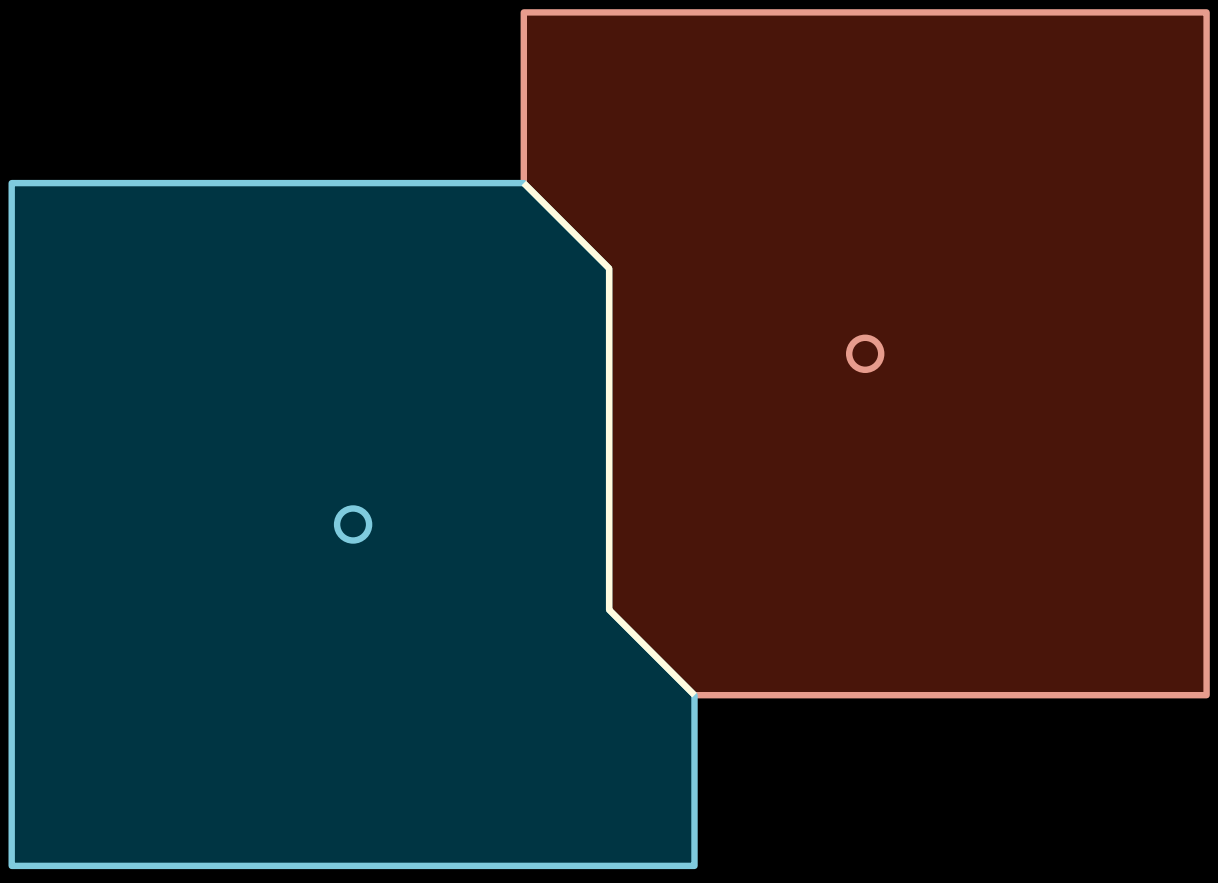


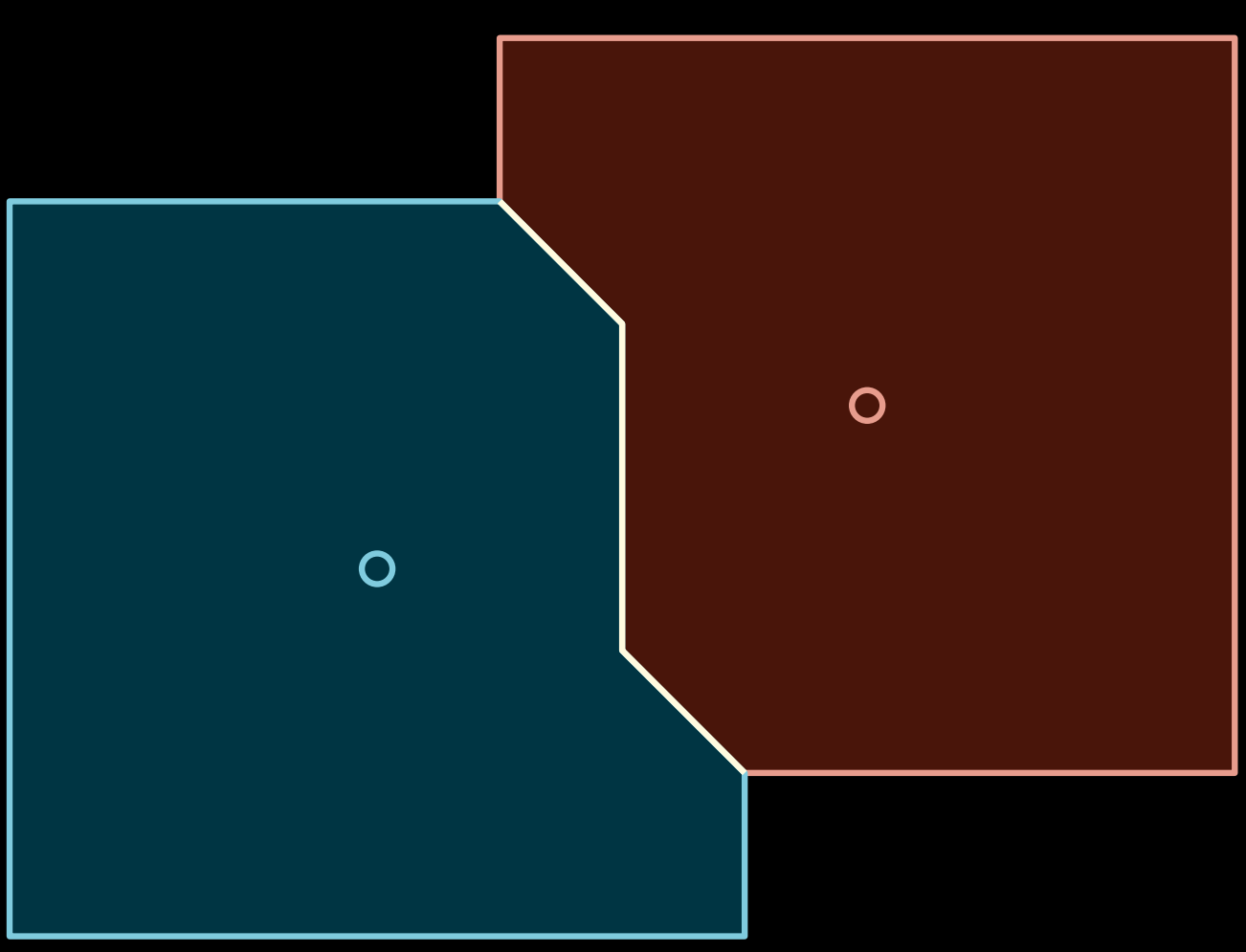


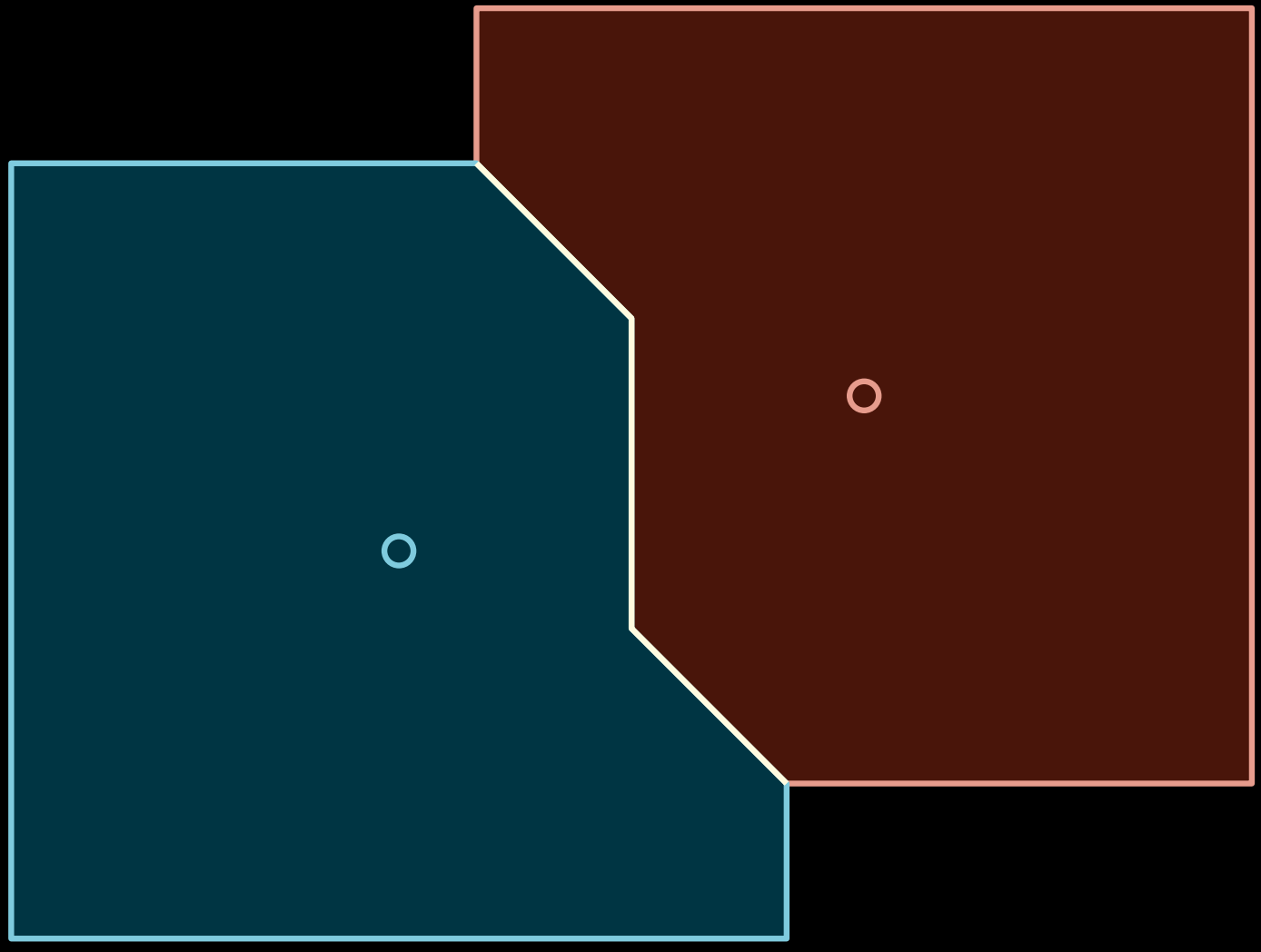


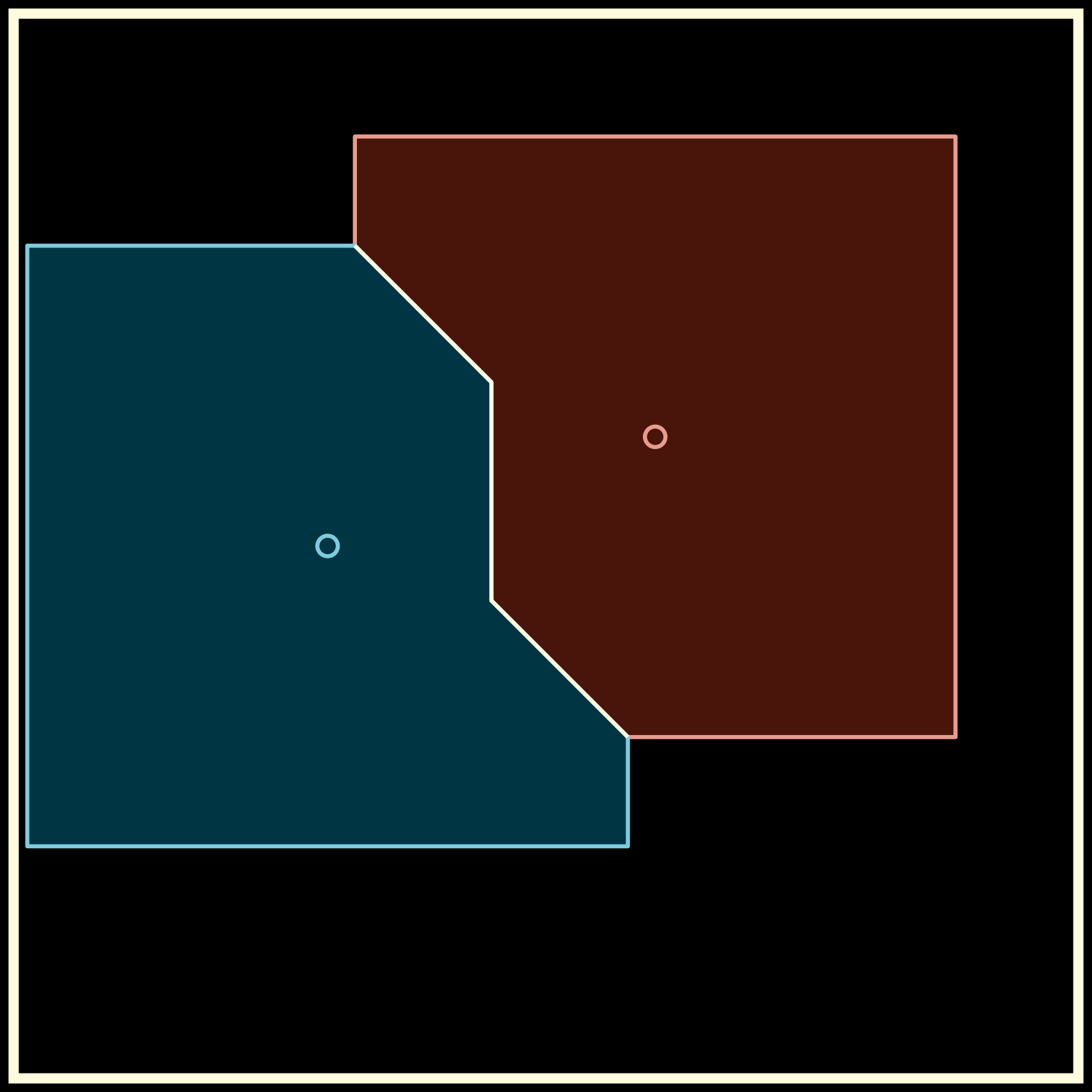


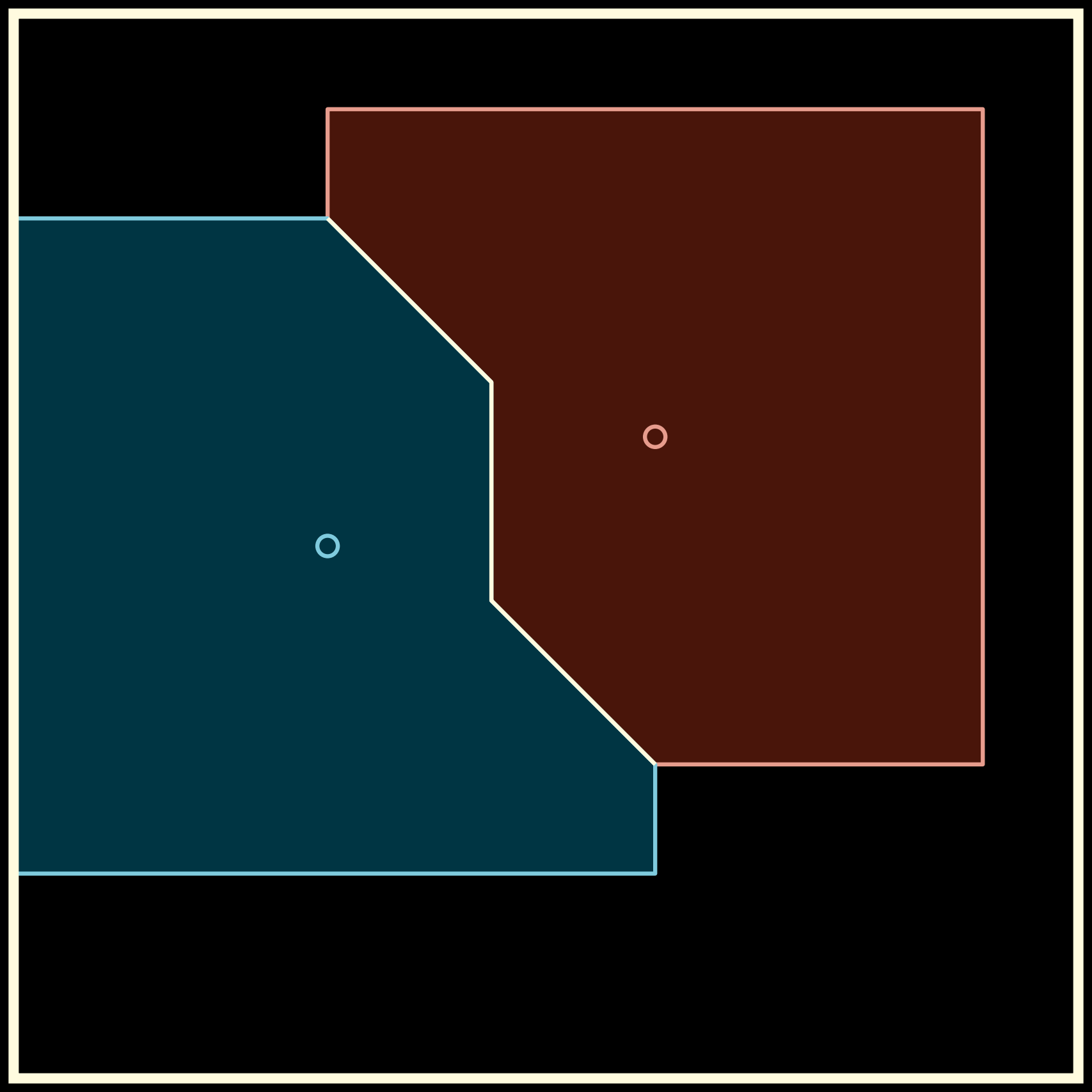


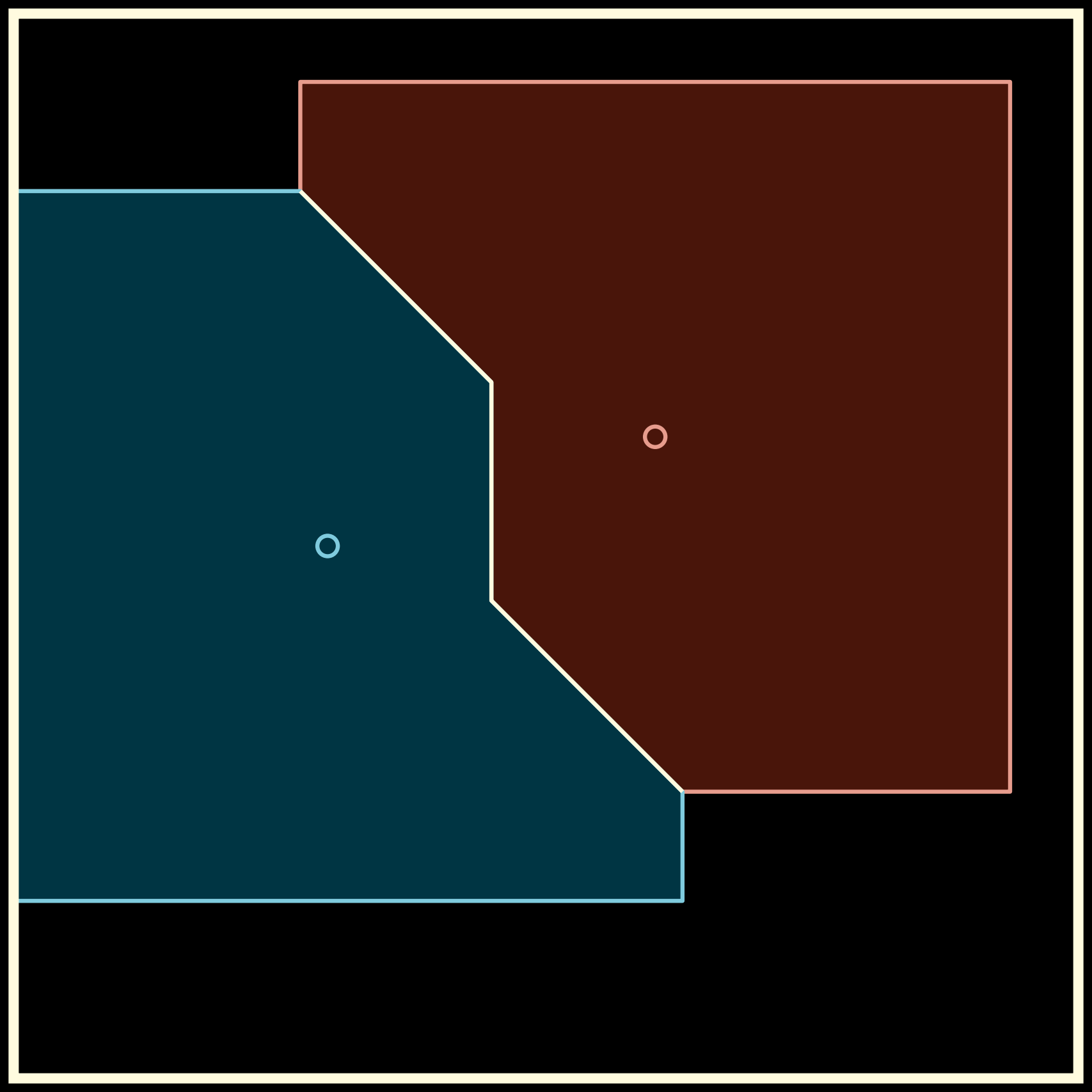


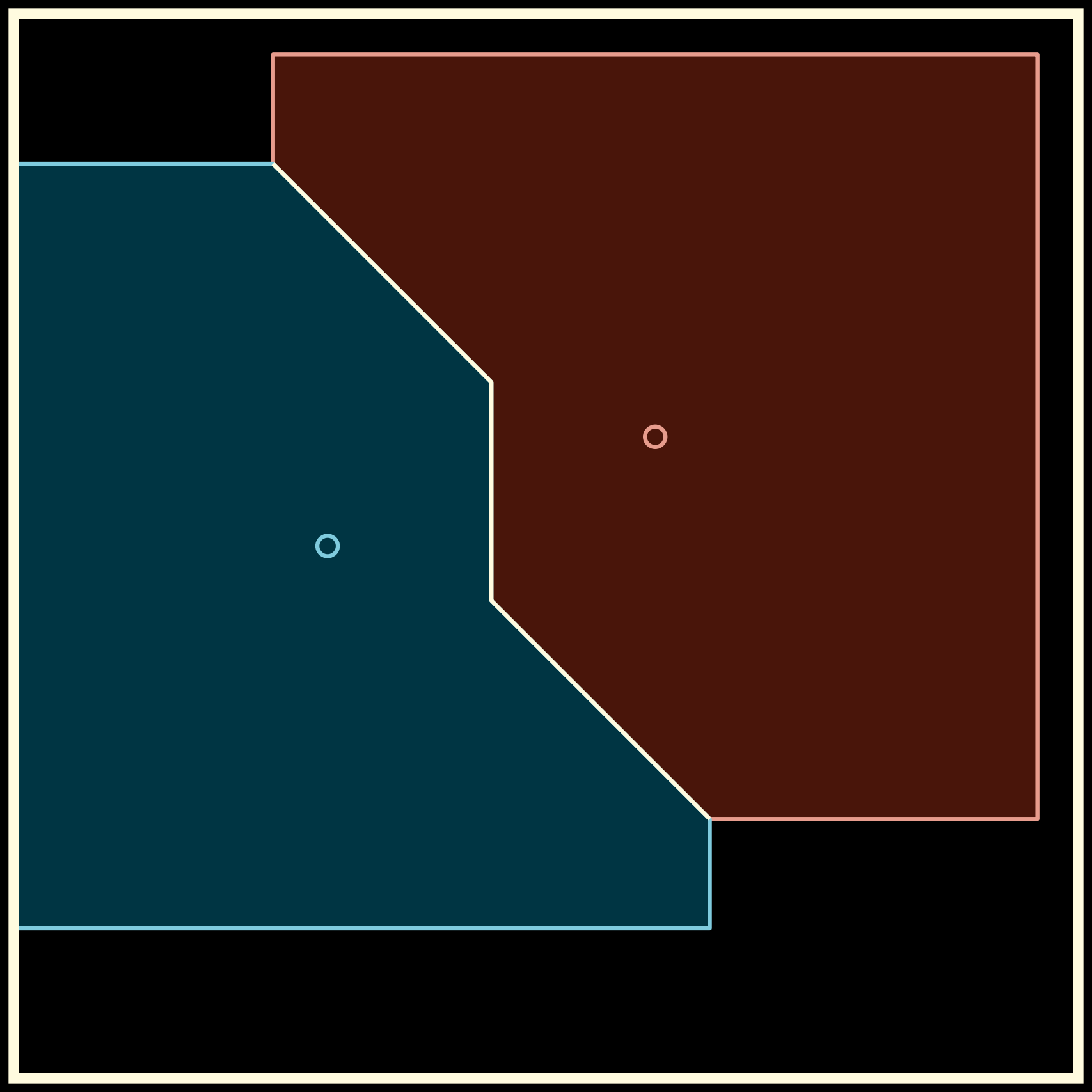


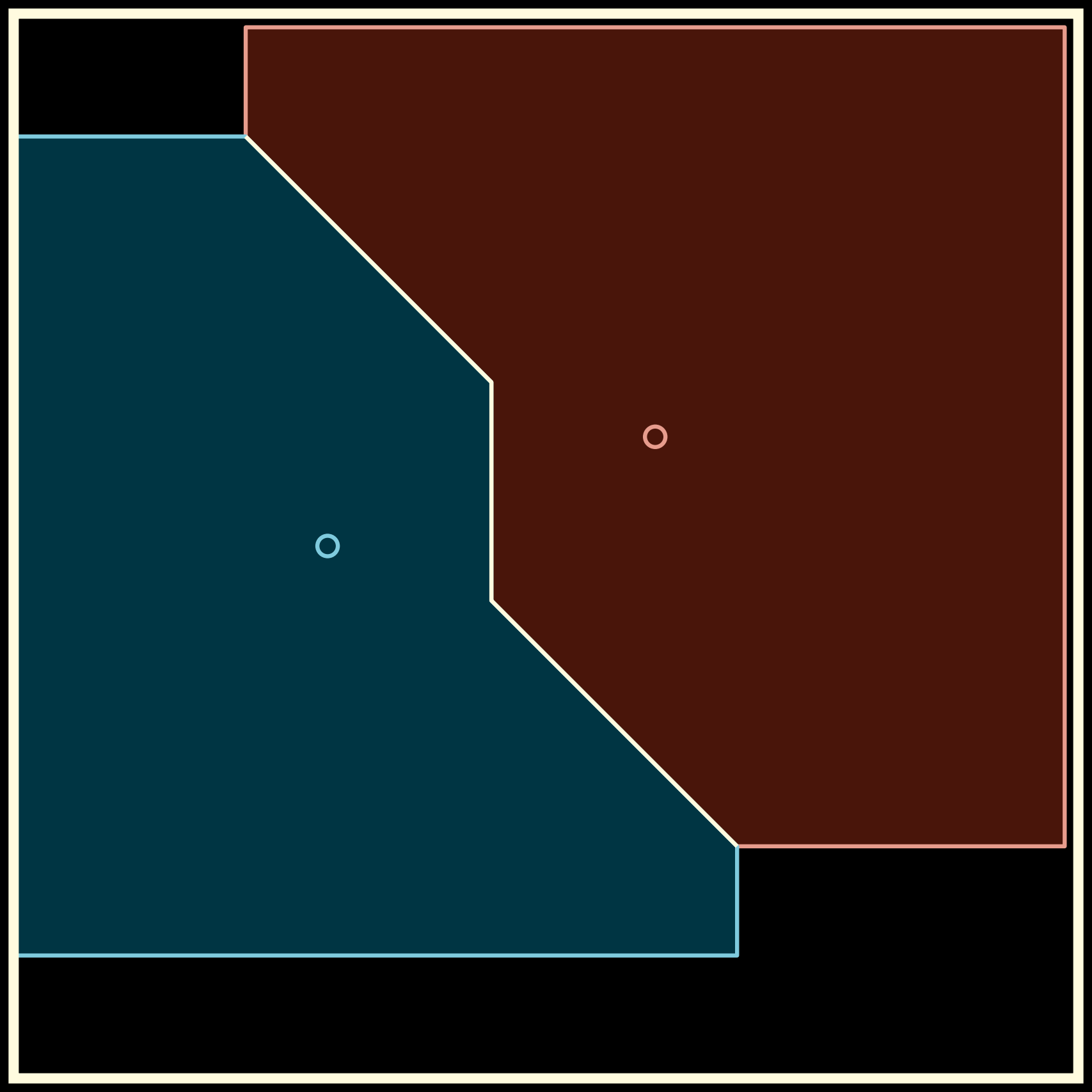


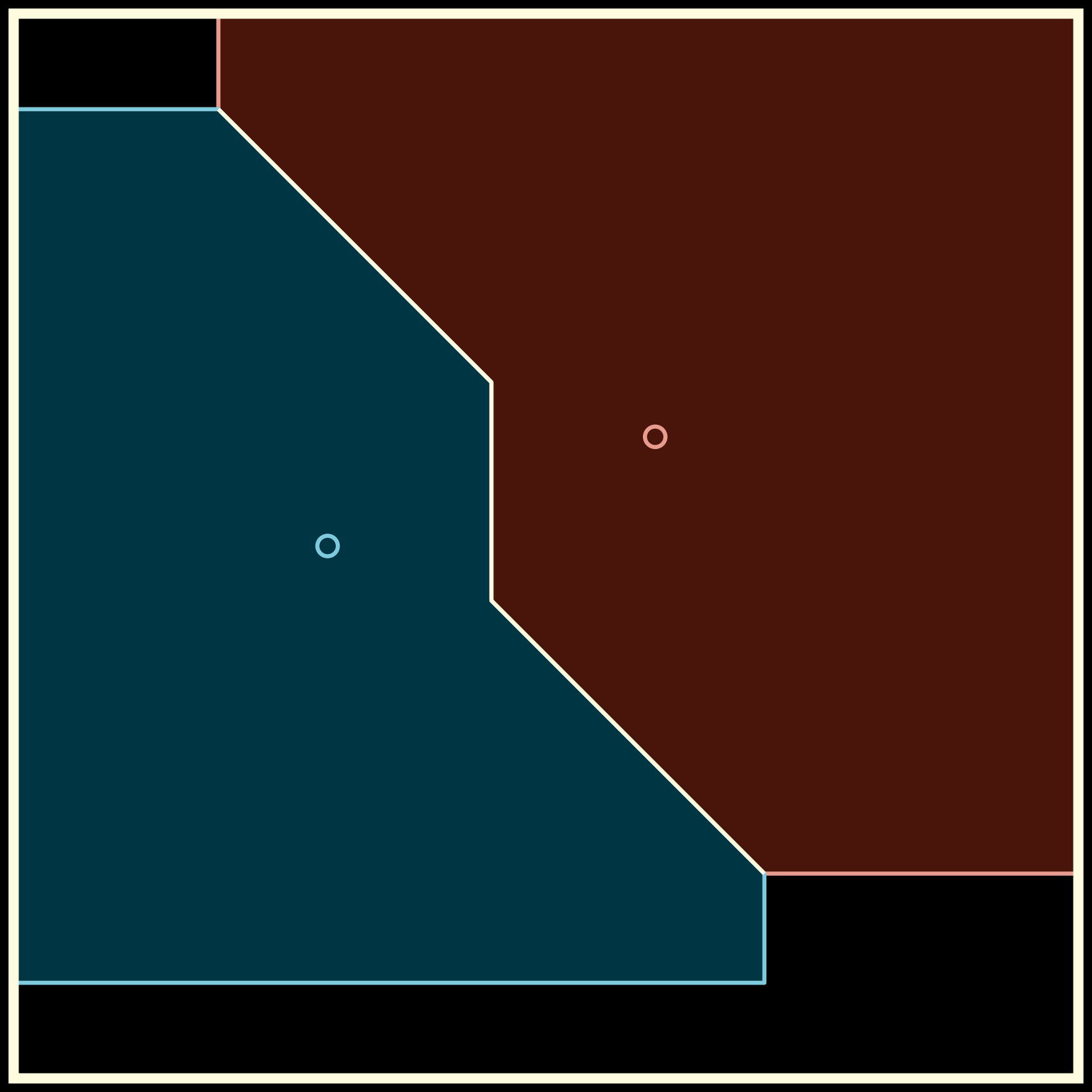


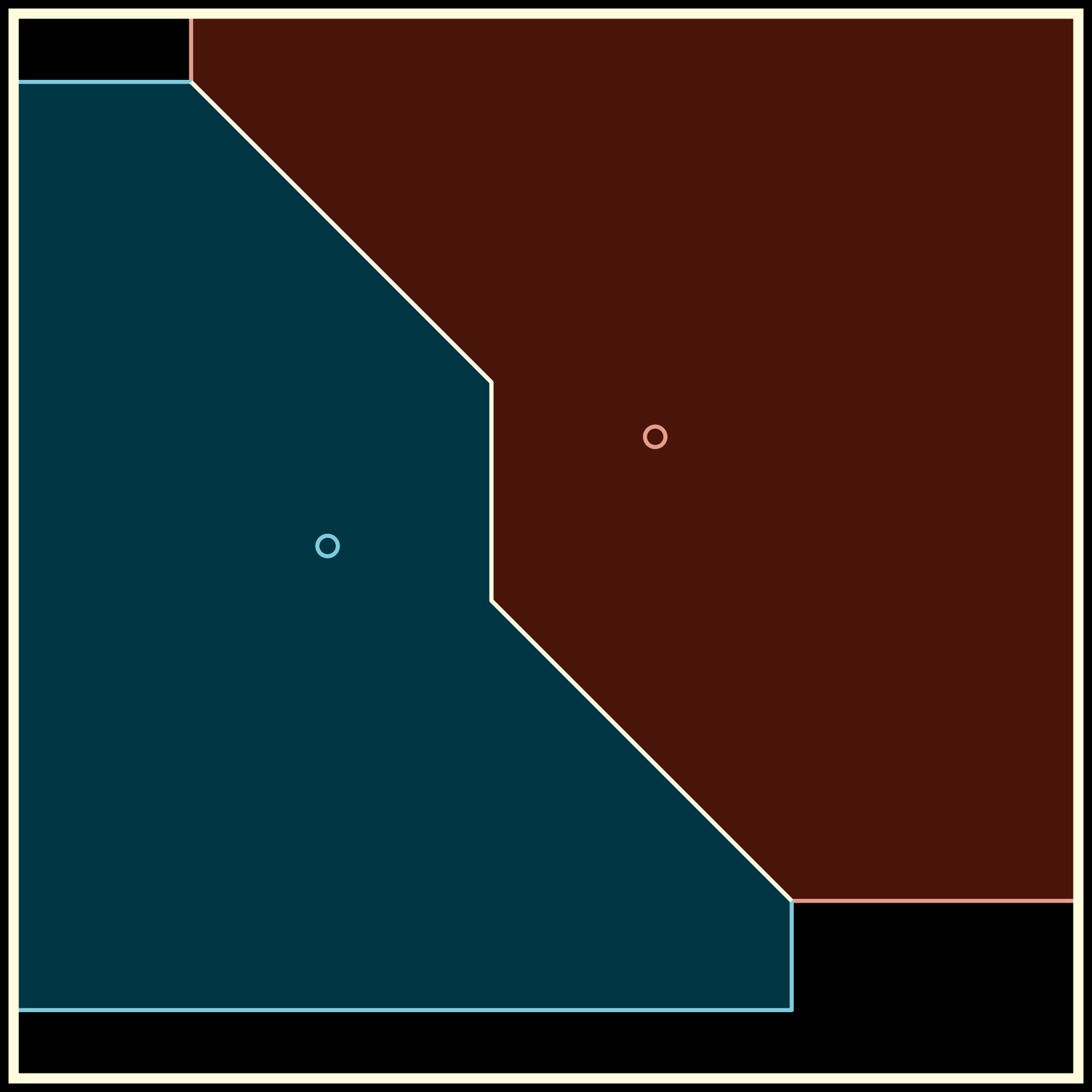


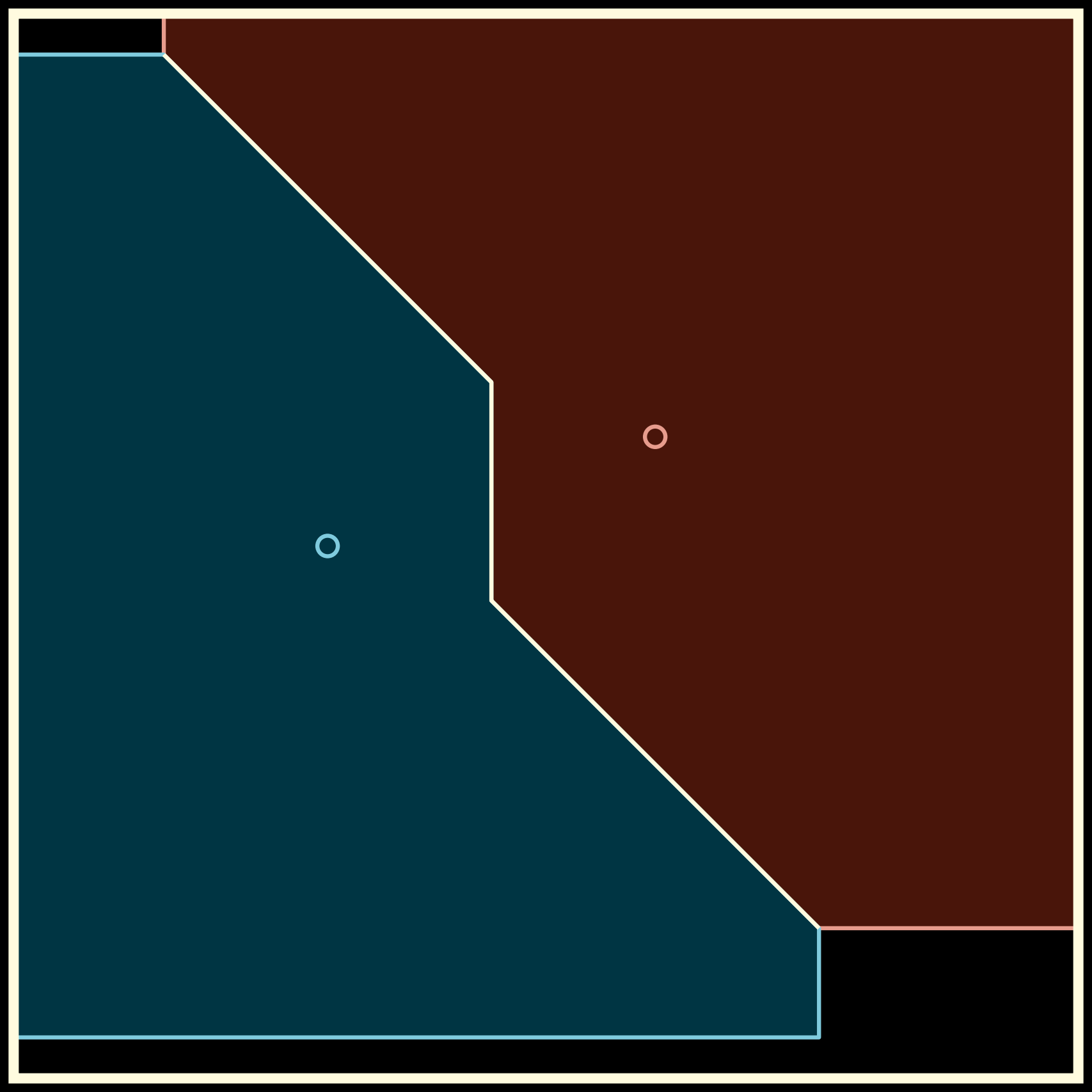


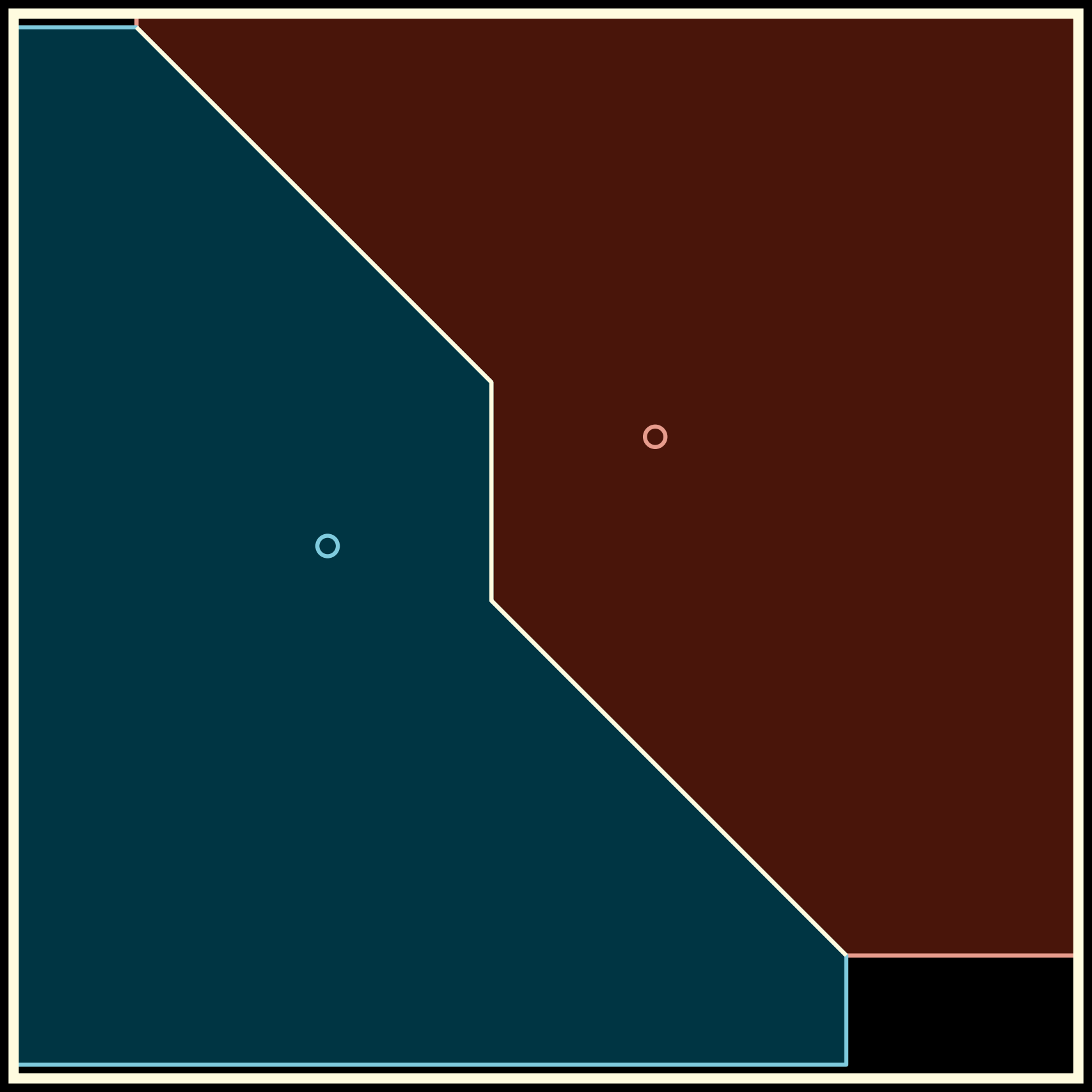


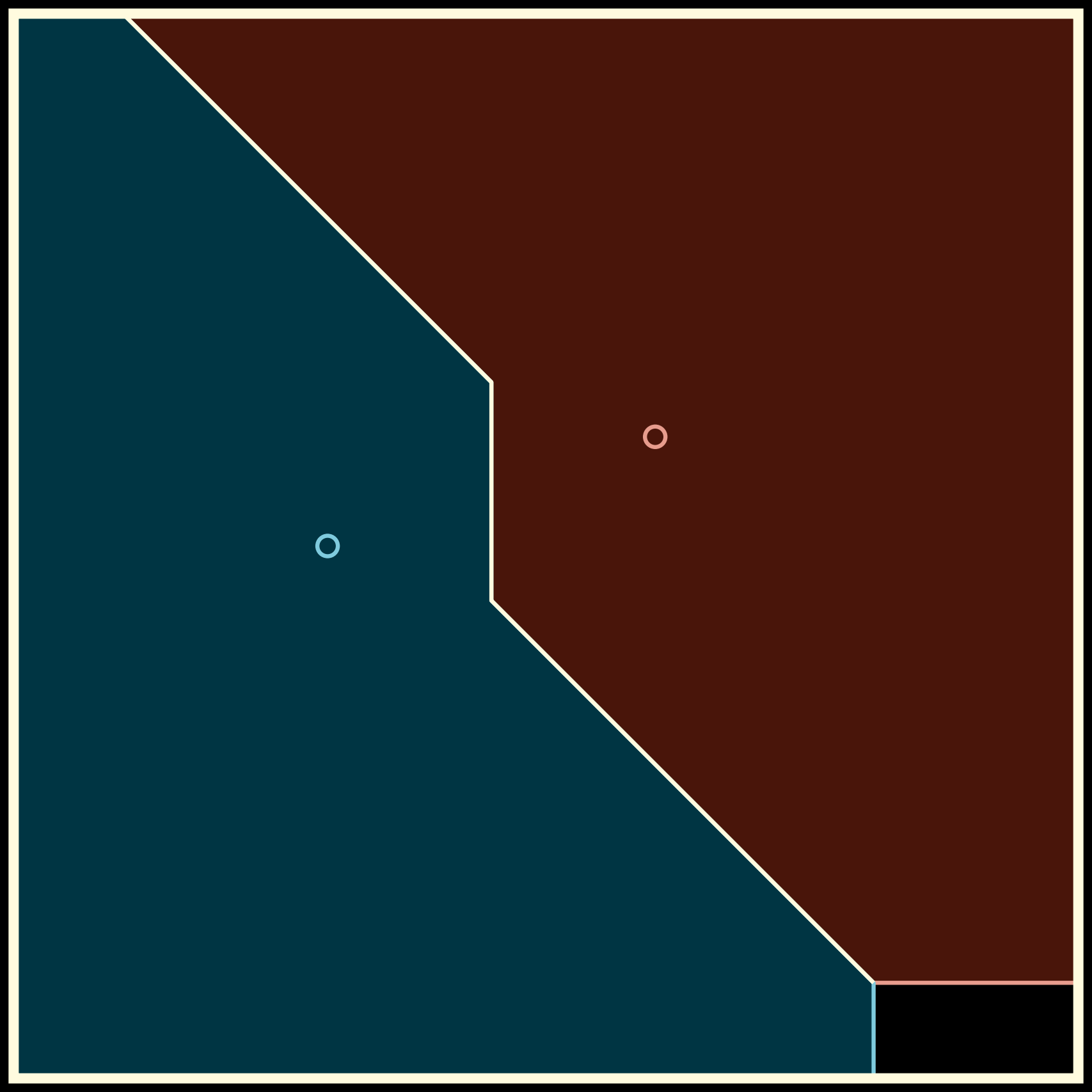


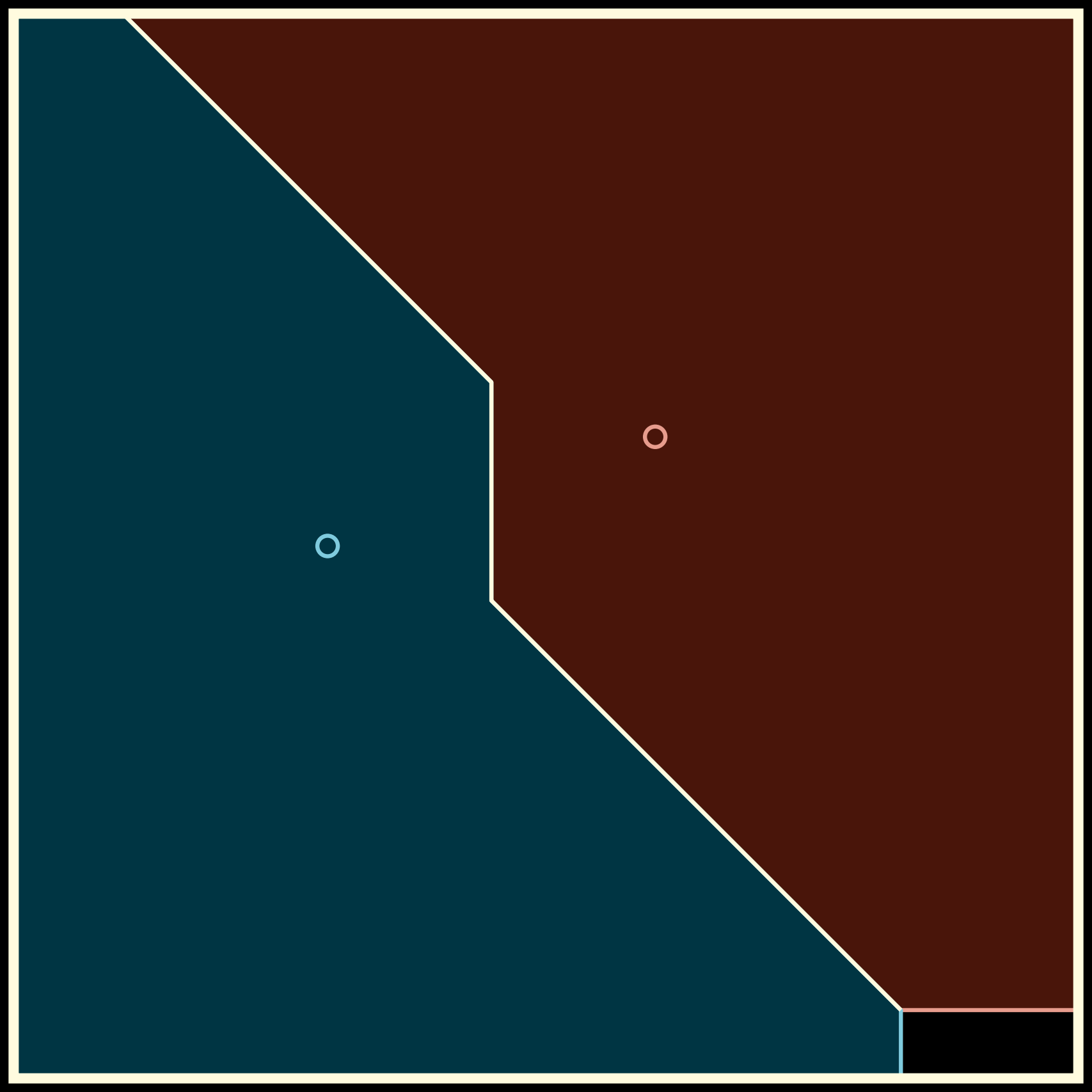


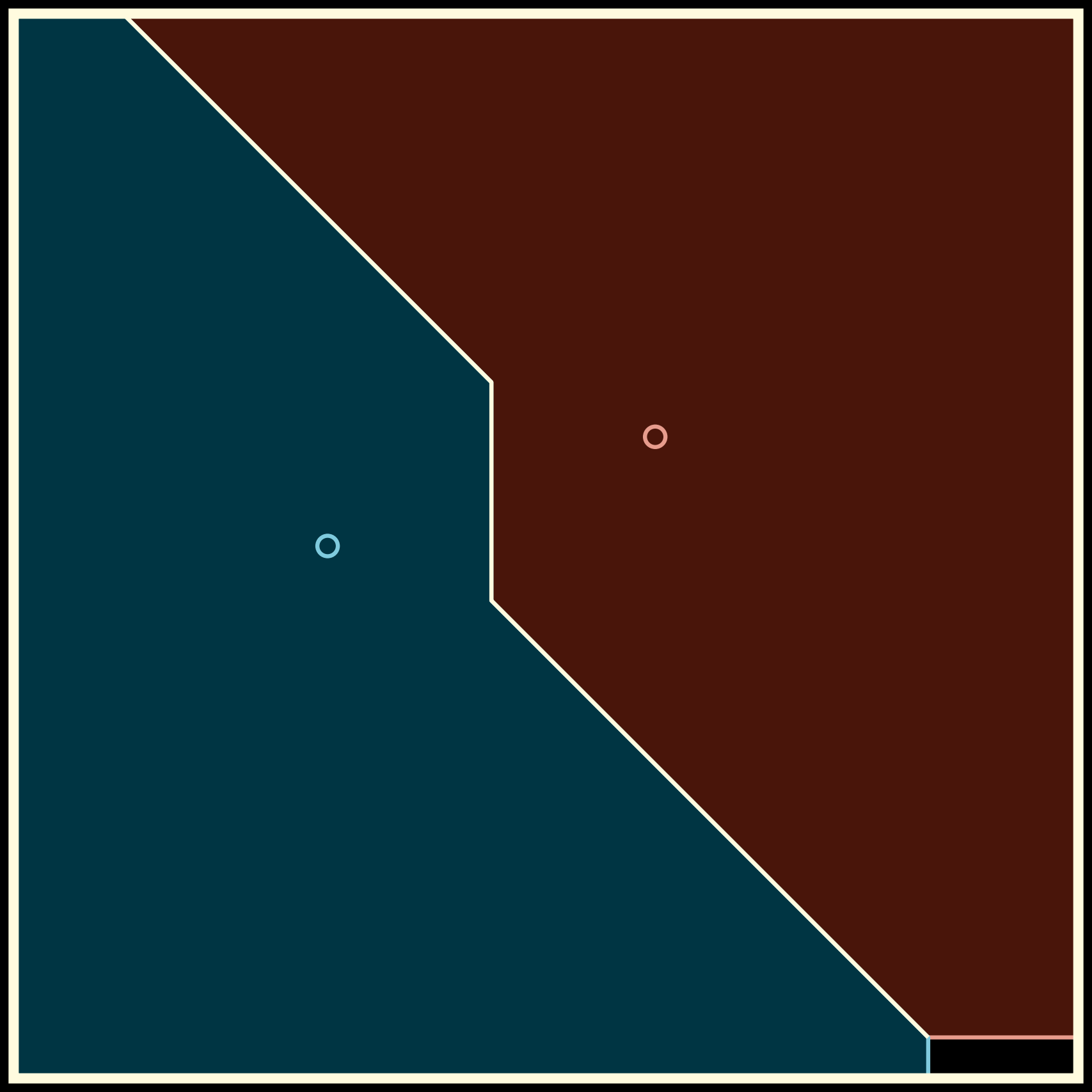


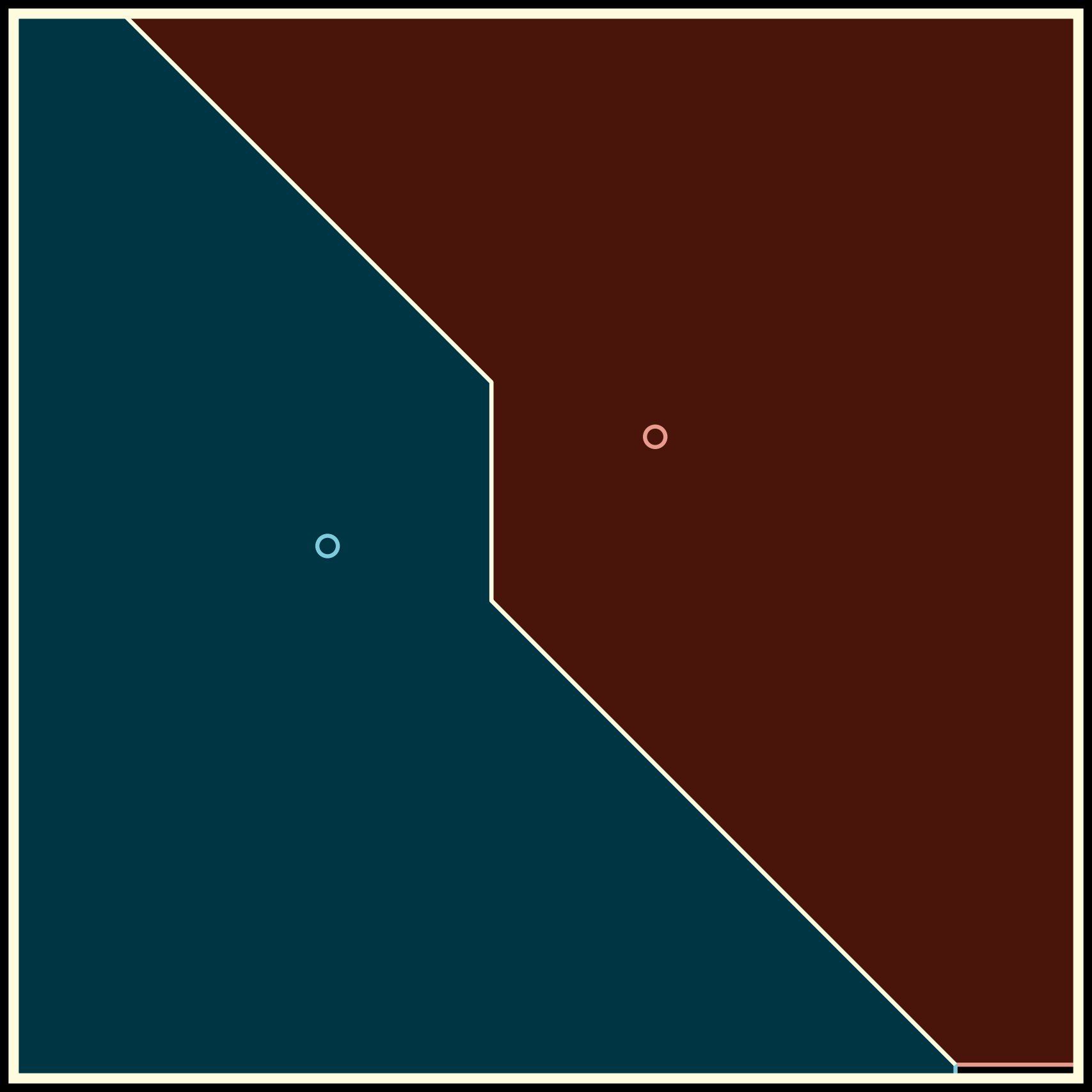


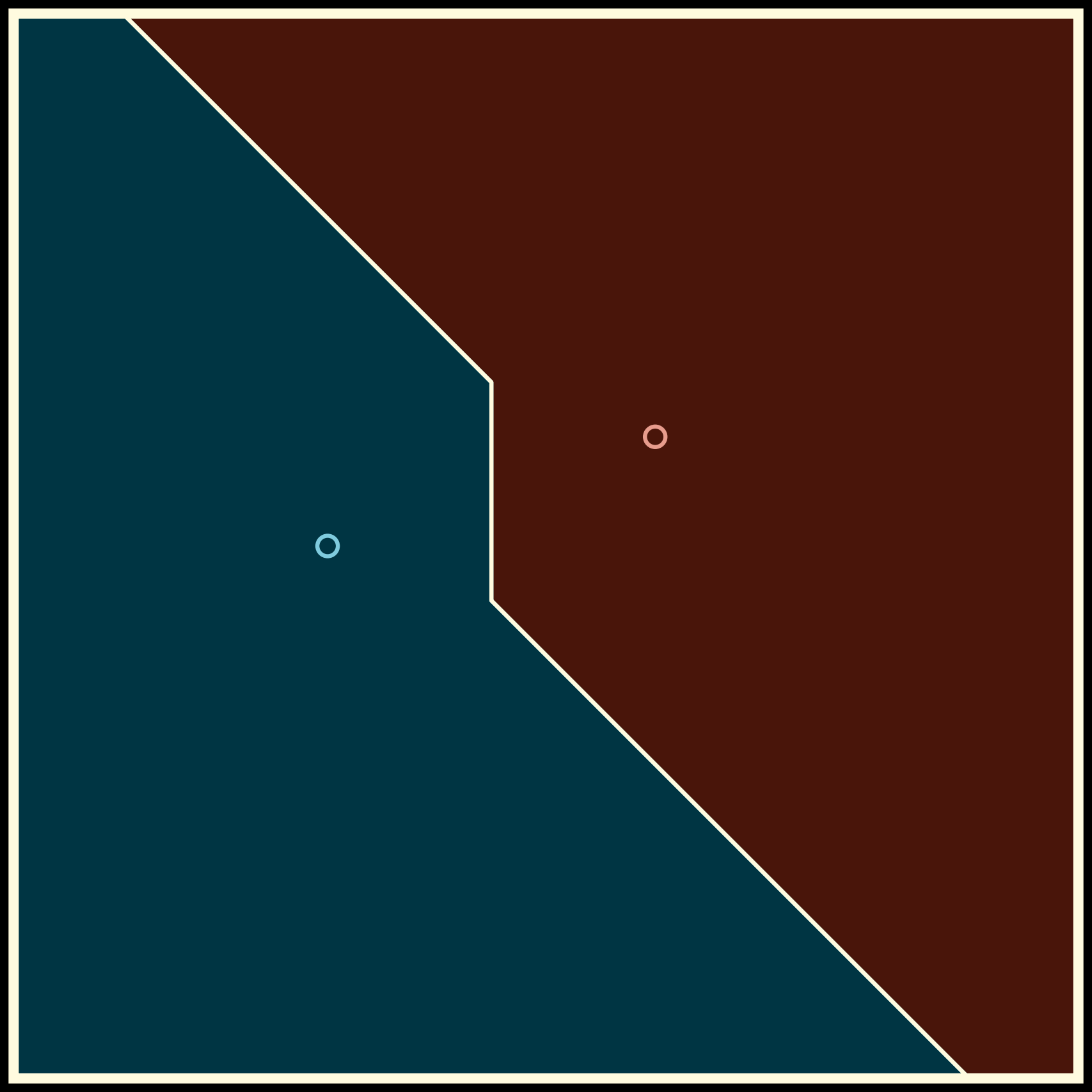


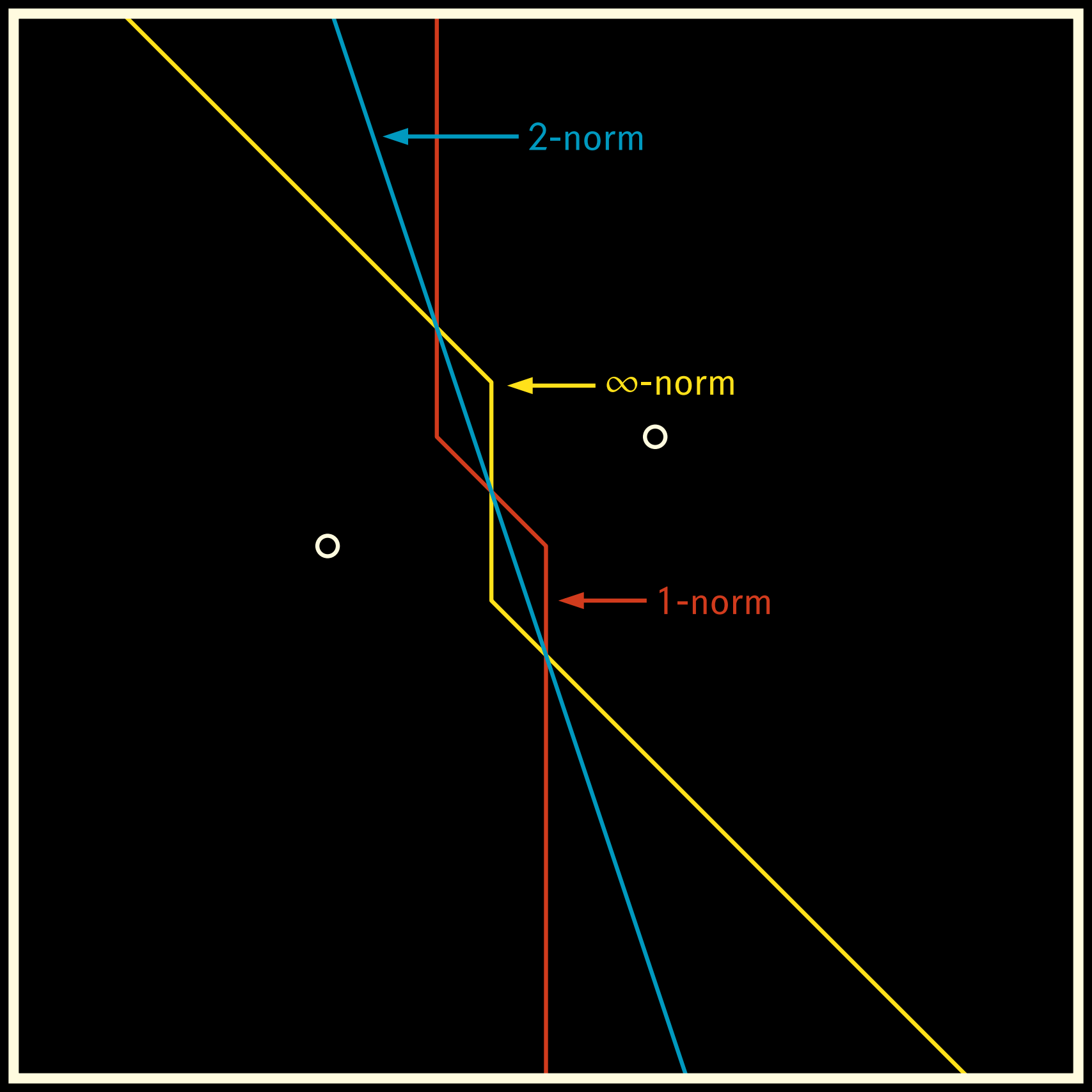


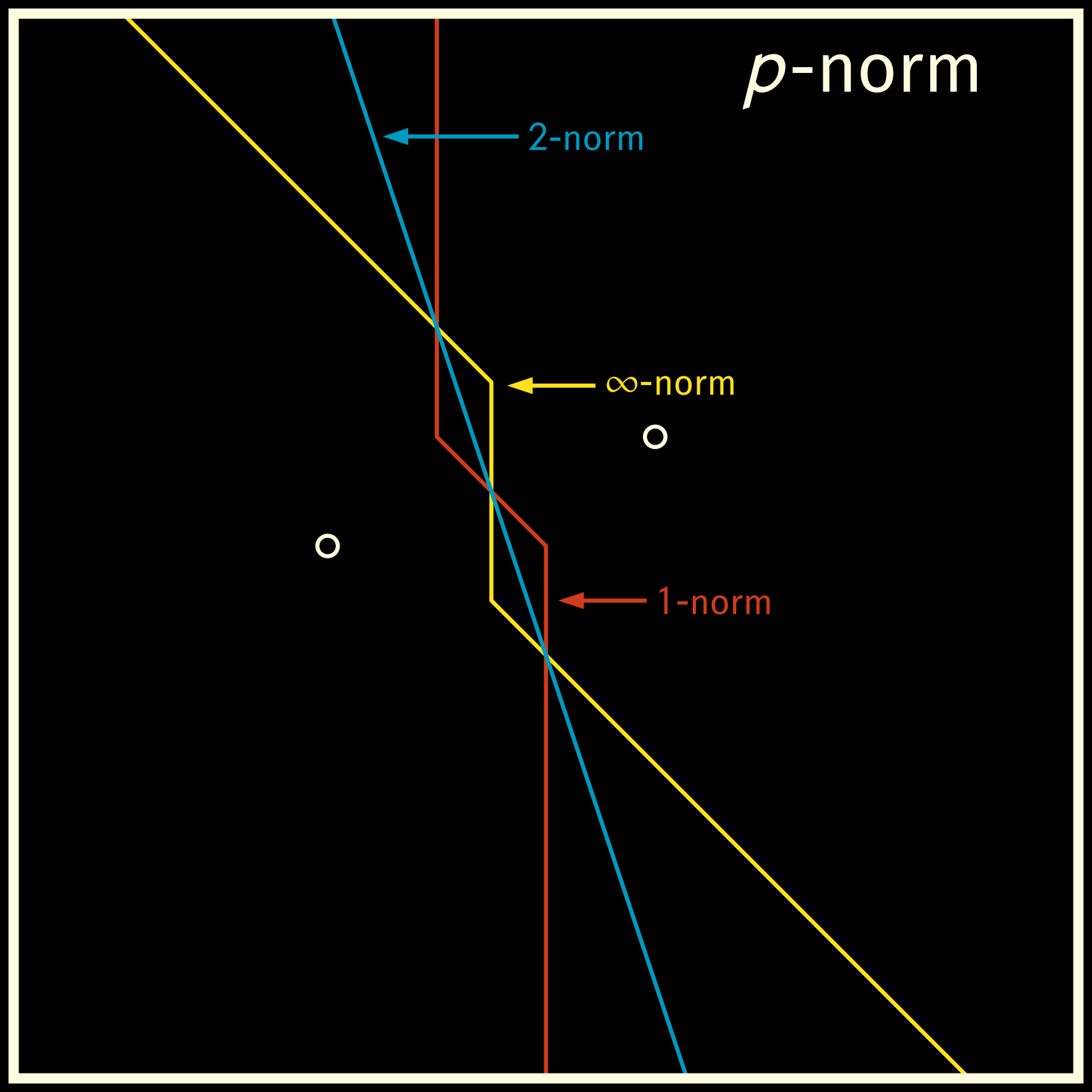


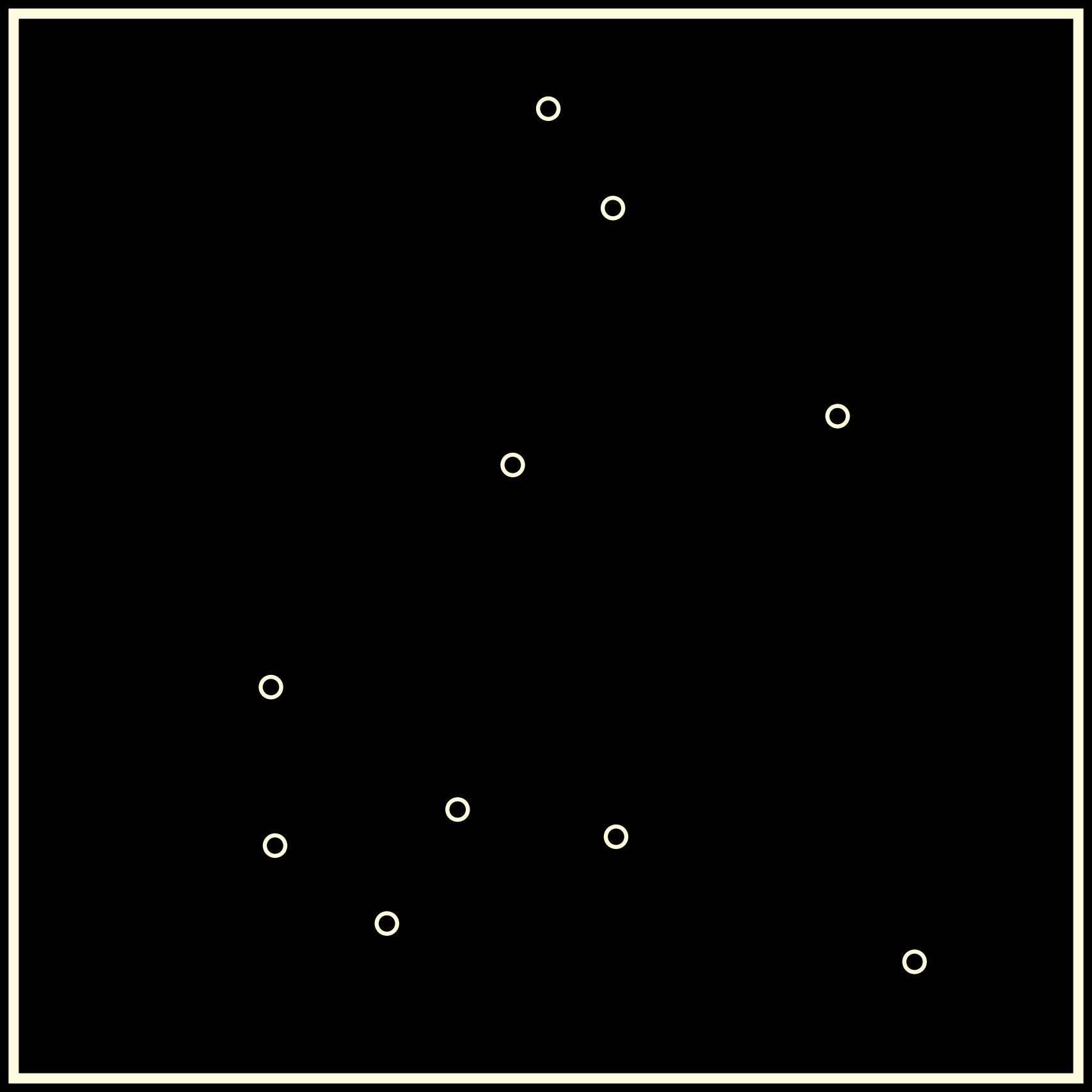


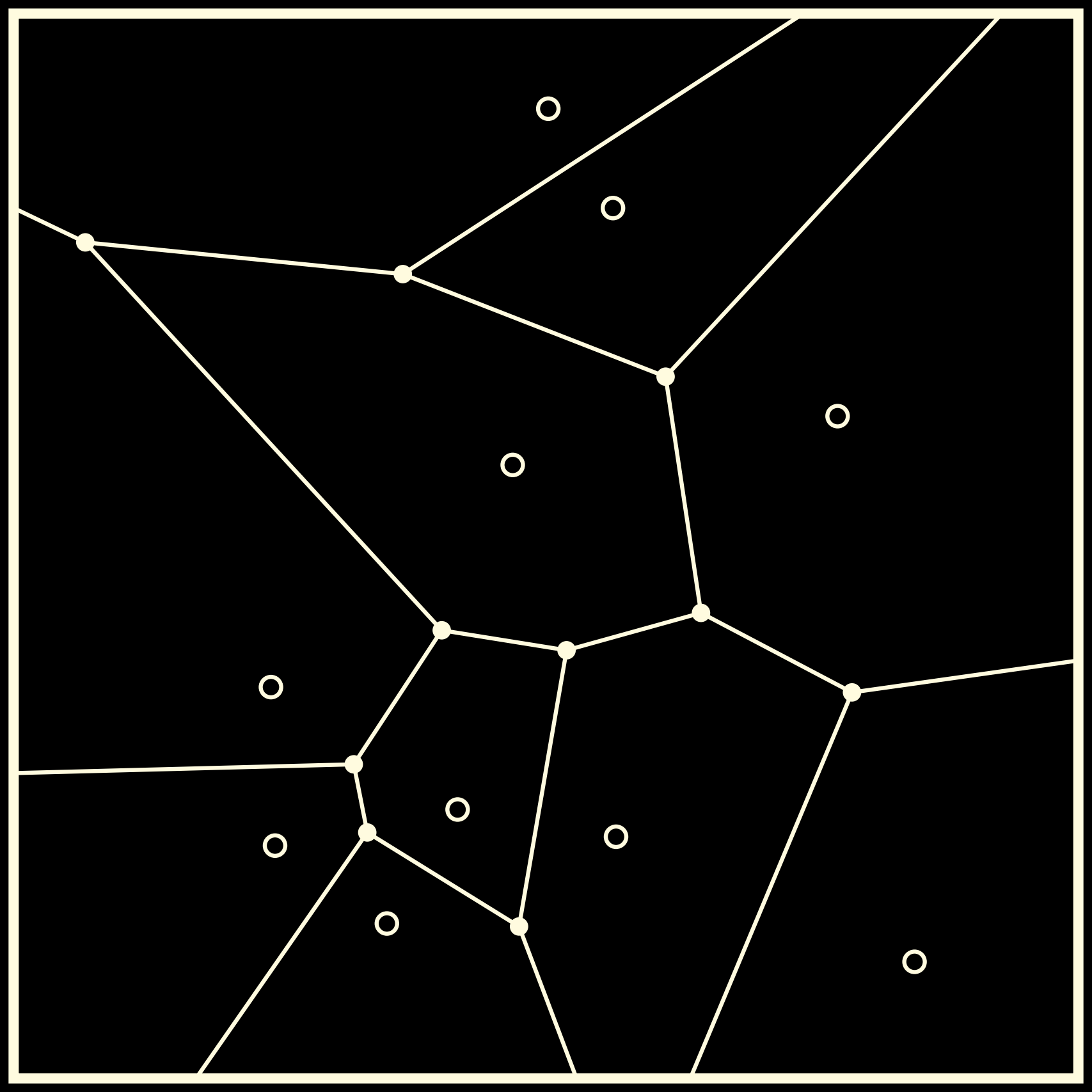


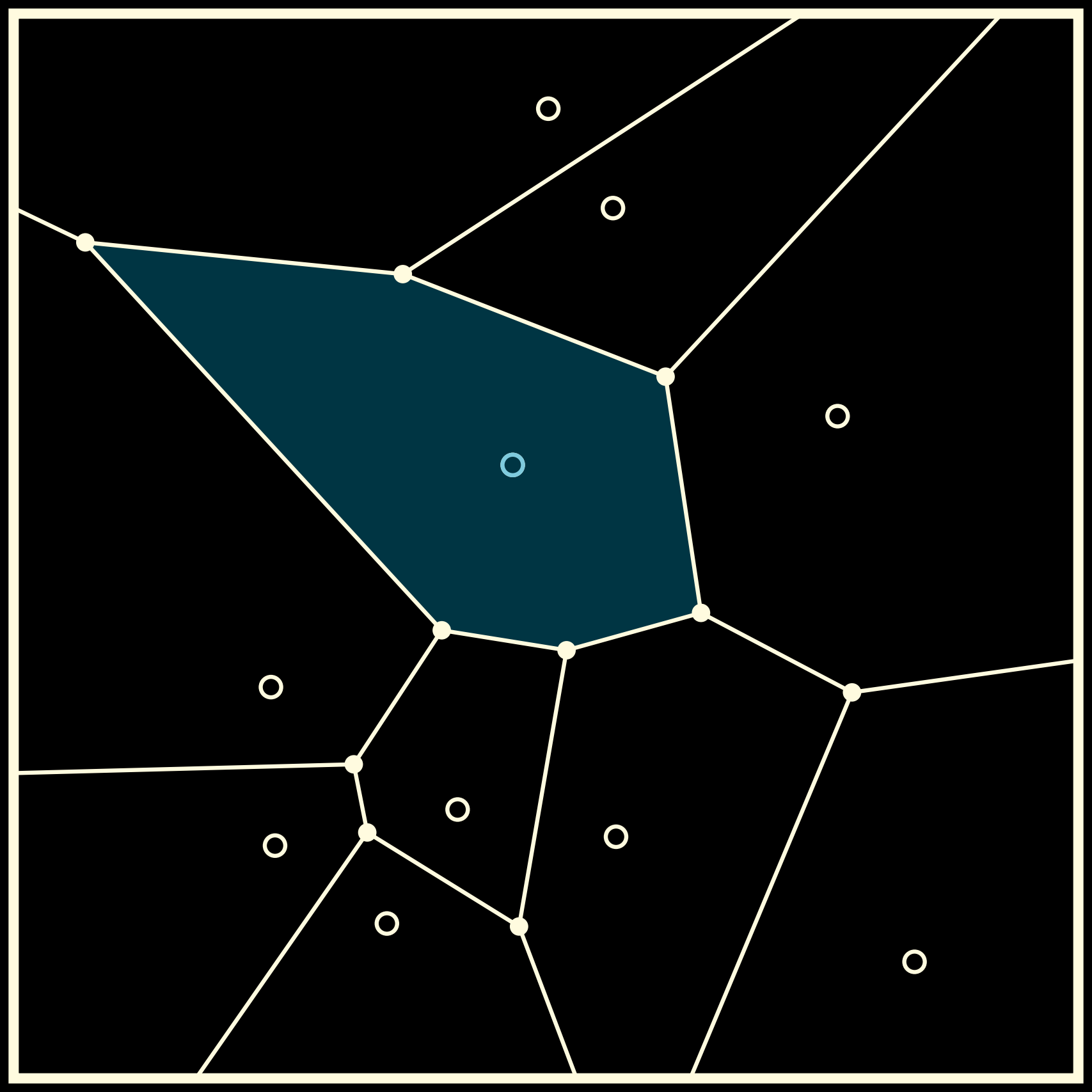


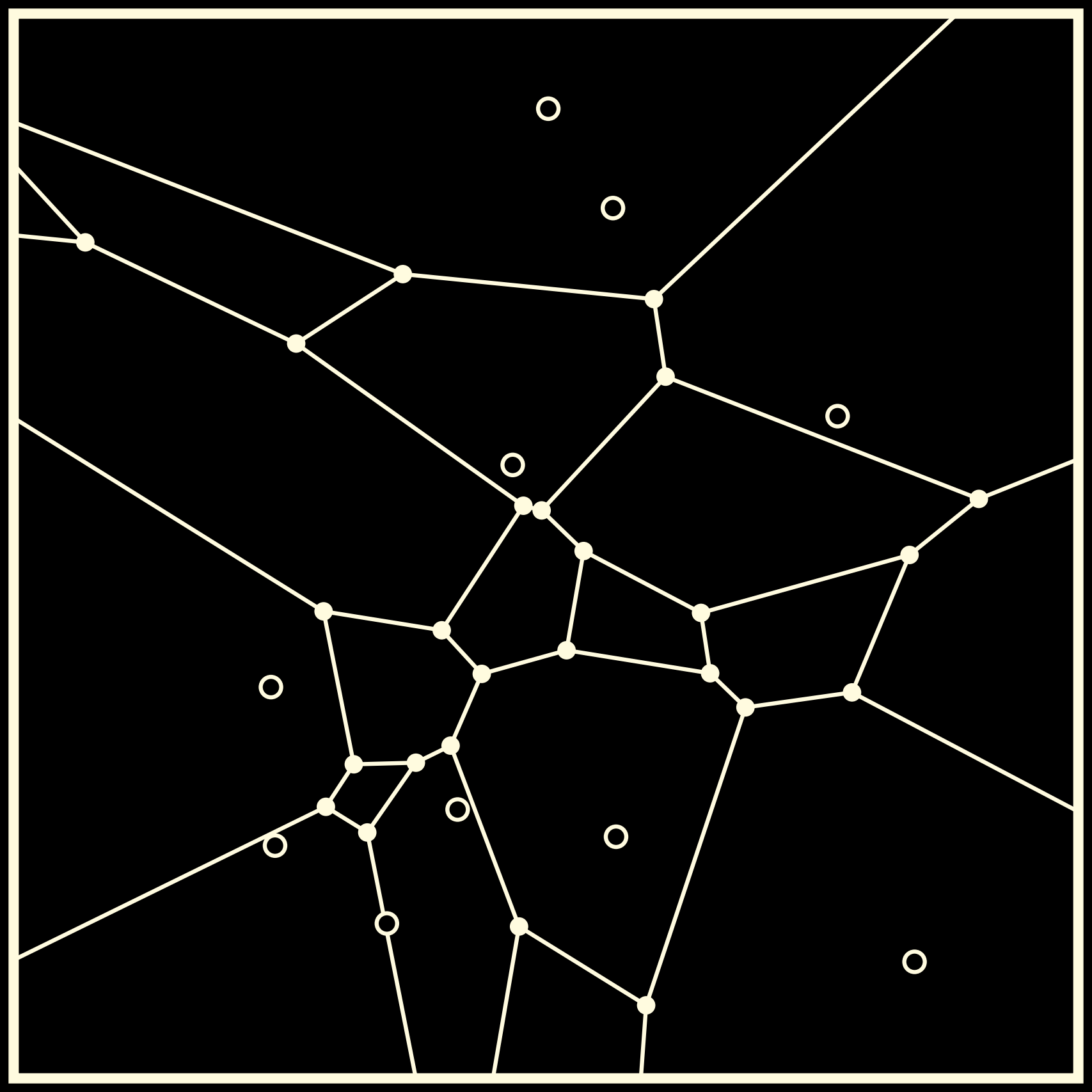

2-norm
∞-norm
1-norm

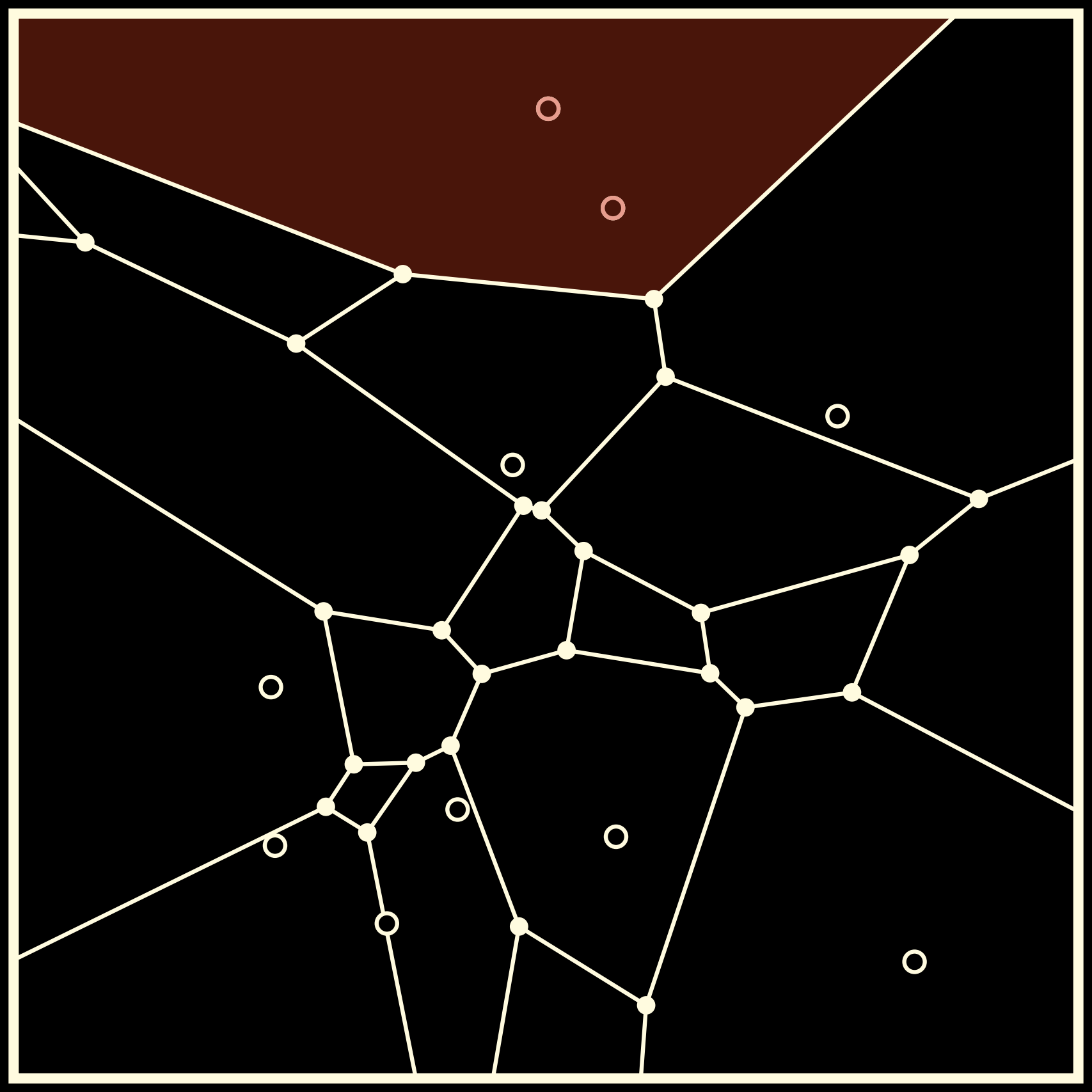

p-norm
2-norm
∞-norm
1-norm

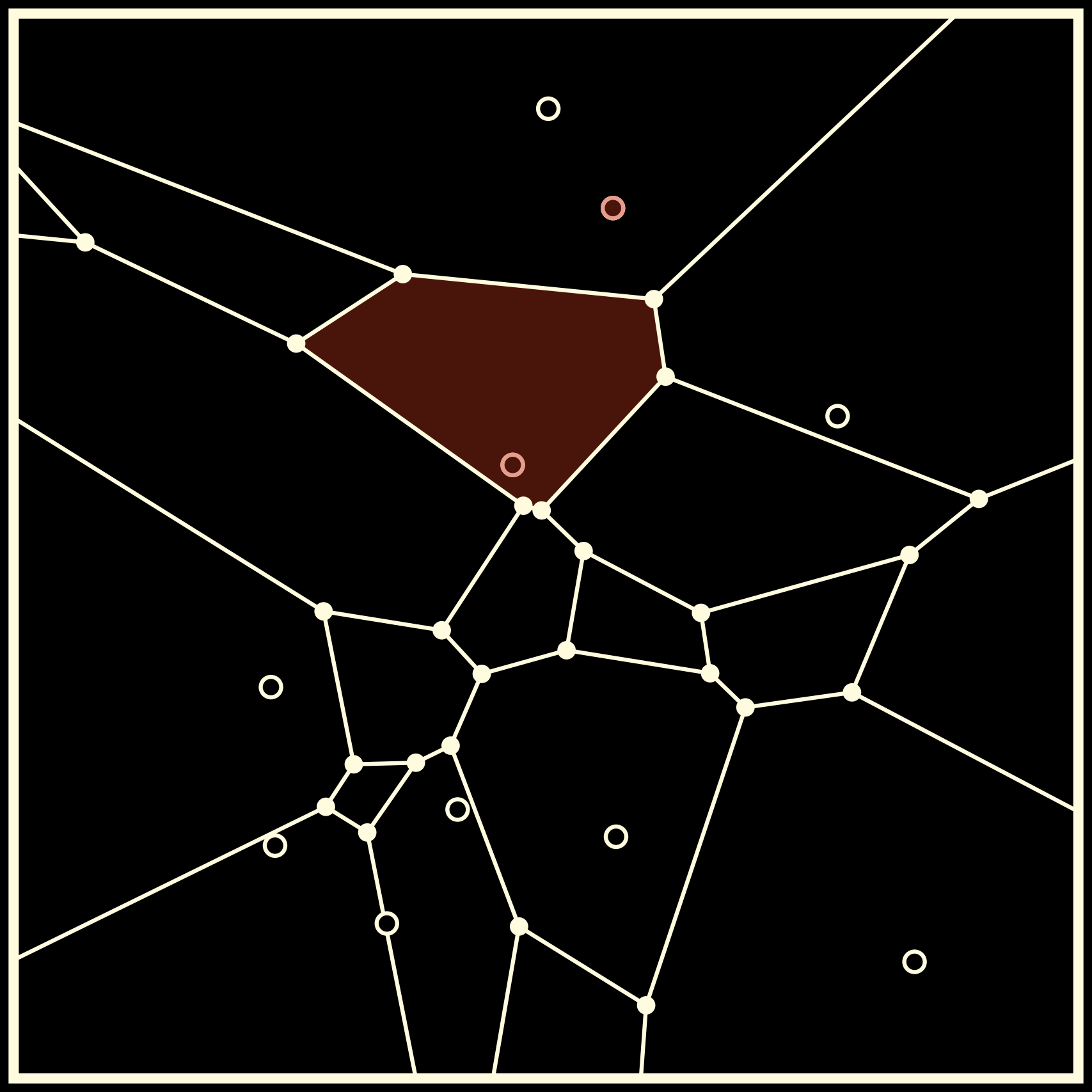


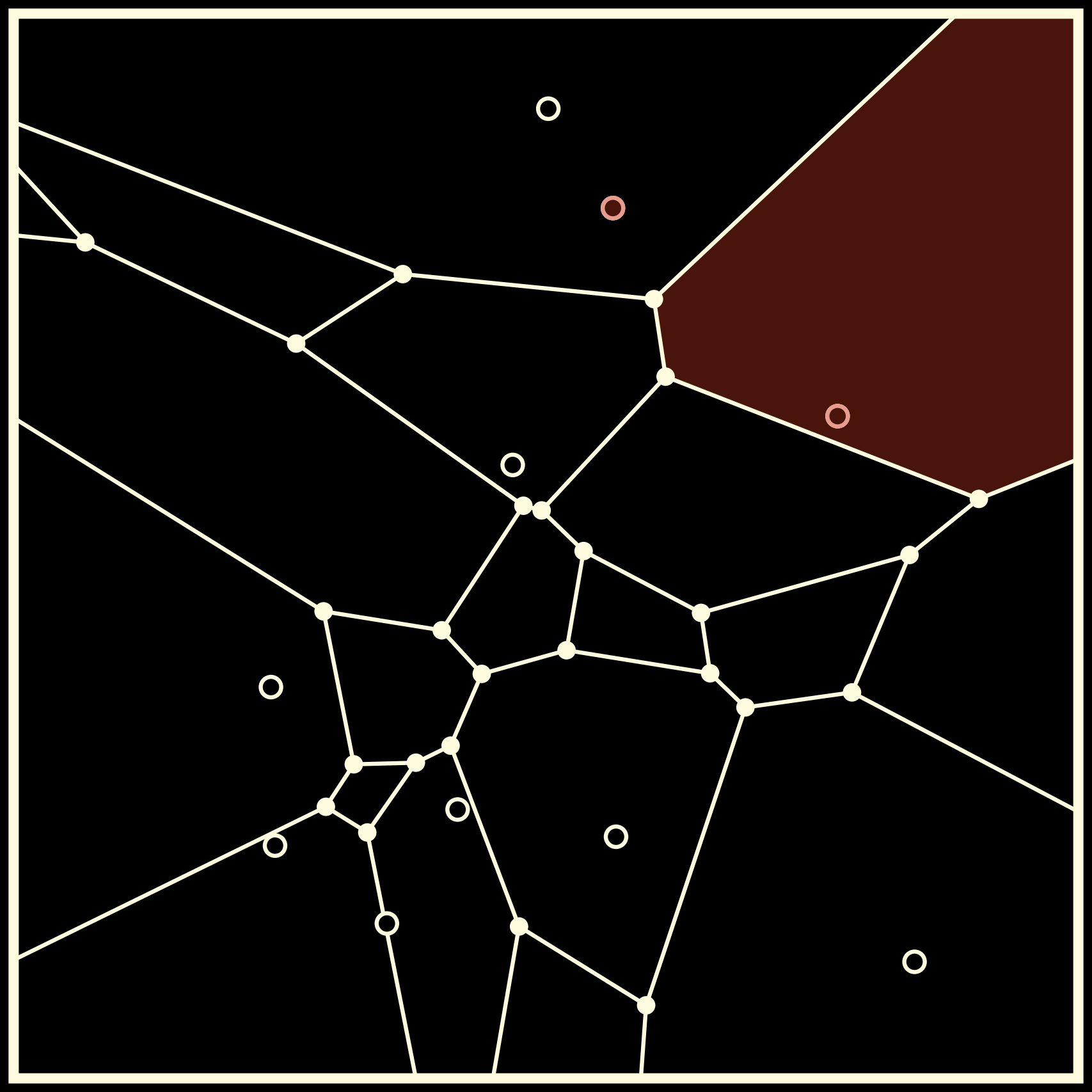


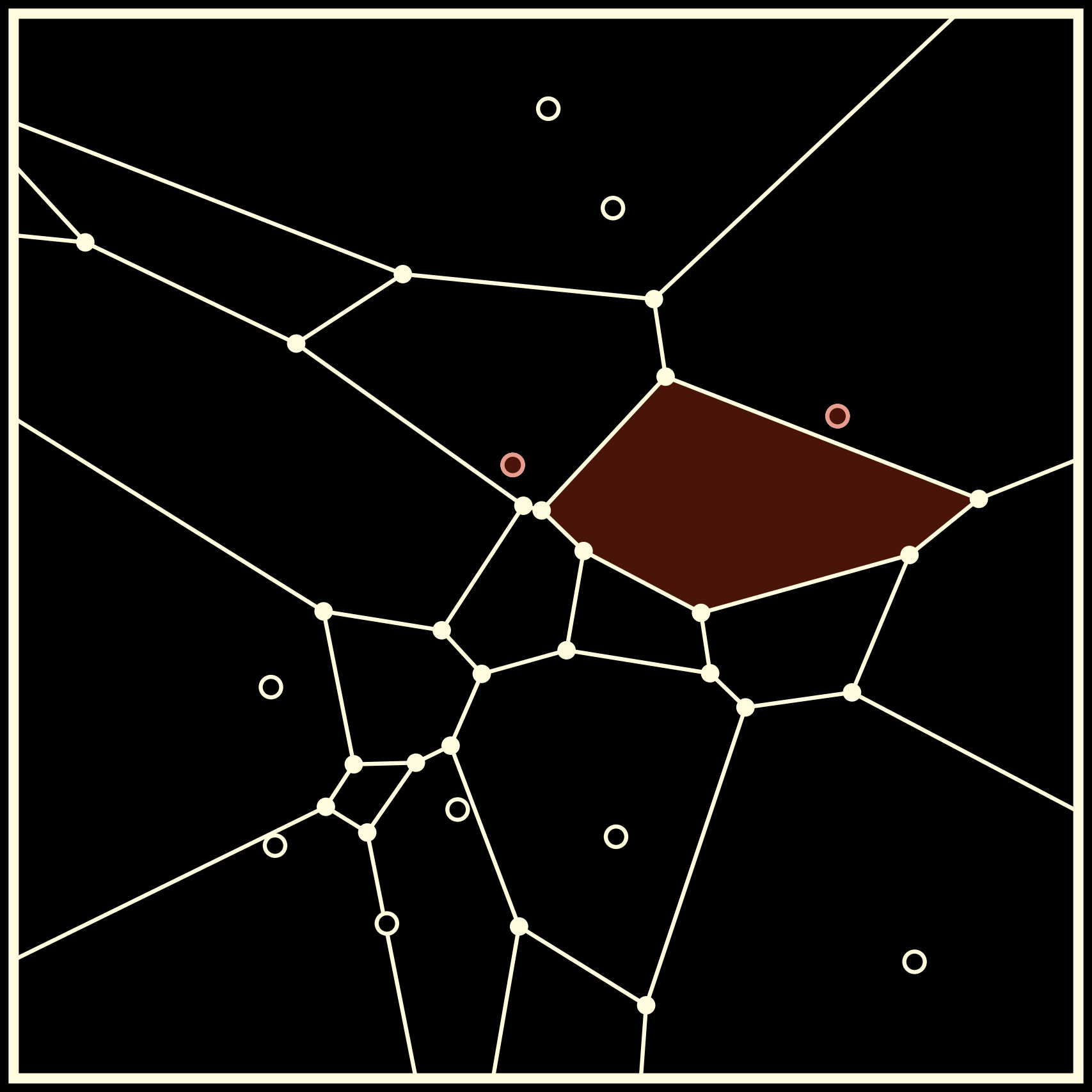


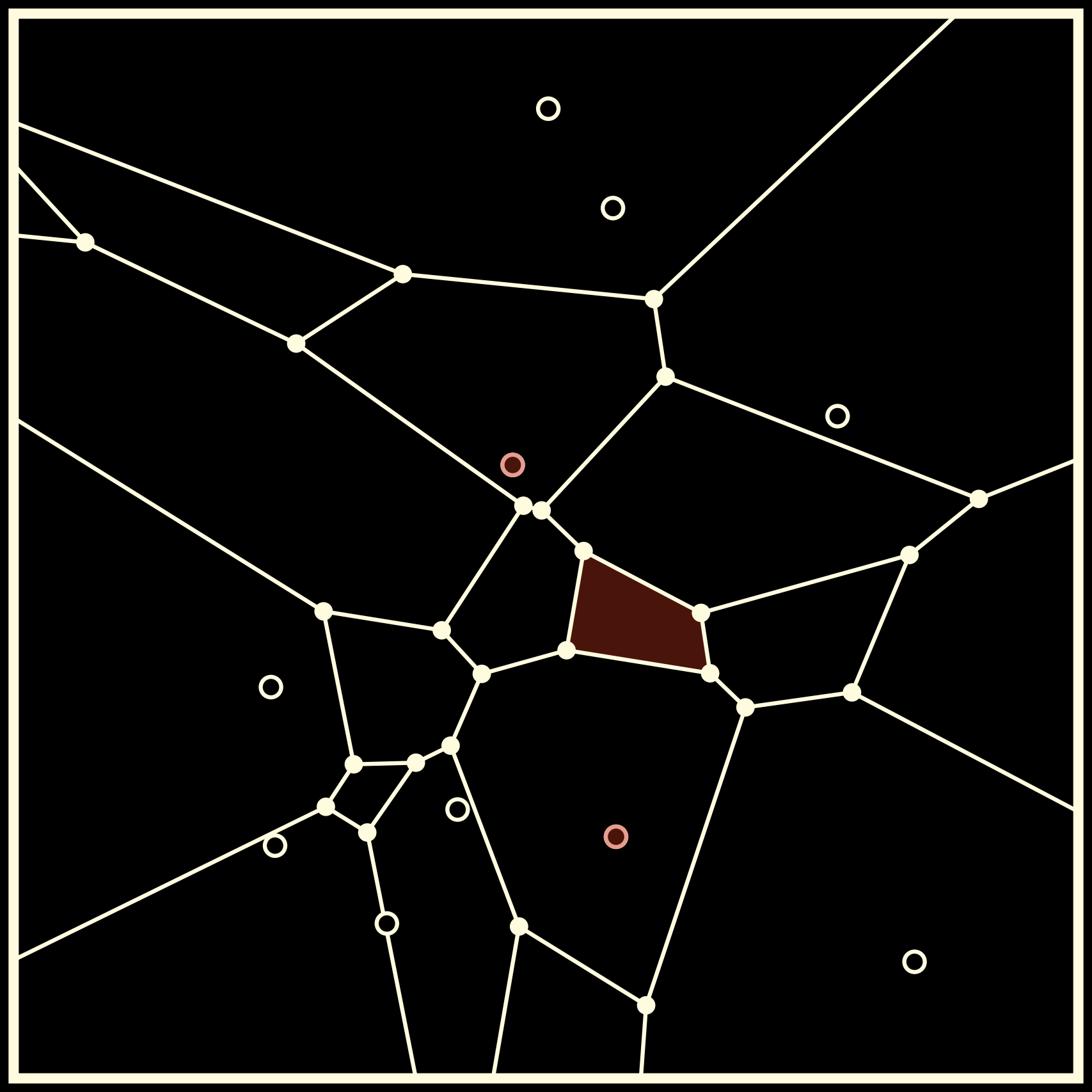


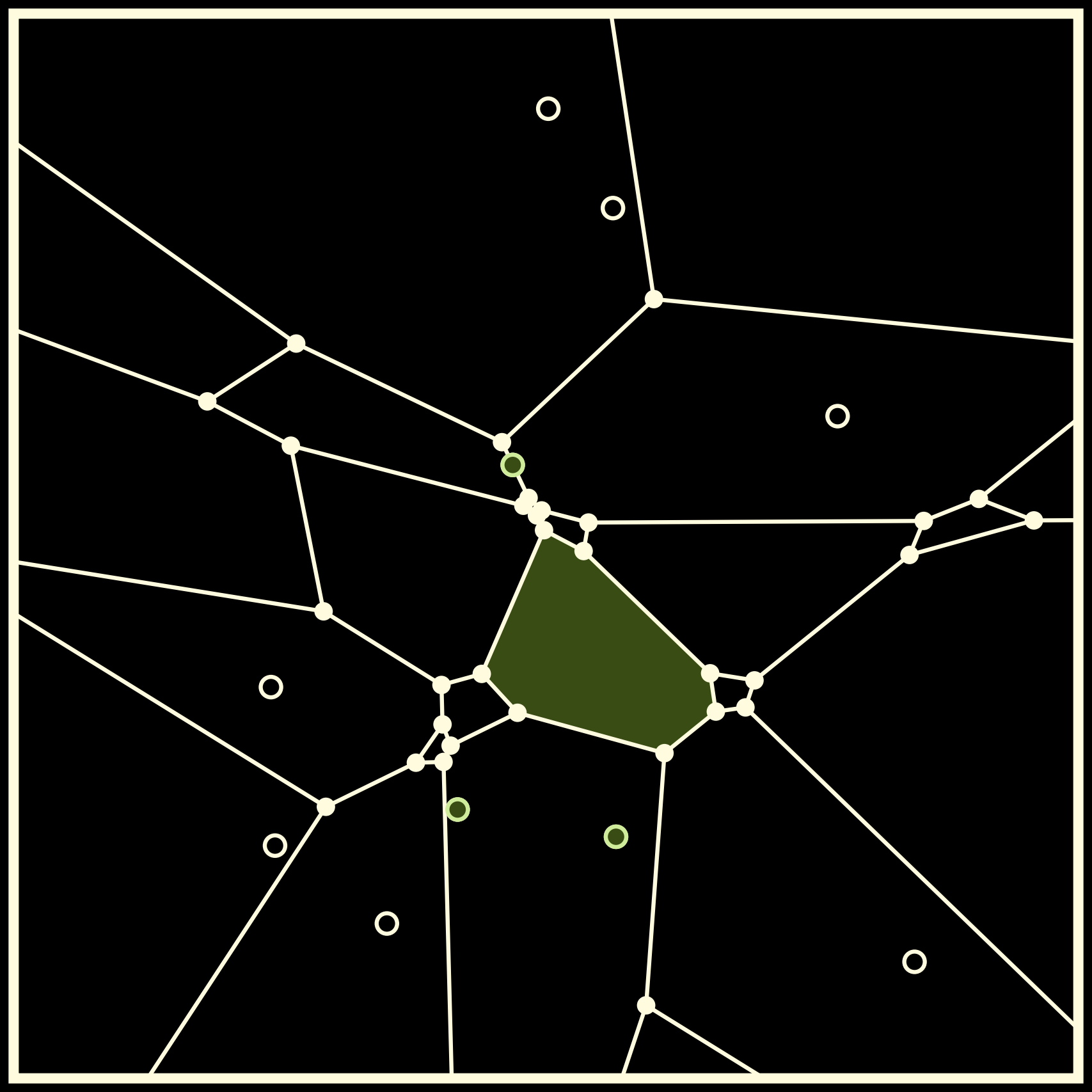


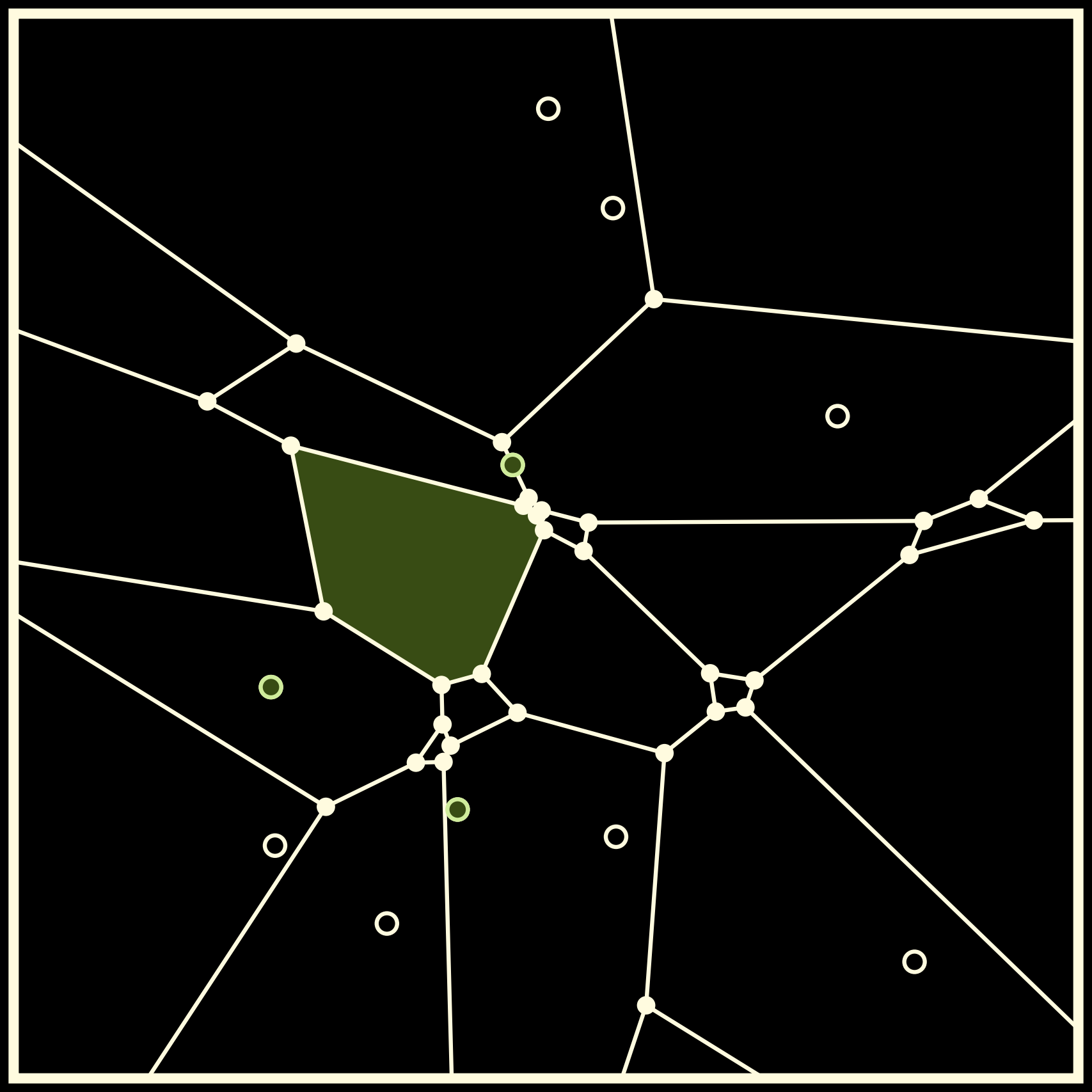


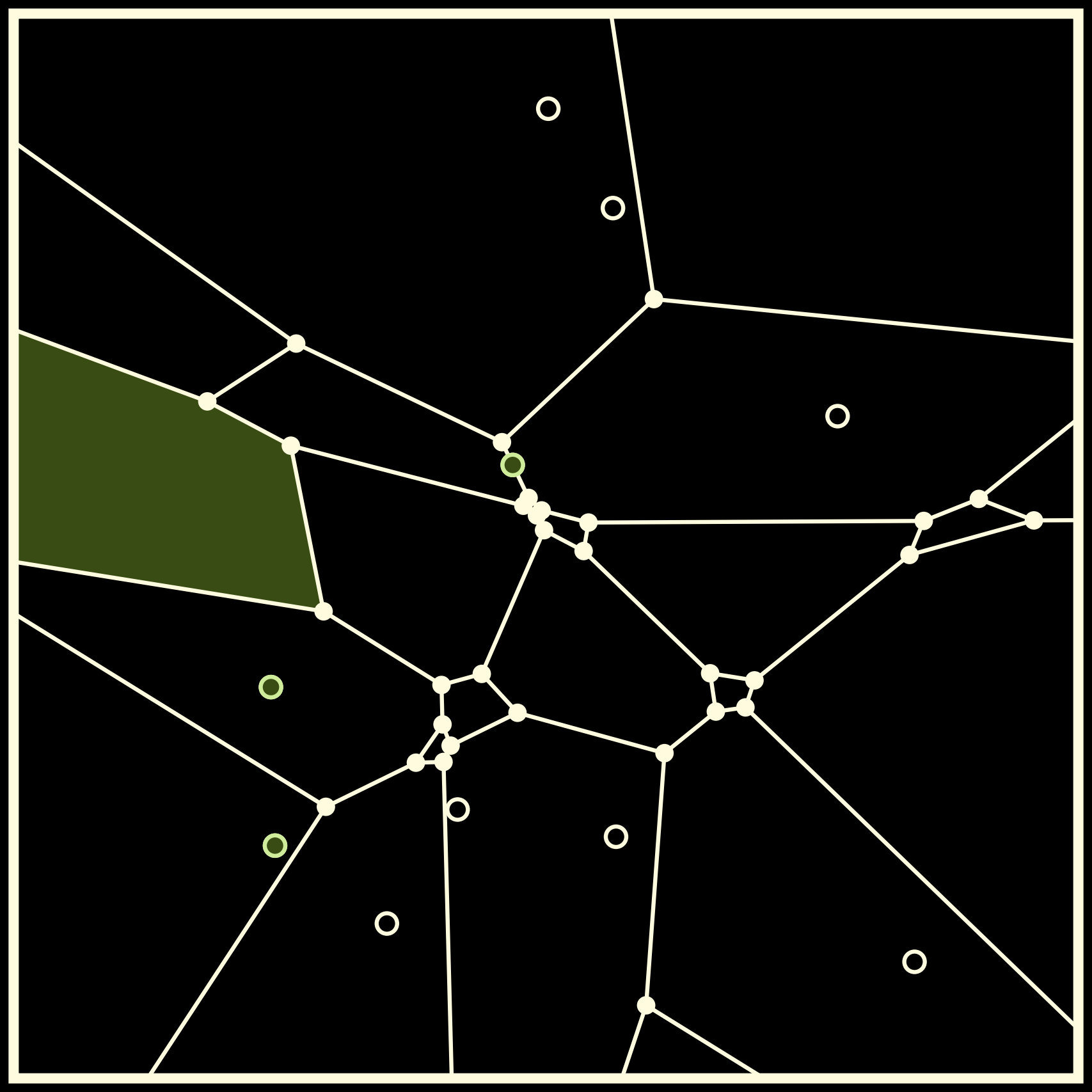


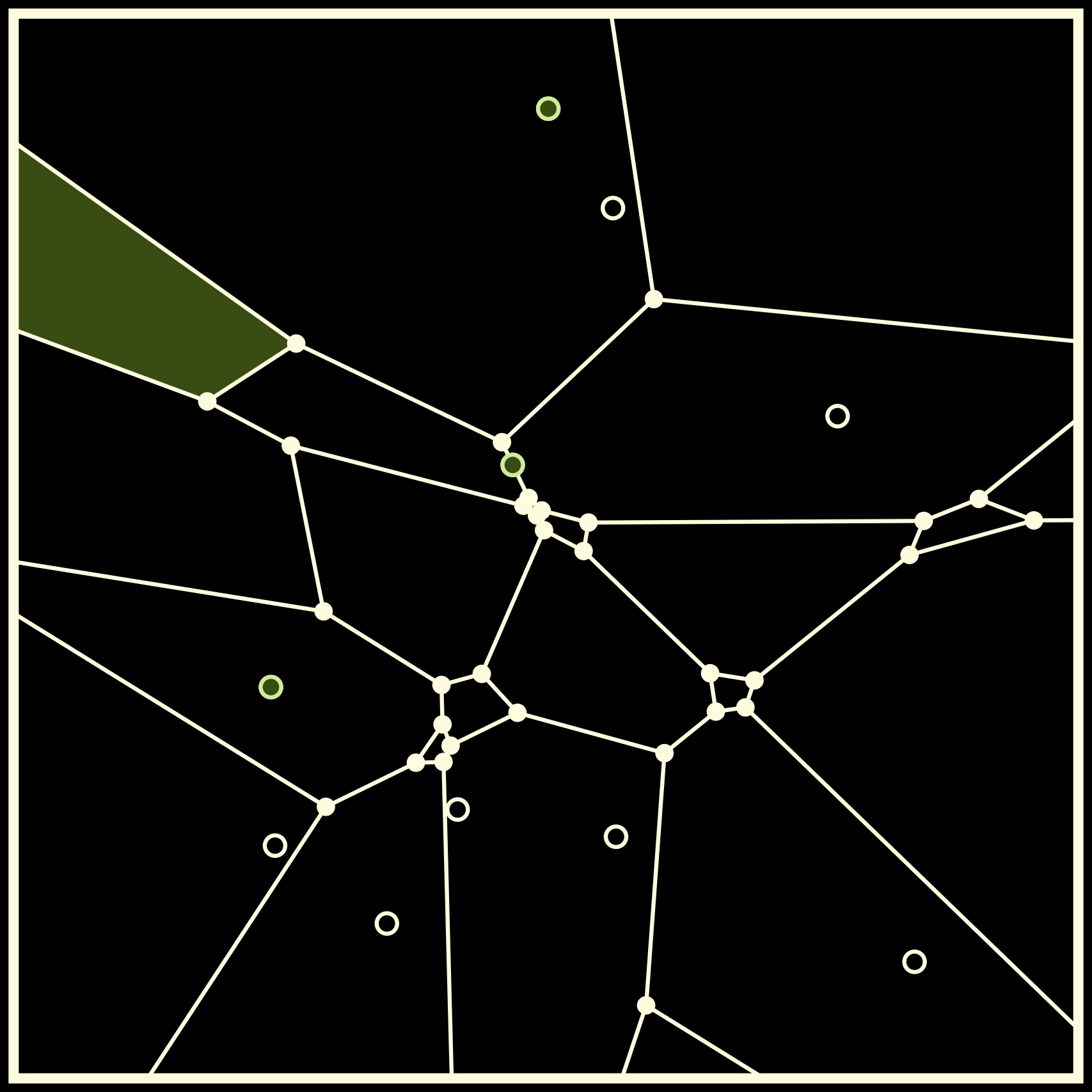


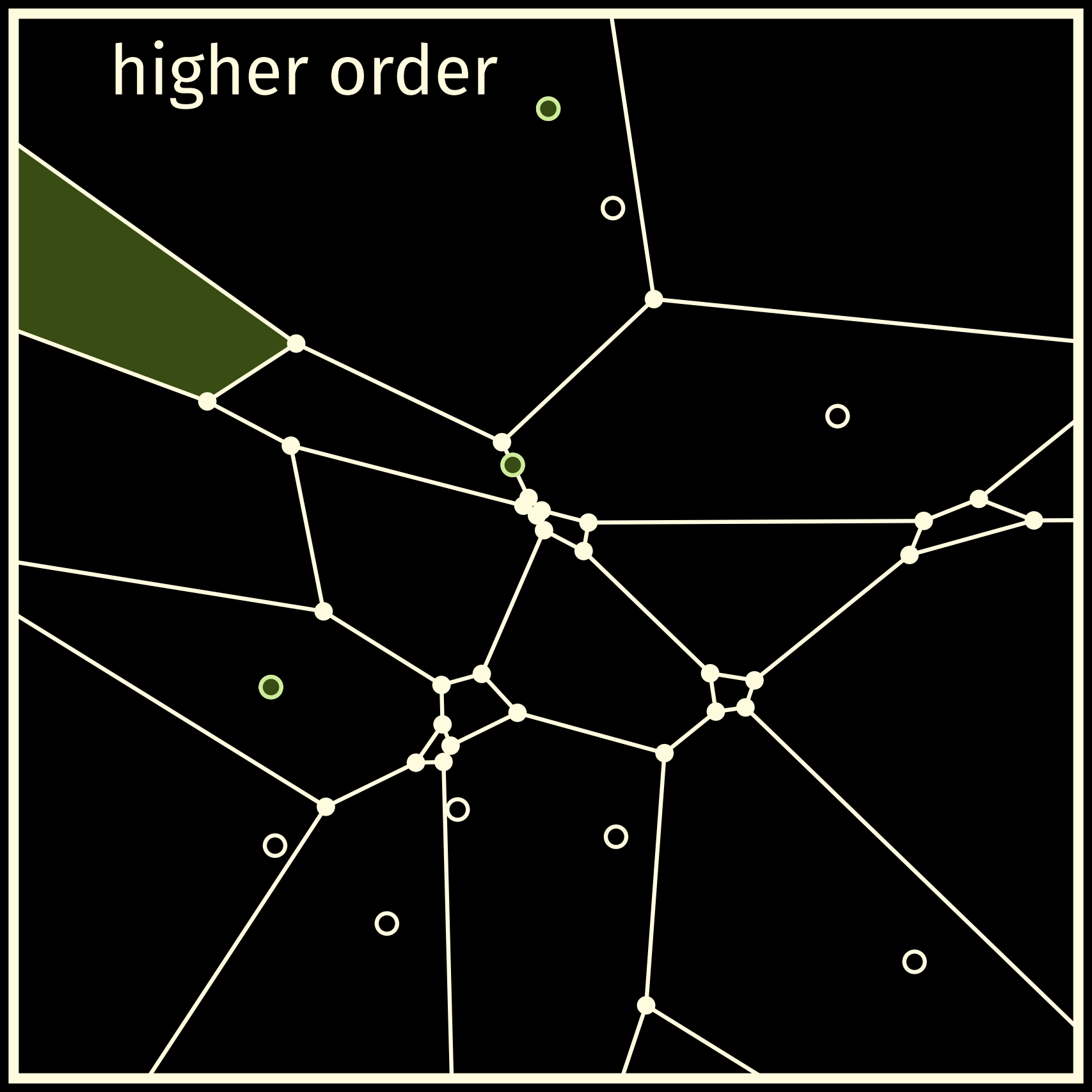


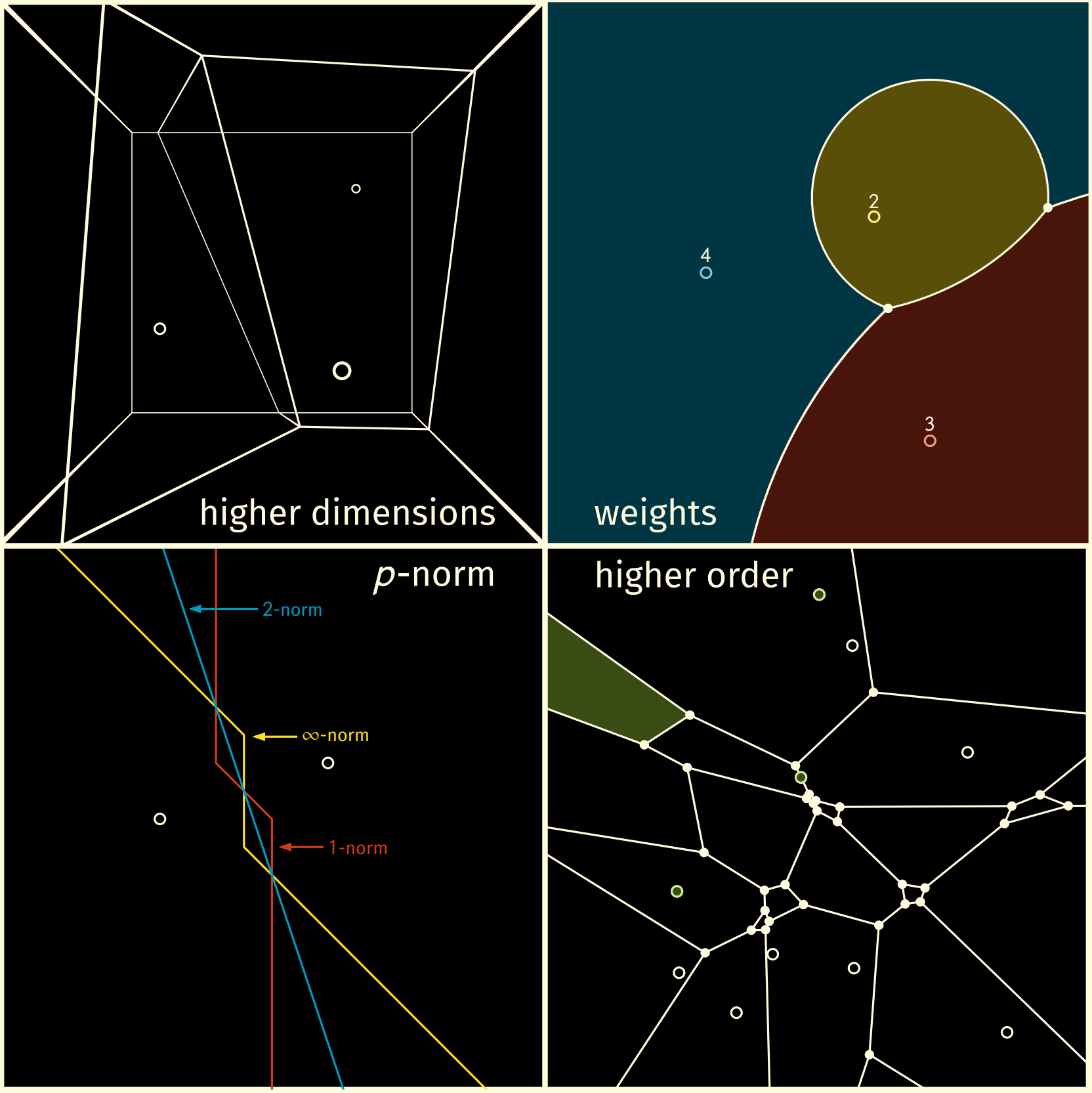




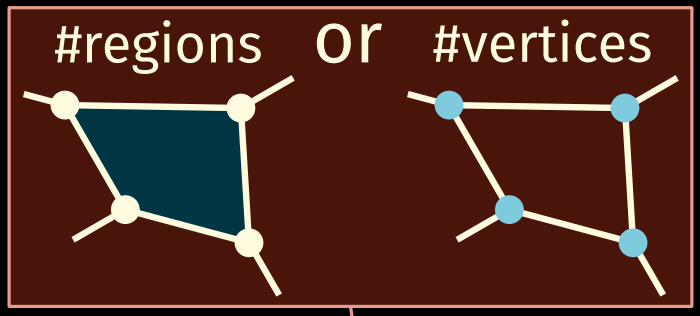




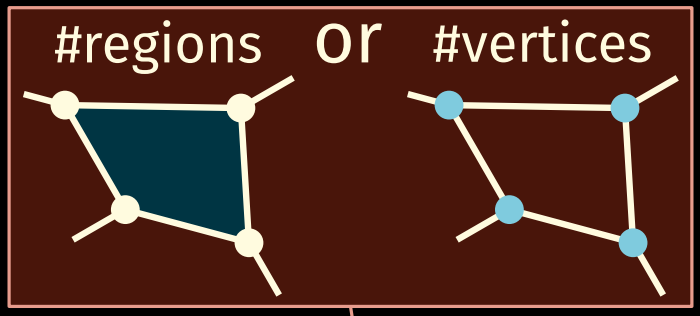

higher order


higher dimensions
weights
2
4
3
p-norm
2-norm
∞-norm
1-norm
higher order

Theorem
random sites $\Longrightarrow$ low complexity Voronoi diagram



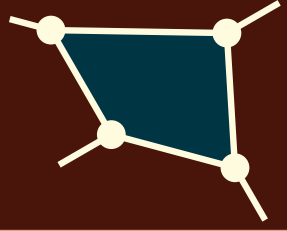
Theorem
random sites $\Longrightarrow$ low complexity Voronoi diagram

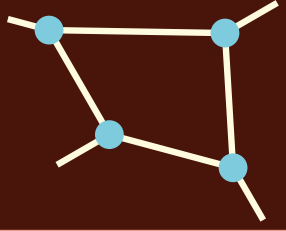


$O(\text{total weight})$
weights ≥ 1

Theorem
random sites $\Longrightarrow$ low complexity Voronoi diagram

#regions or #vertices



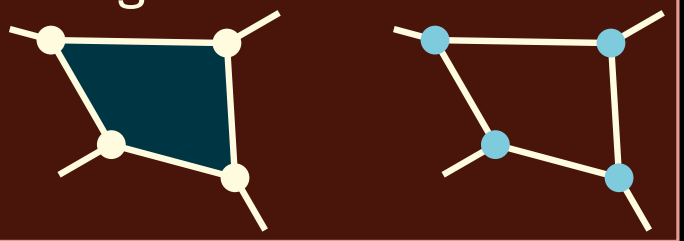


$O(\text{total weight})$
weights ≥ 1

order $\in O(1)$

Theorem
random sites $\Longrightarrow$ low complexity Voronoi diagram

#regions or #vertices



$O(\text{total weight})$
weights ≥ 1

order $\in O(1)$

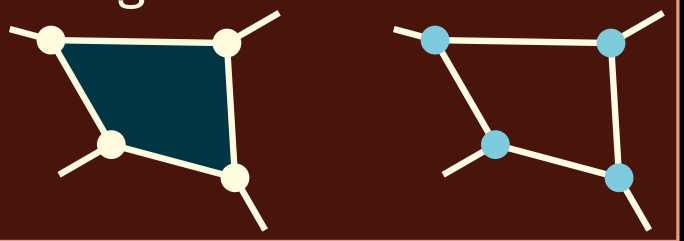
Theorem

random sites $\Longrightarrow$ low complexity Voronoi diagram

hypercube with p-norm
dimension $\in O(1)$

order	dim	norm	weights	complexity (region) (vertex)	
k	2	2	○	$O\left(k(n-k)\right)$	[Lee 82]
k	2	$1, \infty$	○	$O\left(\min\left\{k(n-k), (n-k)^2\right\}\right)$	[Liu, Papadopoulou, Lee 11]
k	2	abstract	○	$\leq 2k(n-k)$	[Bohler, Cheilaris, Klein, Liu, Papadopoulou, Zavershynskyi 15]
k	d	2	○	$O\left(n^{c(d)}\right)$	[Mulmuley 91]
1	3	2	○	$\Theta\left(n^2\right)$	[Klee 80] [Seidel 87]
1	d	p	○	$O\left(n^{c(d)}\right)$	[Lê 96]
1	d	∞	○	$\Theta\left(n^{\lceil d/2 \rceil}\right)$	[Boissonnat, Sharir, Tagansky, Yvinec 98]
1	d	1	○	$\Theta\left(n^2\right)$	[Boissonnat, Sharir, Tagansky, Yvinec 98]
1	d	2	○	expected $O(n)$	[Bienkowski, Damerow, Meyer auf der Heide, Sohler 05]
1	2	2	✓	$\Omega\left(n^2\right)$	[Aurenhammer, Edelsbrunner 84]
1	2	2	✓ random	expected $O(n \operatorname{polylog} n)$	[Har-Peled, Raichel 15]

#regions or #vertices



$O(\text{total weight})$
weights ≥ 1

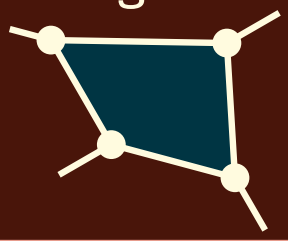
order $\in O(1)$

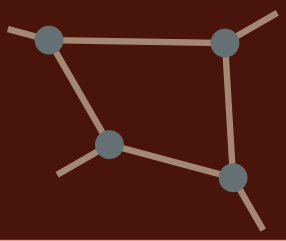
Theorem

random sites $\Longrightarrow$ low complexity Voronoi diagram

hypercube with p-norm
dimension $\in O(1)$

#regions or #vertices





$O(\text{total weight})$
weights ≥ 1

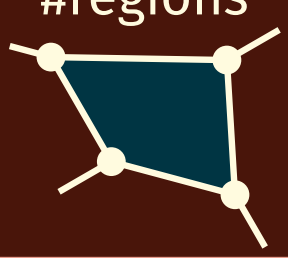
order $\in O(1)$

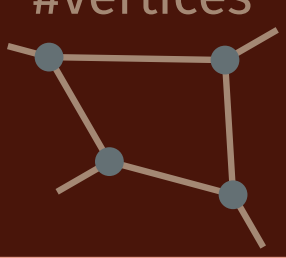
Theorem

random sites $\Longrightarrow$ low complexity Voronoi diagram

hypercube with p-norm
dimension $\in O(1)$

#regions or #vertices





$O(\text{total weight})$
weights ≥ 1

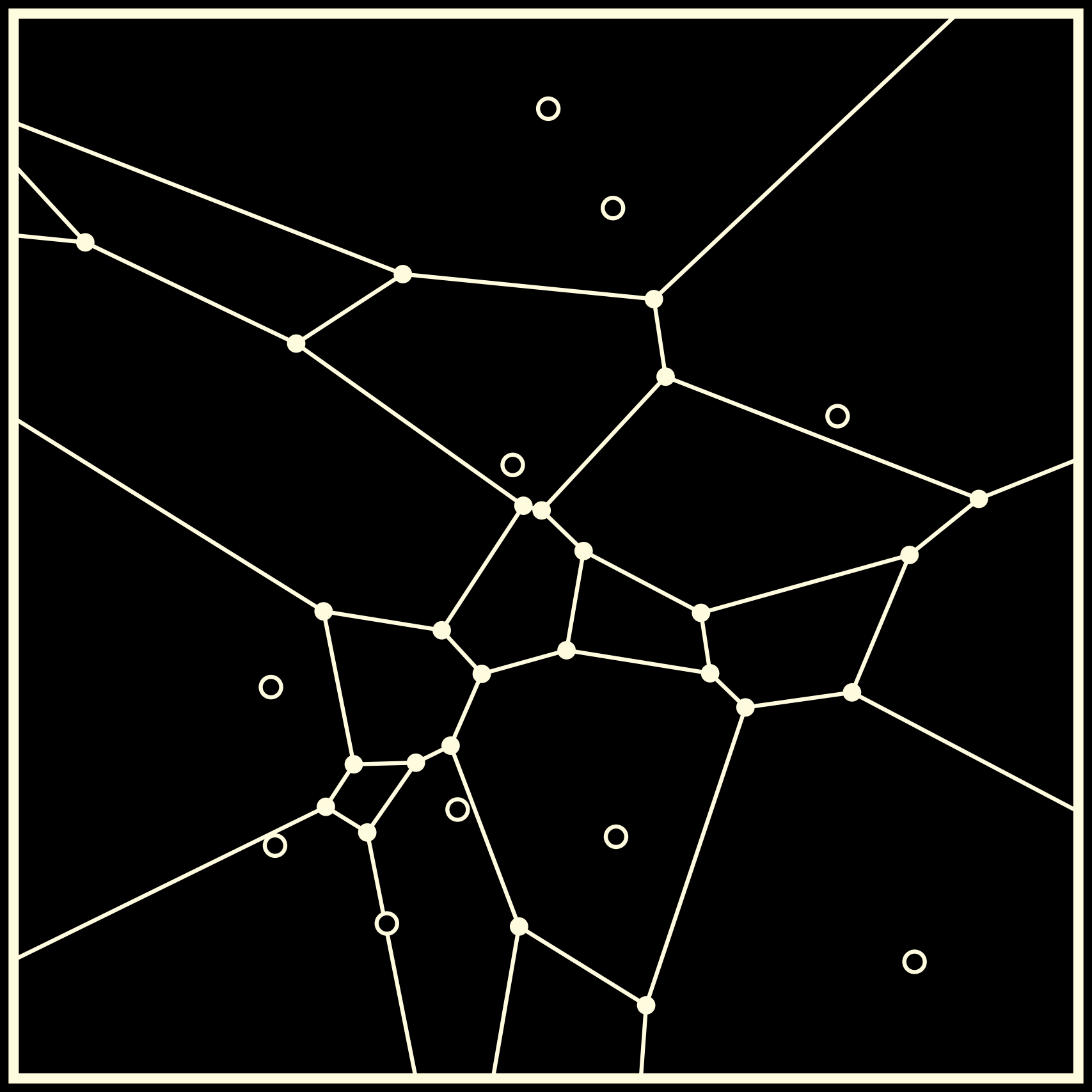
order $\in O(1)$

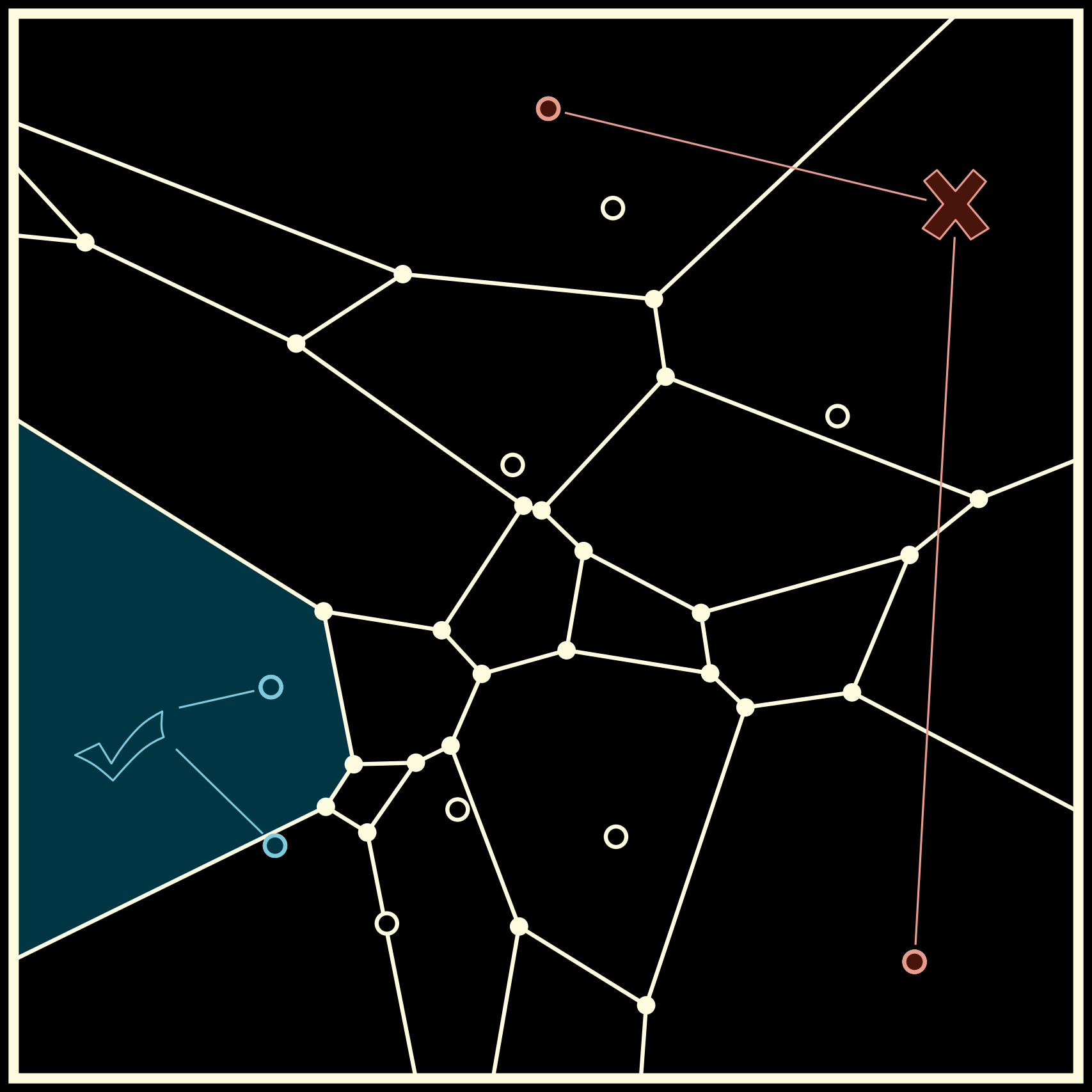
Theorem

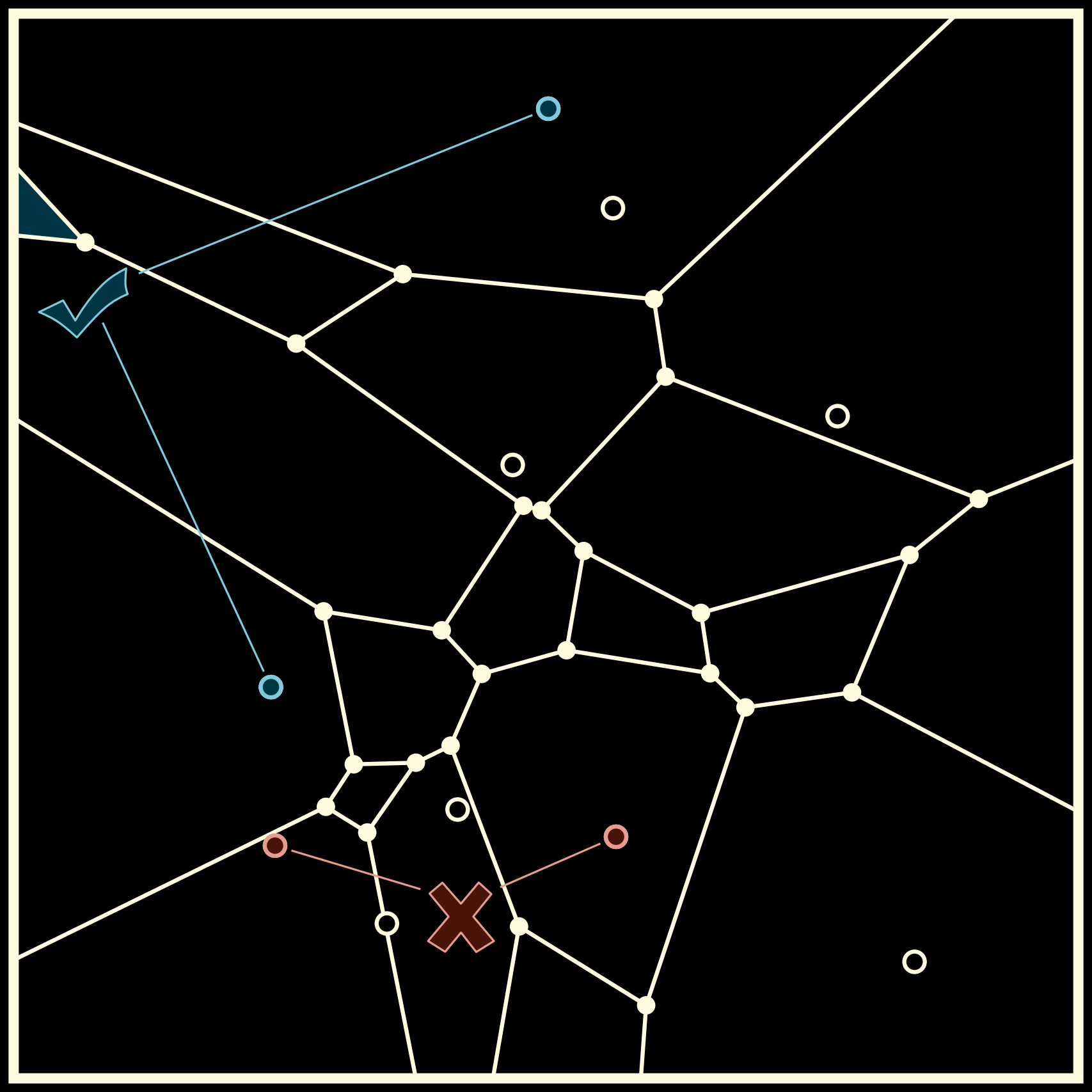
random sites $\Longrightarrow$ low complexity Voronoi diagram

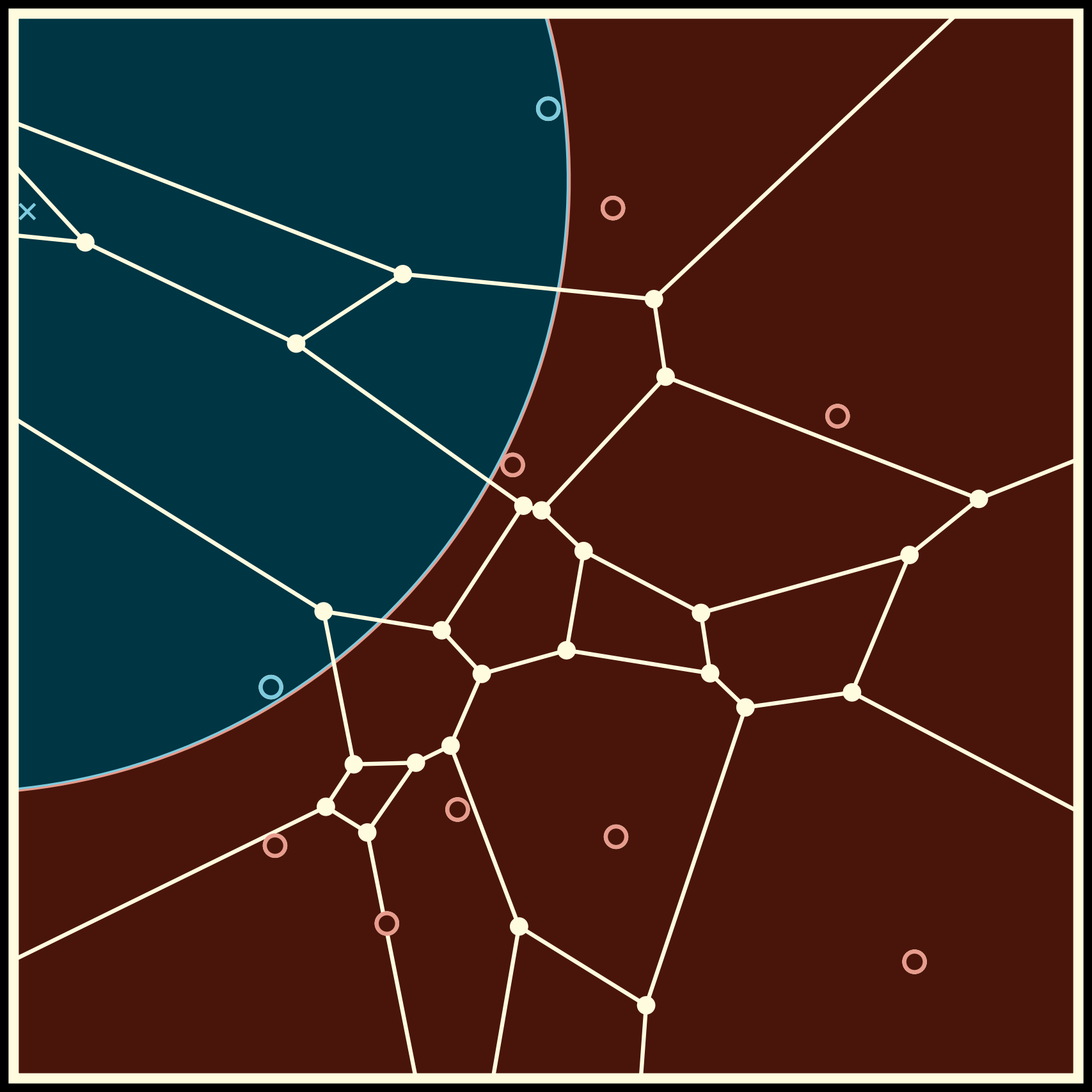
hypercube with p-norm
dimension $\in O(1)$

$$\mathbb{E}\left[\#\text{regions}\right] = \sum_{\substack{A \subseteq S \\ |A|=k}} \Pr\left[A \text{ has Voronoi region}\right]$$

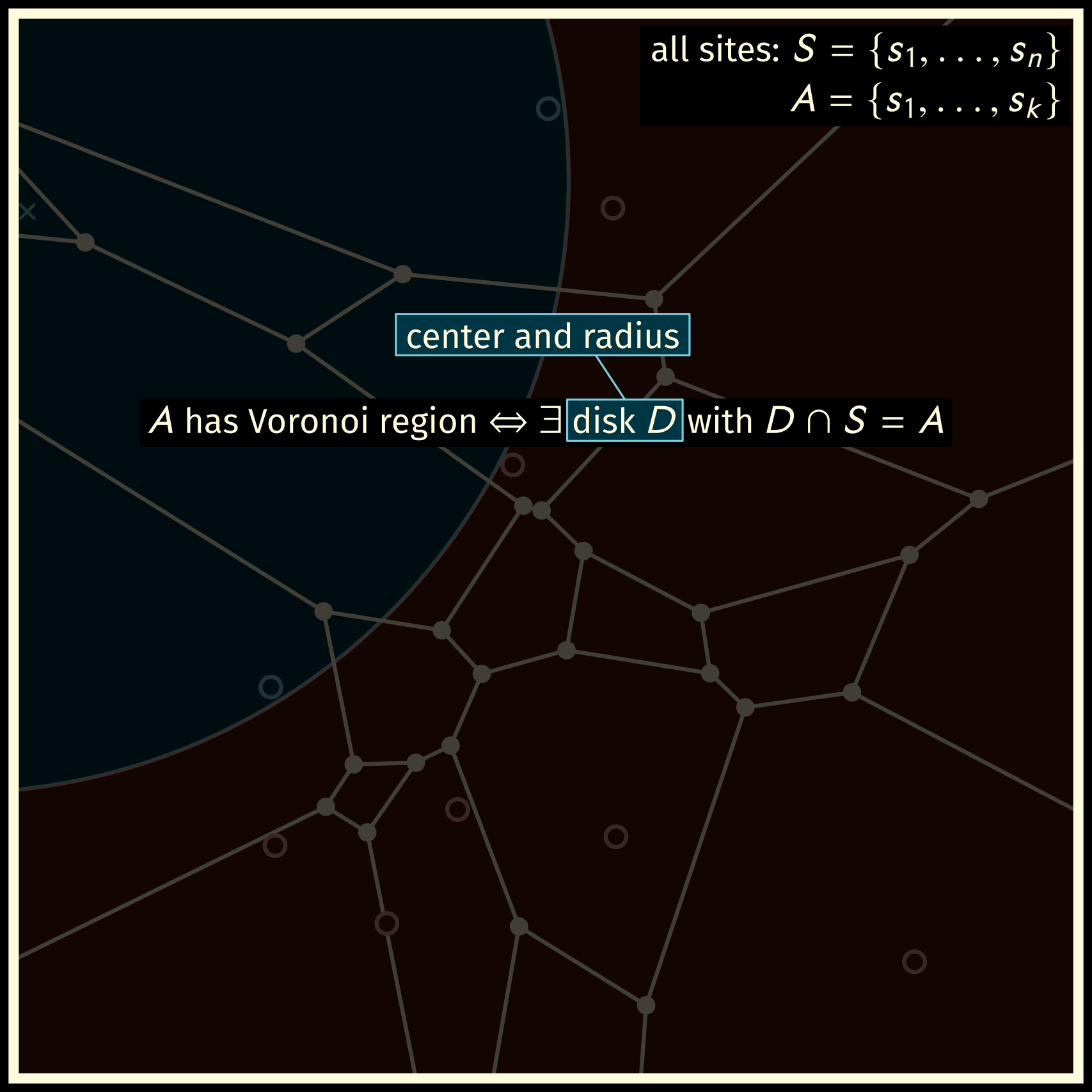








all sites: $S = \{s_1, \ldots, s_n\}$
$A = \{s_1, \ldots, s_k\}$
A has Voronoi region $\Leftrightarrow \exists$ disk D with $D \cap S = A$


all sites: $S = \{s_1, \ldots, s_n\}$
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center and radius
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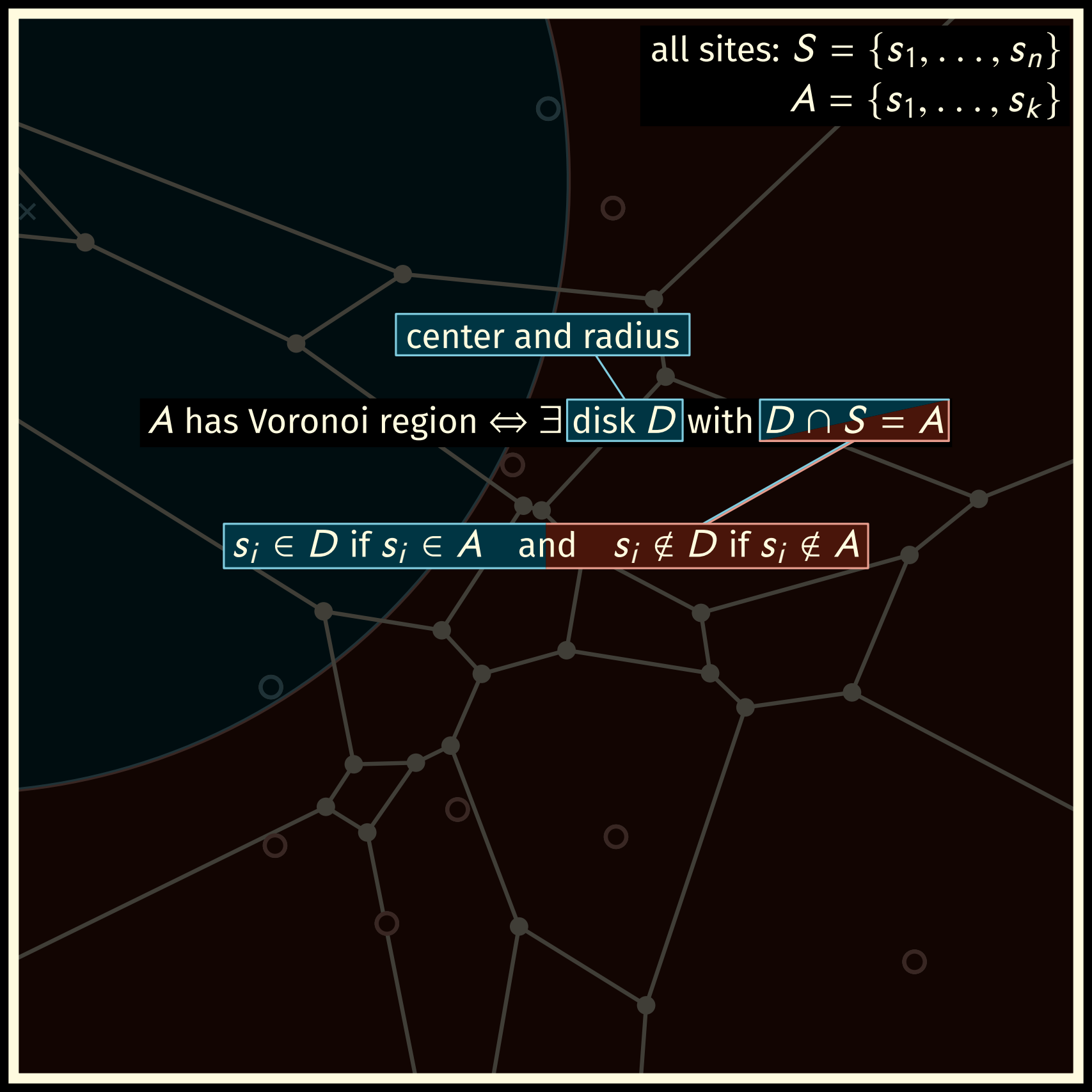
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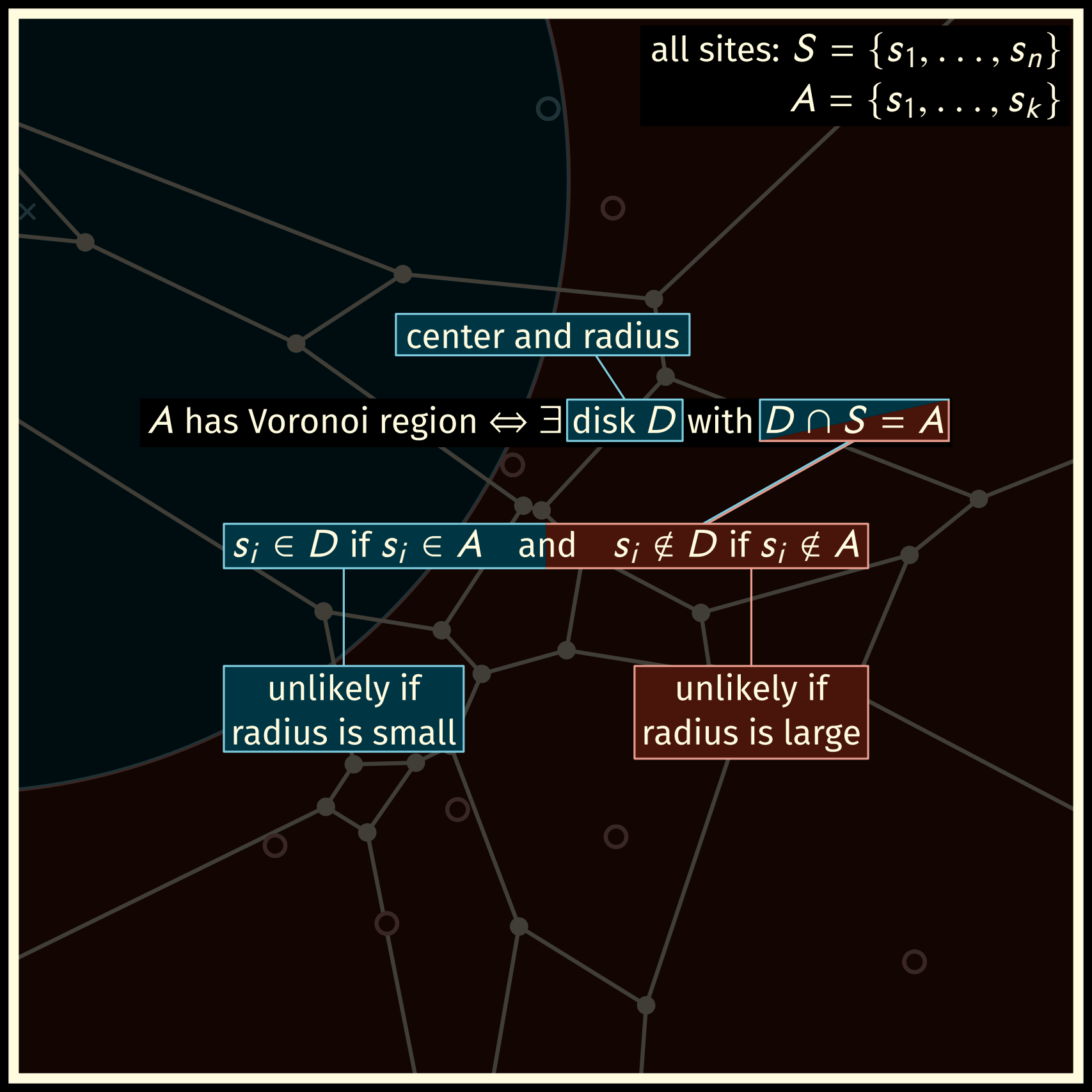
$s_i \in D$ if $s_i \in A$ and $s_i \notin D$ if $s_i \notin A$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center and radius
A has Voronoi region $\Leftrightarrow \exists$ disk D with $D \cap S = A$
$s_i \in D$ if $s_i \in A$ and $s_i \notin D$ if $s_i \notin A$
unlikely if radius is small
unlikely if radius is large

all sites: $S = \{s_1, \ldots, s_n\}$

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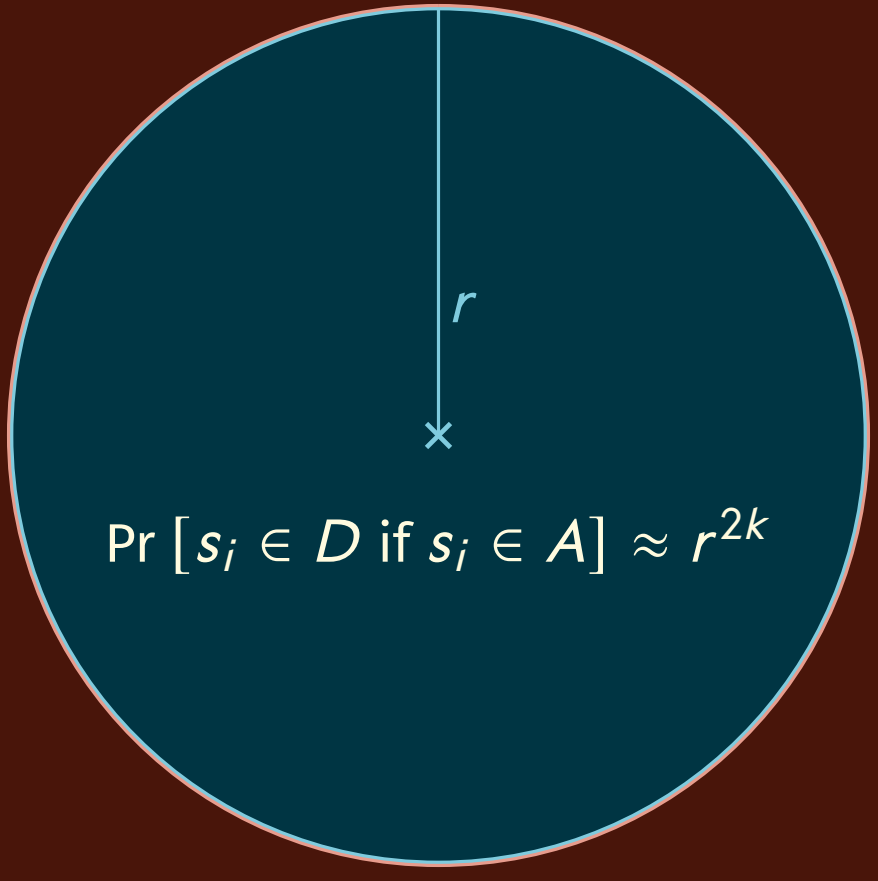

$$\Pr\left[s_i \notin D \text{ if } s_i \notin A\right] \approx \left(1 - r^2\right)^{n-k}$$

disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$

$$\Pr[A \text{ has Voronoi region}]$$

$$= \Pr\left[\exists r\ \exists p\colon \boxed{s_i \in D \text{ if } s_i \in A} \wedge \boxed{s_i \notin D \text{ if } s_i \notin A}\right]$$

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smallest r s.t. $\exists p\colon\ s_i \in D$ if $s_i \in A$

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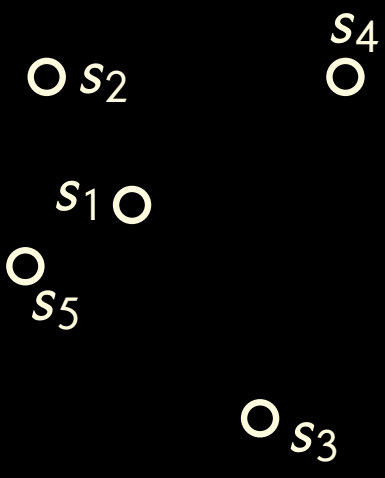

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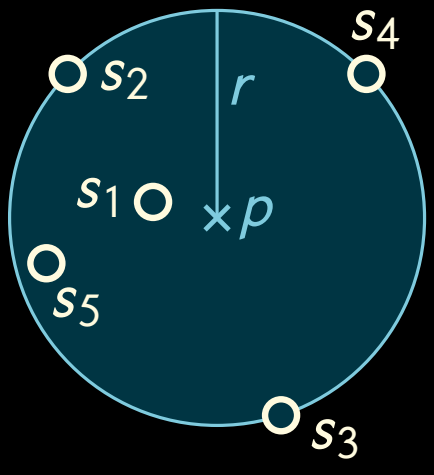

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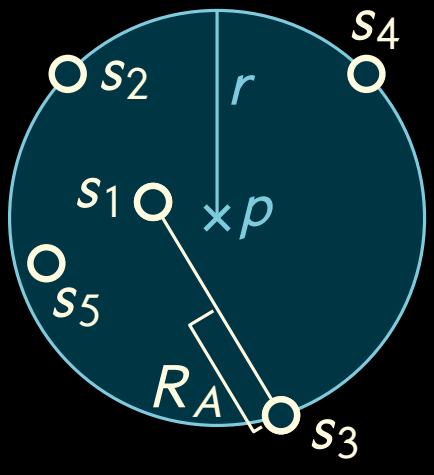

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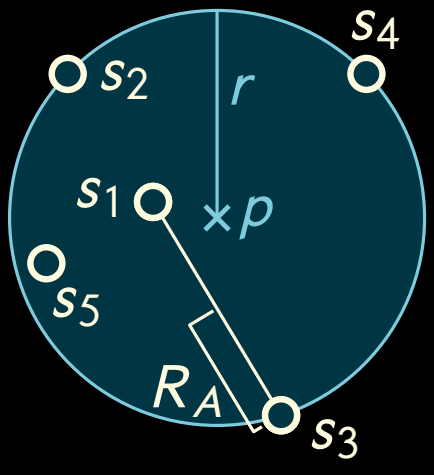

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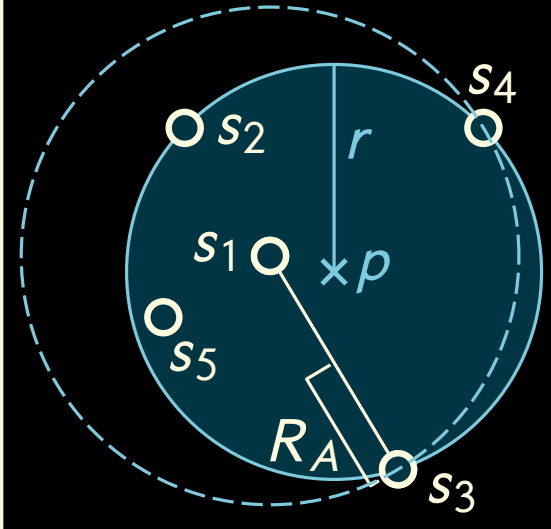

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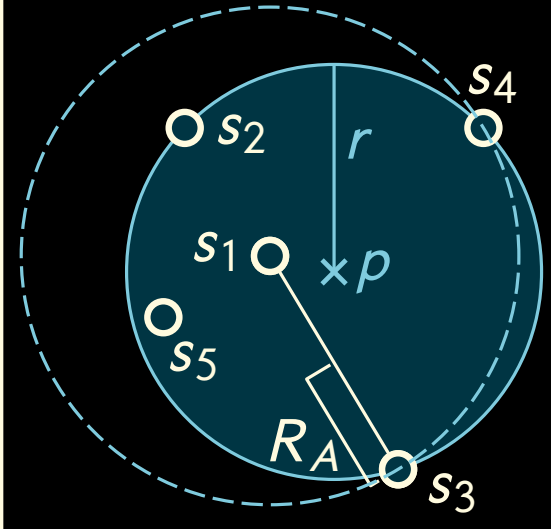

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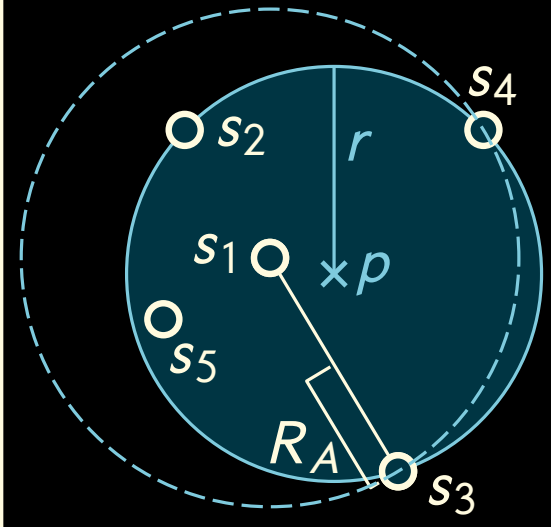

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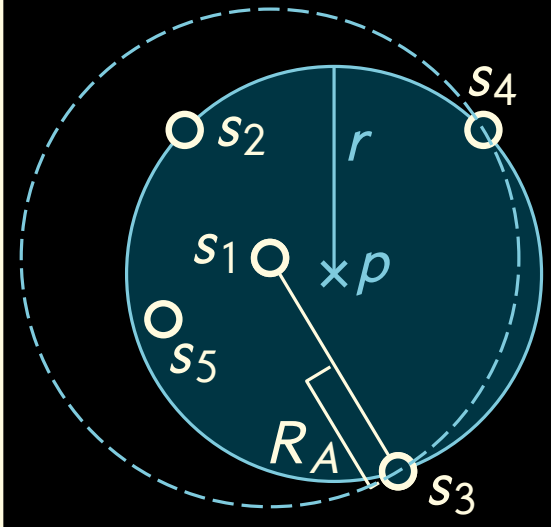

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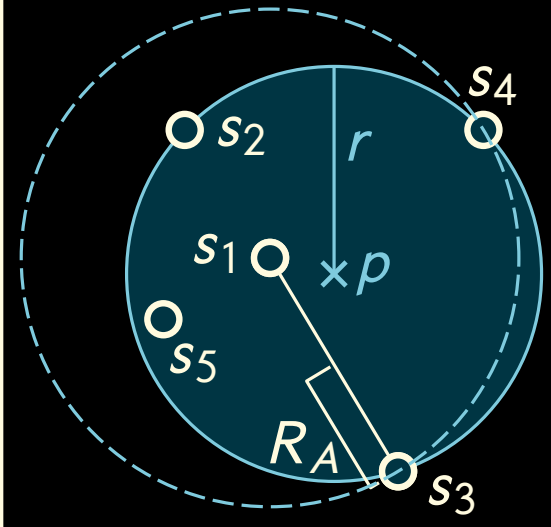

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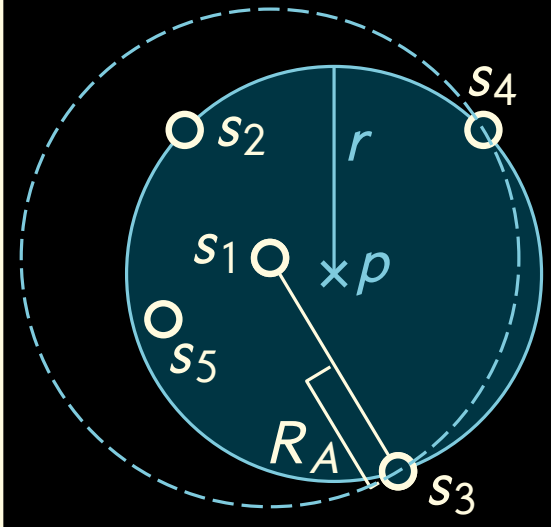

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disk D: center p and radius r ——— all sites: $S = \{s_1, \ldots, s_n\}$

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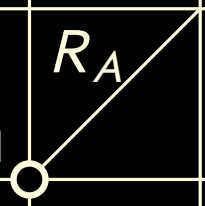

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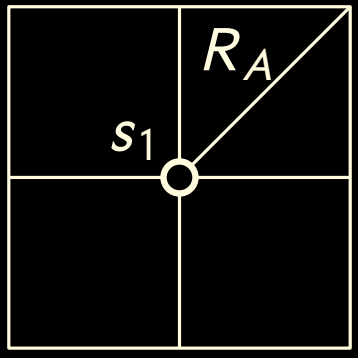

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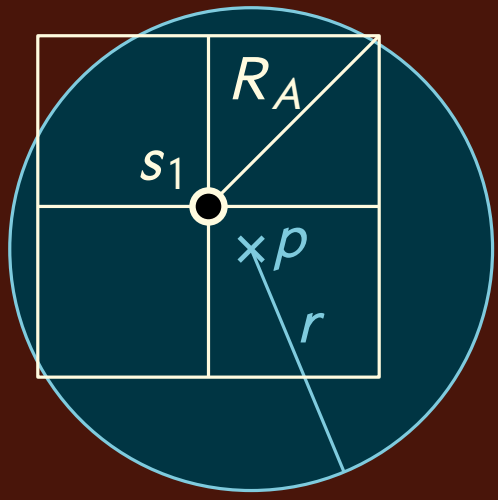

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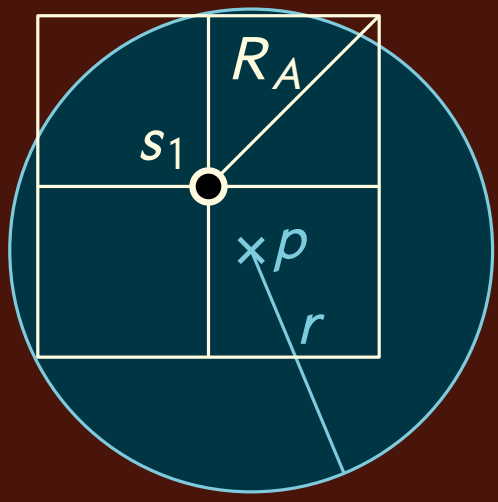

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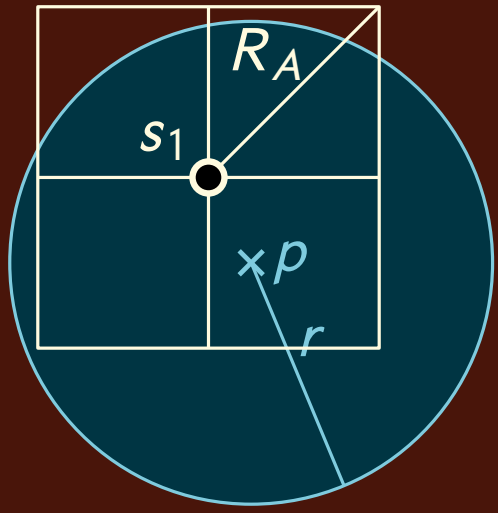

$$\Pr\left[A \text{ has Voronoi region} \mid R_A\right]$$

$$= \Pr\left[\exists r\ \exists p\colon \boxed{s_i \in D \text{ if } s_i \in A} \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid R_A\right]$$

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disk D: center p and radius r

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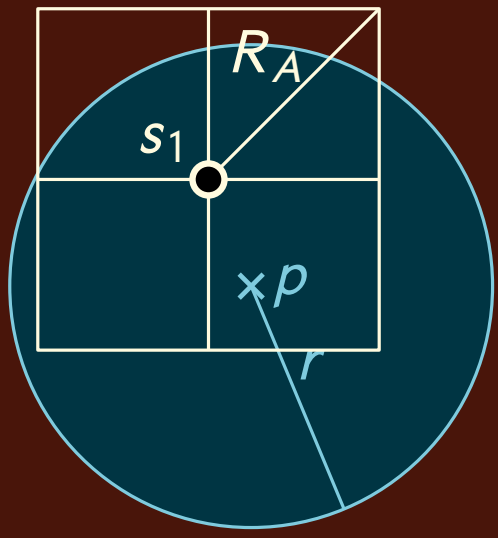

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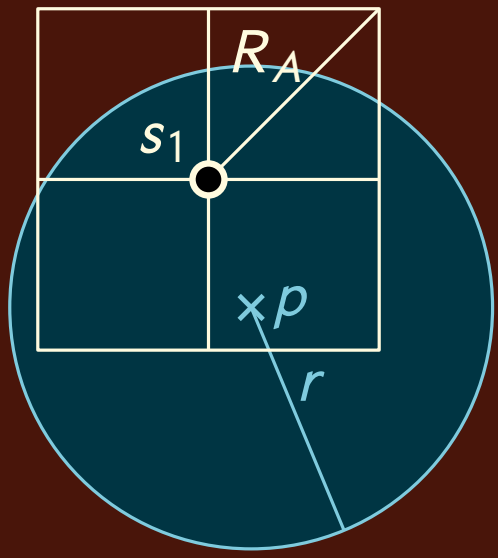

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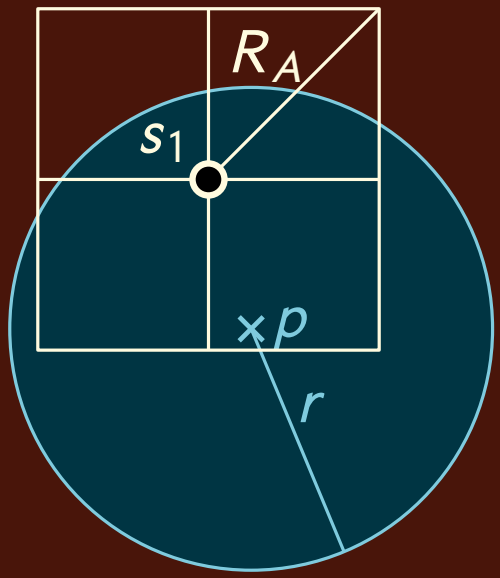

$$\begin{aligned}
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\leq\ &\Pr\left[\exists r \geq R_A\ \exists p\colon\ s_1 \in D \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]
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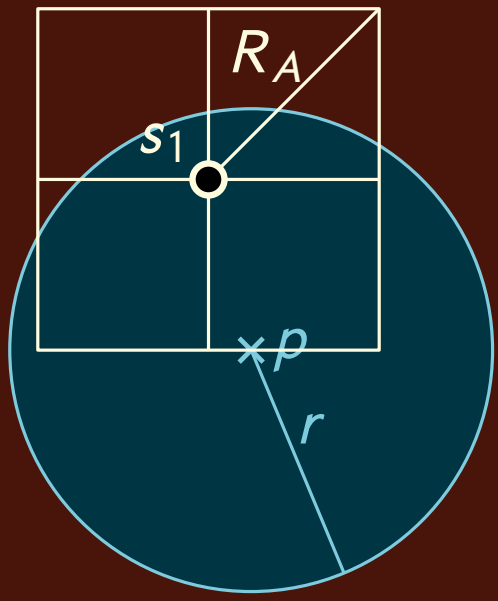

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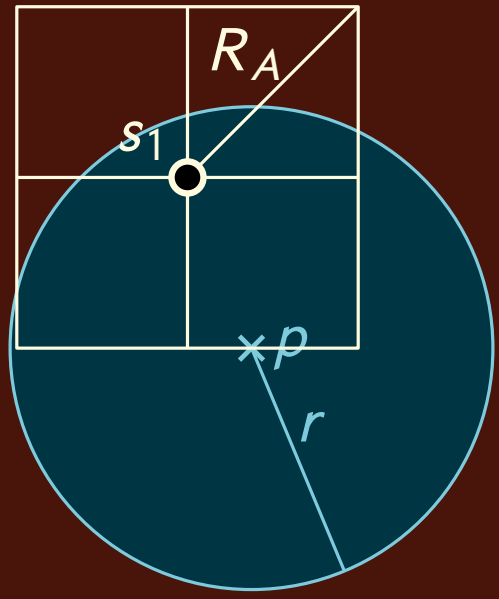

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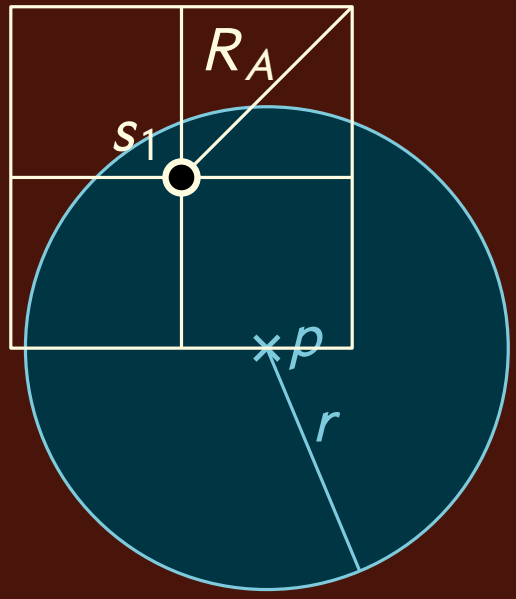

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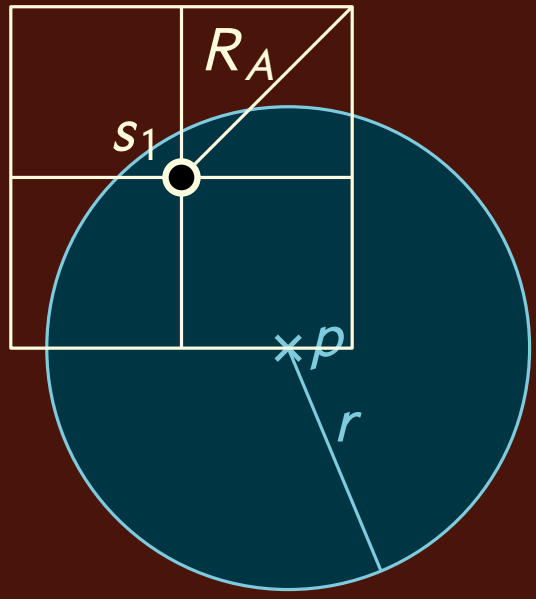

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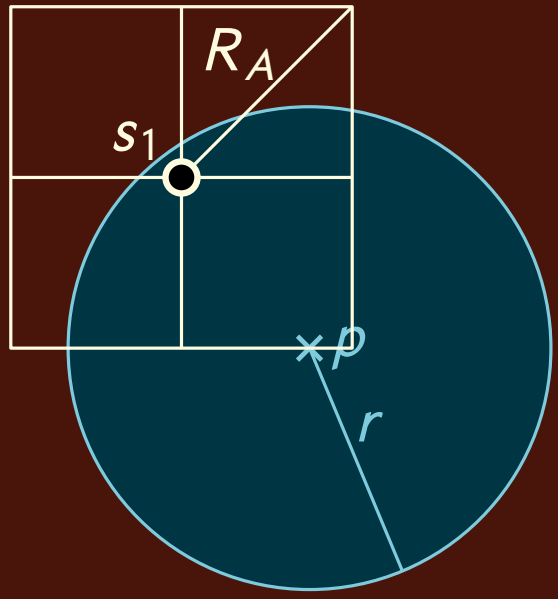

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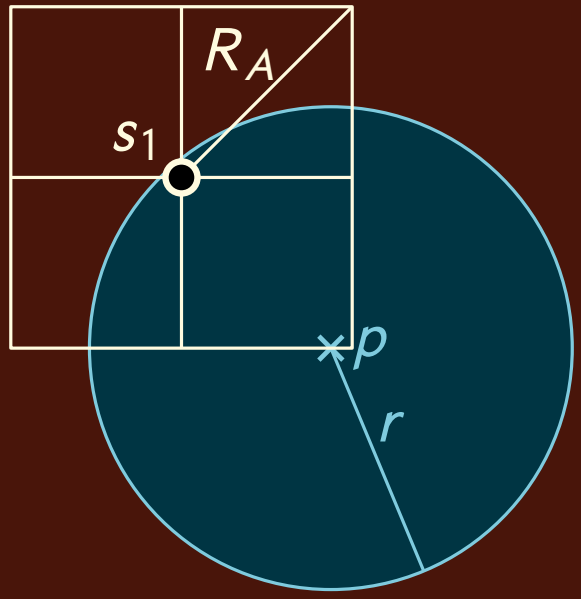

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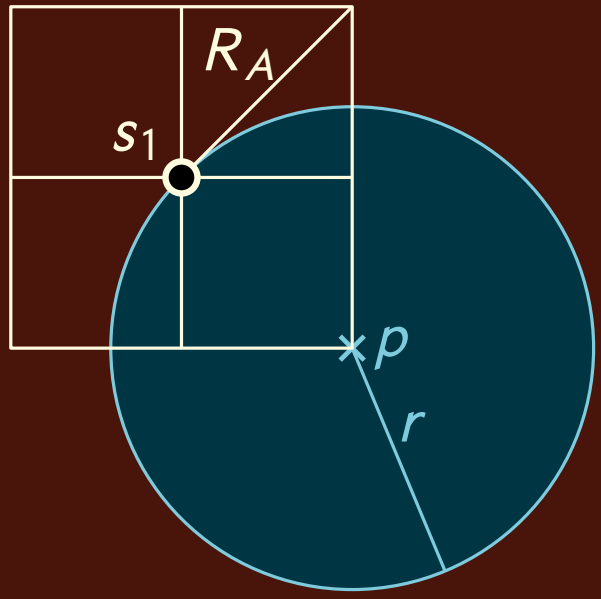

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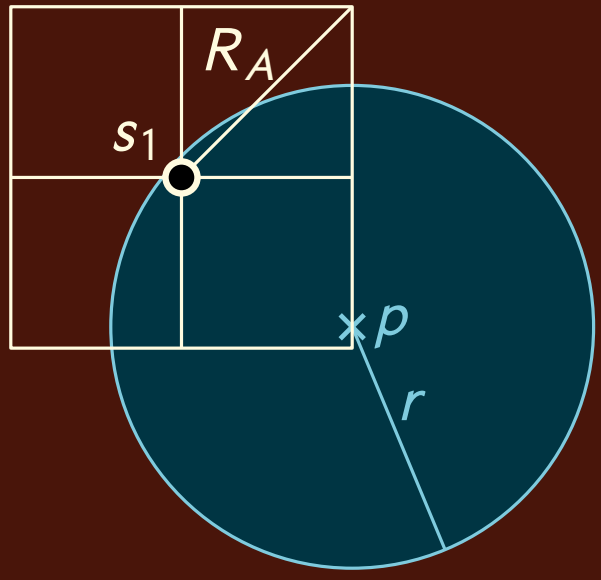

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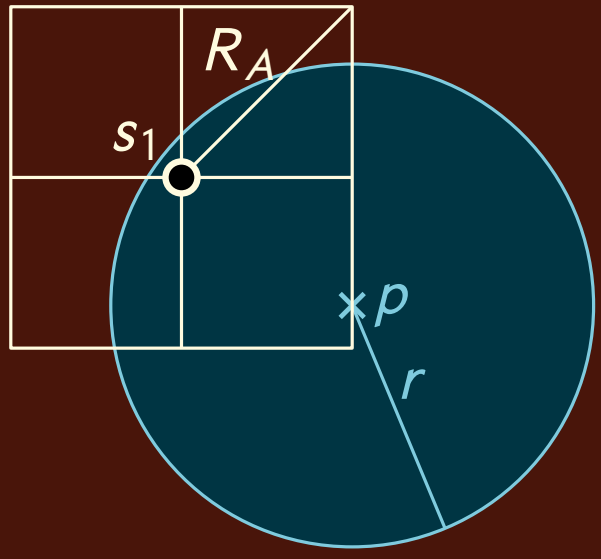

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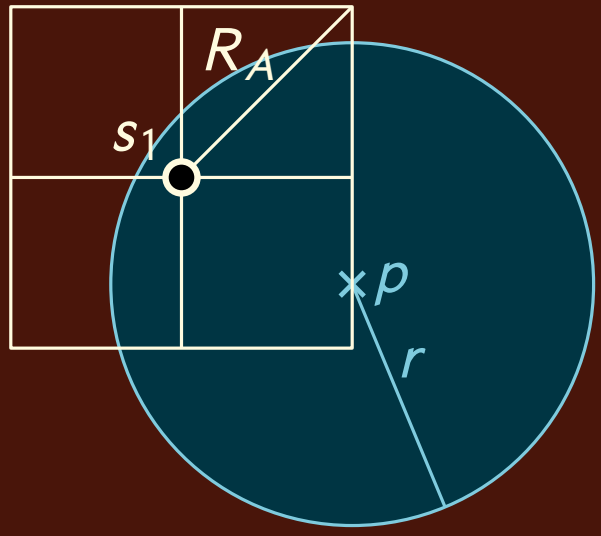

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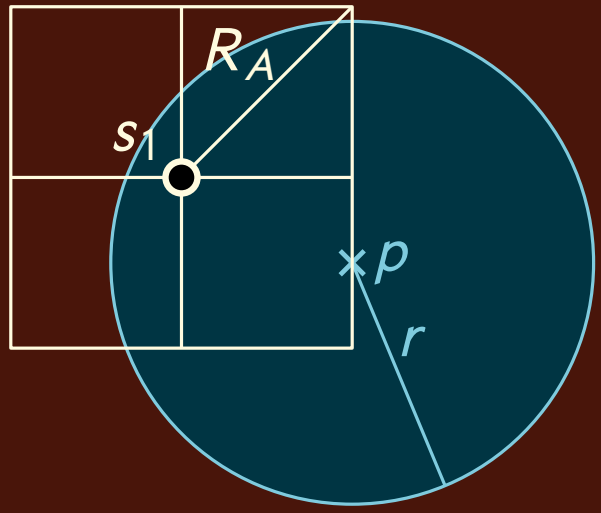

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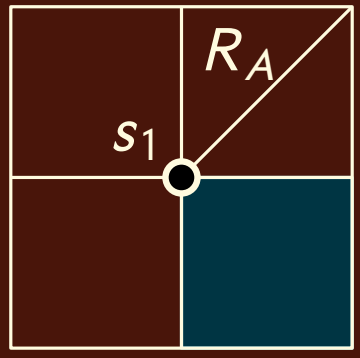

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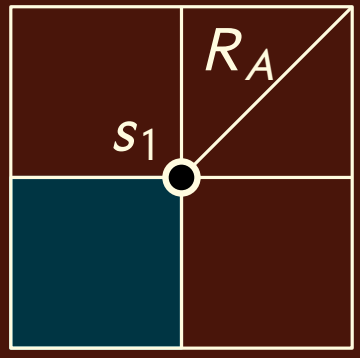

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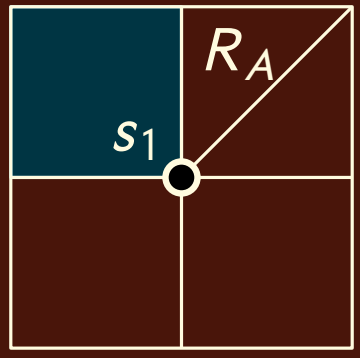

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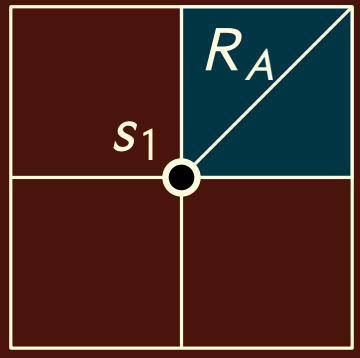

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$$\leq 4 \cdot \Pr\left[s_i \notin \text{ grid cell if } s_i \notin A\right] +$$

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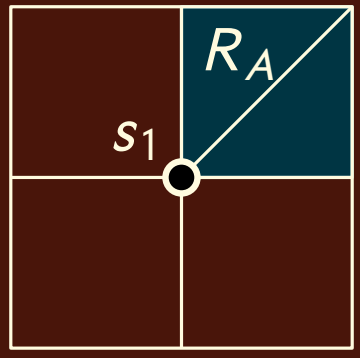

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$$= \Pr\left[\exists r \geq R_A\ \exists p\colon \boxed{s_i \in D \text{ if } s_i \in A} \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid R_A\right]$$

$$\leq \Pr\left[\exists \boxed{r \geq R_A}\ \exists p\colon\ s_1 \in D \quad \wedge \boxed{s_i \notin D \text{ if } s_i \notin A} \mid \boxed{R_A}\right]$$

$$\leq 4 \cdot \Pr\left[s_i \notin \text{ grid cell if } s_i \notin A\right] + \text{ something small}$$

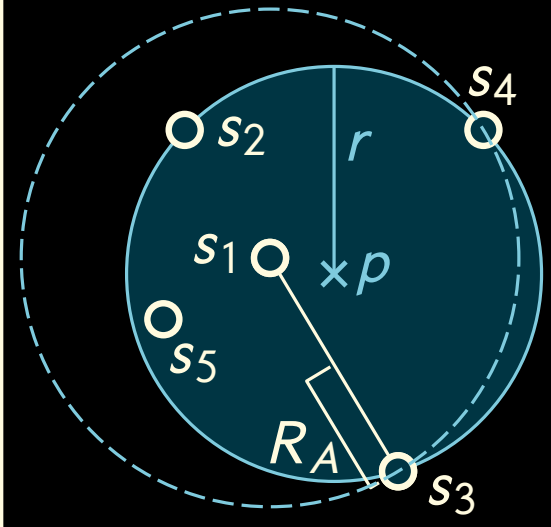
disk D: center p and radius r

all sites: $S = \{s_1, \ldots, s_n\}$

$A = \{s_1, \ldots, s_k\}$

$$\Pr[A \text{ has Voronoi region}]$$
$$= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A\right]$$

smallest r s.t. $\exists p\colon\ s_i \in D$ if $s_i \in A \Rightarrow r \geq R_A$



$$\Pr[A \text{ has Voronoi region}]$$
$$= \int_r \Pr\left[A \text{ has region} \mid R_A = r\right] f_{R_A}(r)\,\mathrm{d}r$$

$$\Pr\left[A \text{ has Voronoi region} \mid R_A\right]$$
$$= \Pr\left[\exists r\ \exists p\colon\ s_i \in D \text{ if } s_i \in A \ \wedge\ s_i \notin D \text{ if } s_i \notin A \mid R_A\right]$$
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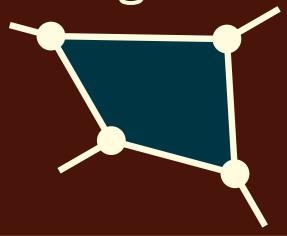
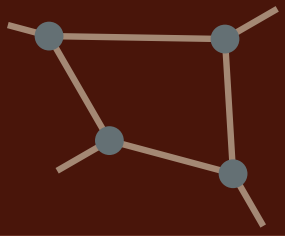
$$\leq 4 \cdot \Pr[s_i \notin \text{ grid cell if } s_i \notin A] + \text{ something small}$$

#regions or #vertices





O(total weight)
weights ≥ 1

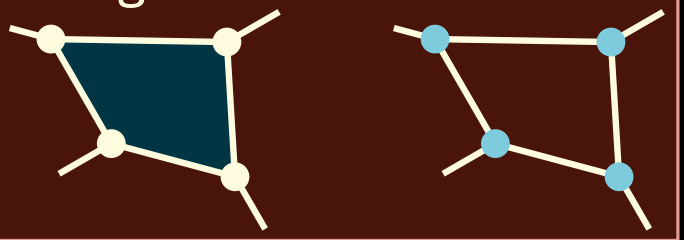
order $\in O(1)$

Theorem

random sites $\Longrightarrow$ low complexity Voronoi diagram

hypercube with p-norm
dimension $\in O(1)$

#regions or #vertices



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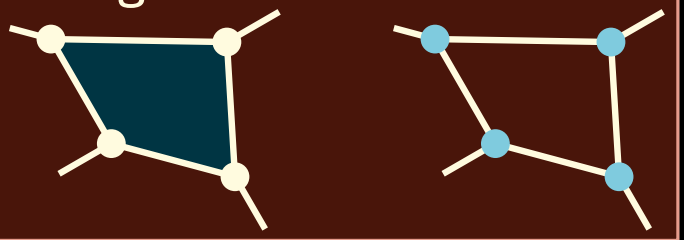
Theorem

many vertices $\Longrightarrow$ many regions in higher order

#regions or #vertices



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ℓ

$\Omega(\ell)$

Theorem

many vertices $\Longrightarrow$ many regions in higher order

dim d, order $k \longrightarrow$ order $(k + d)$

Theorem
many vertices $\Longrightarrow$ many regions in higher order

Theorem

many vertices ℓ $\Longrightarrow$ many regions $\Omega(\ell)$ in higher order

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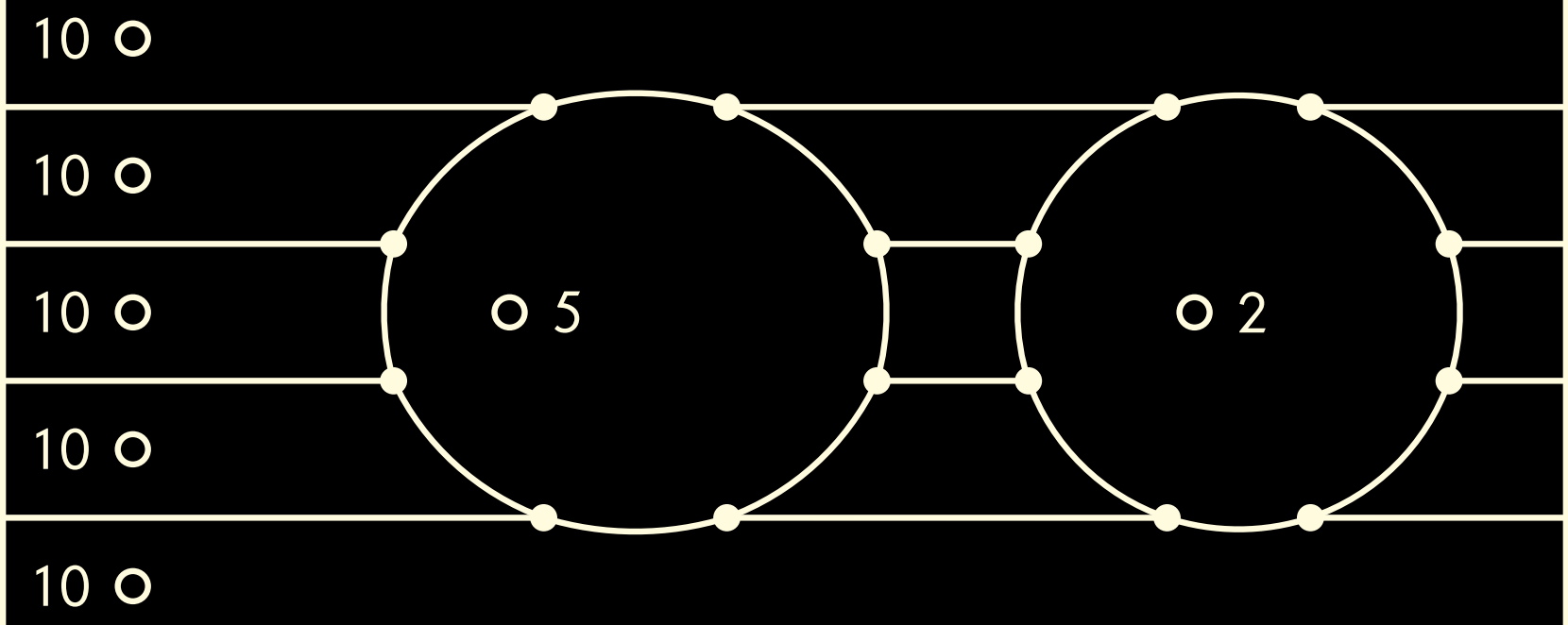

Theorem

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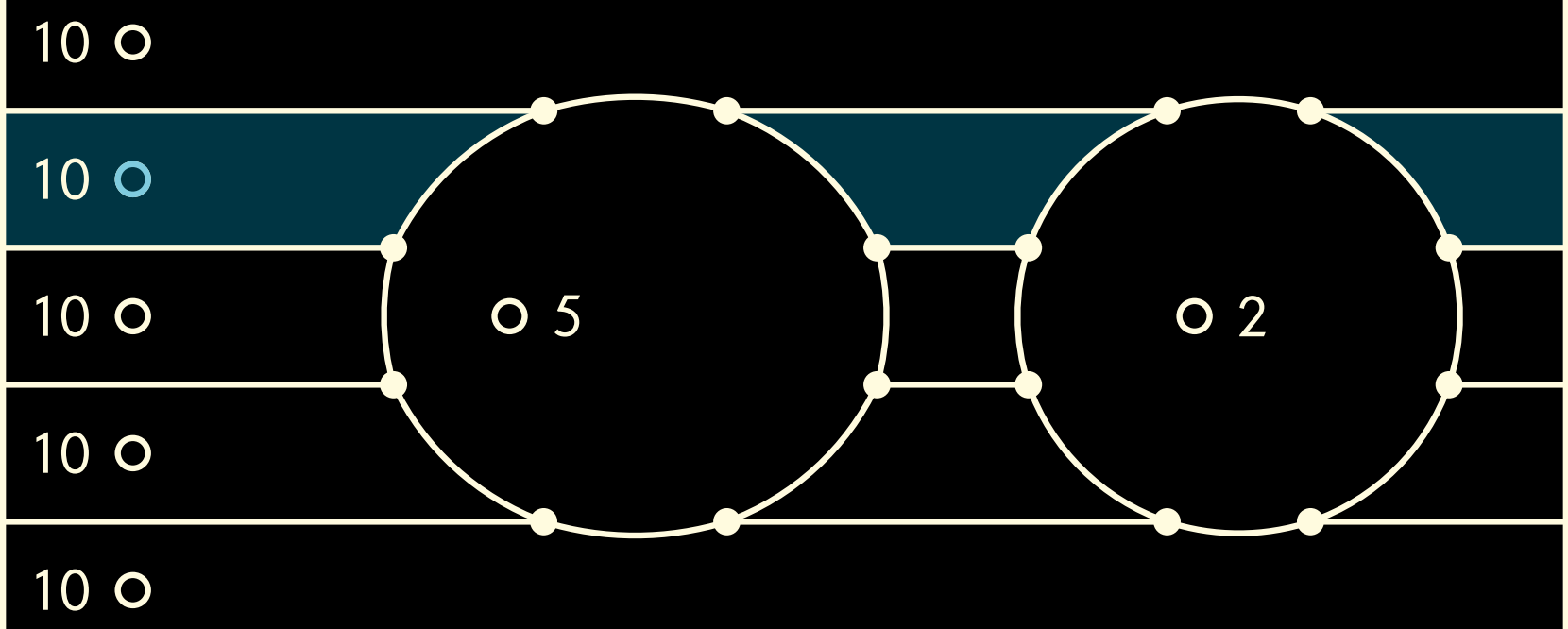

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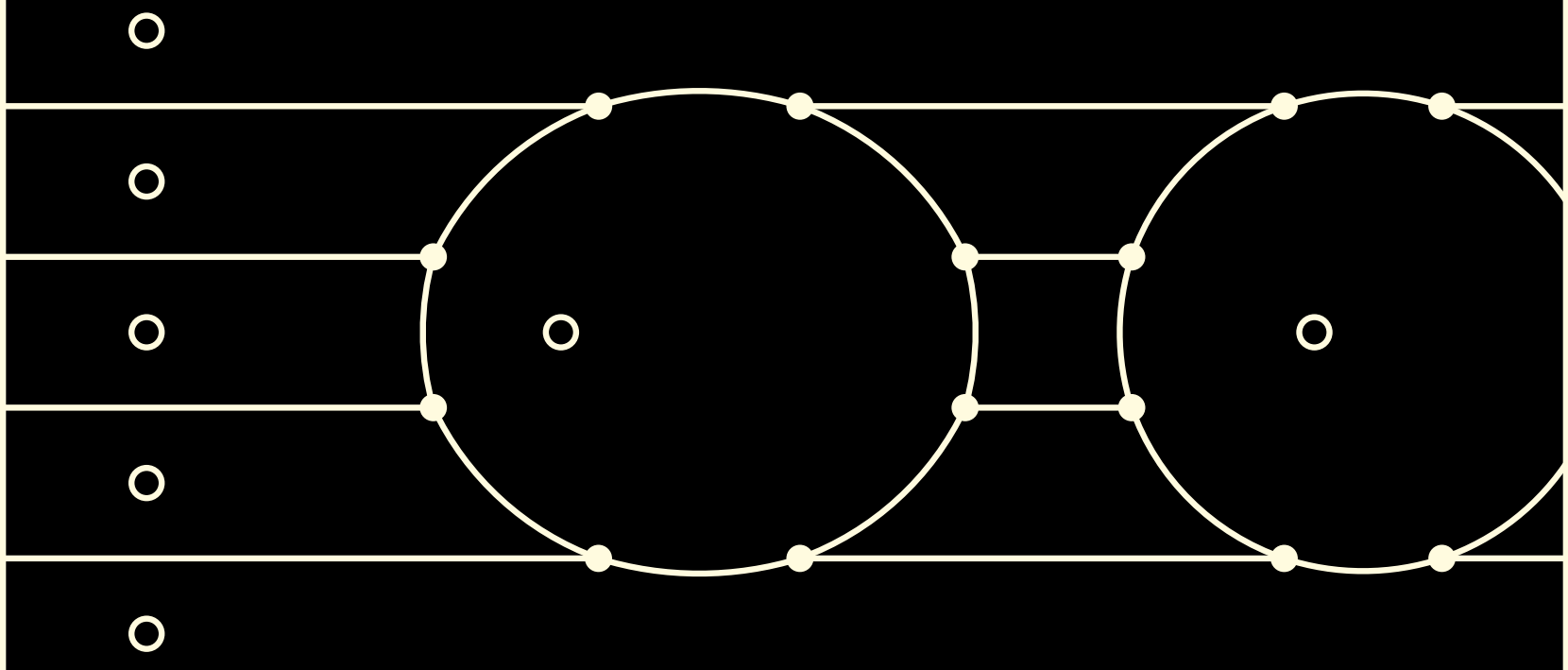

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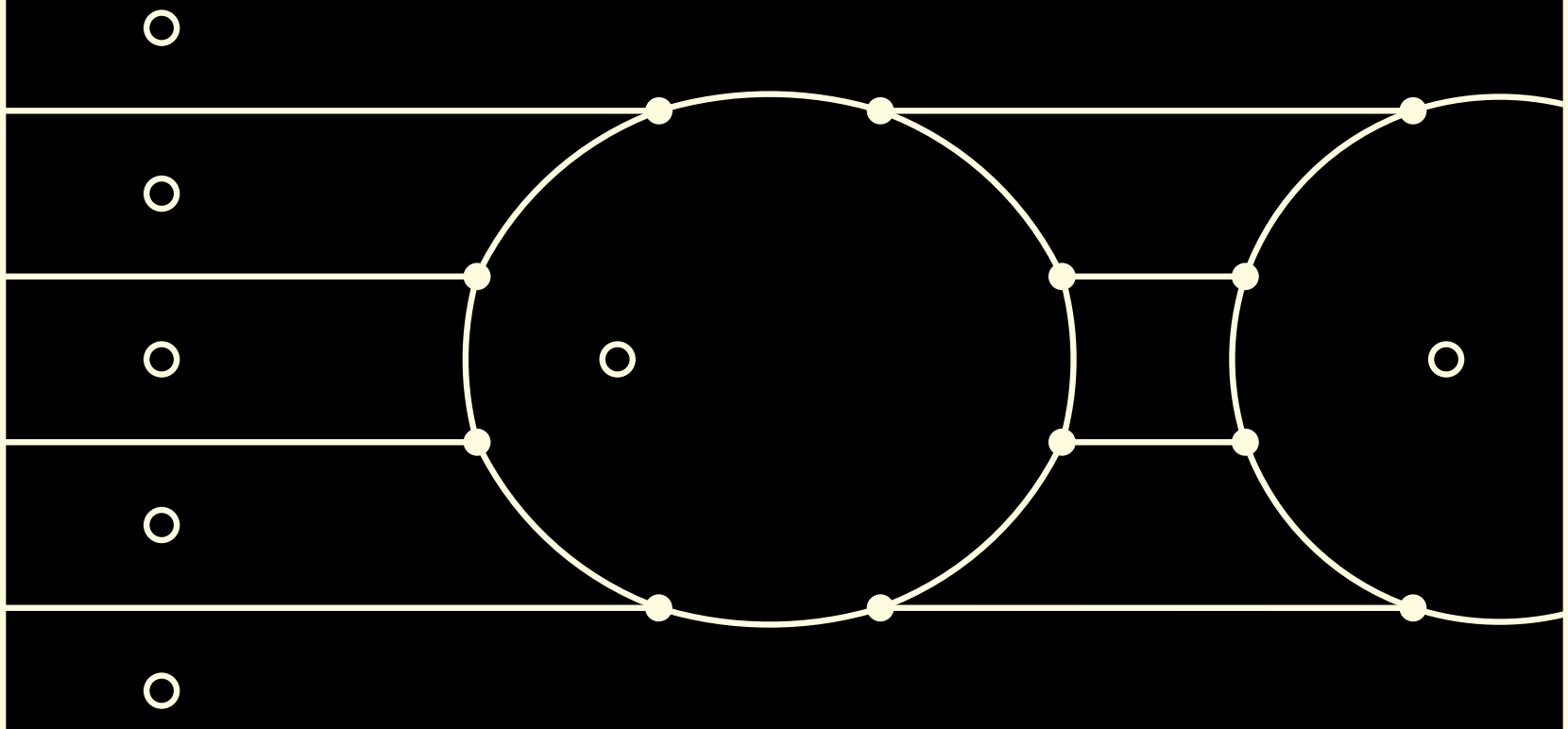

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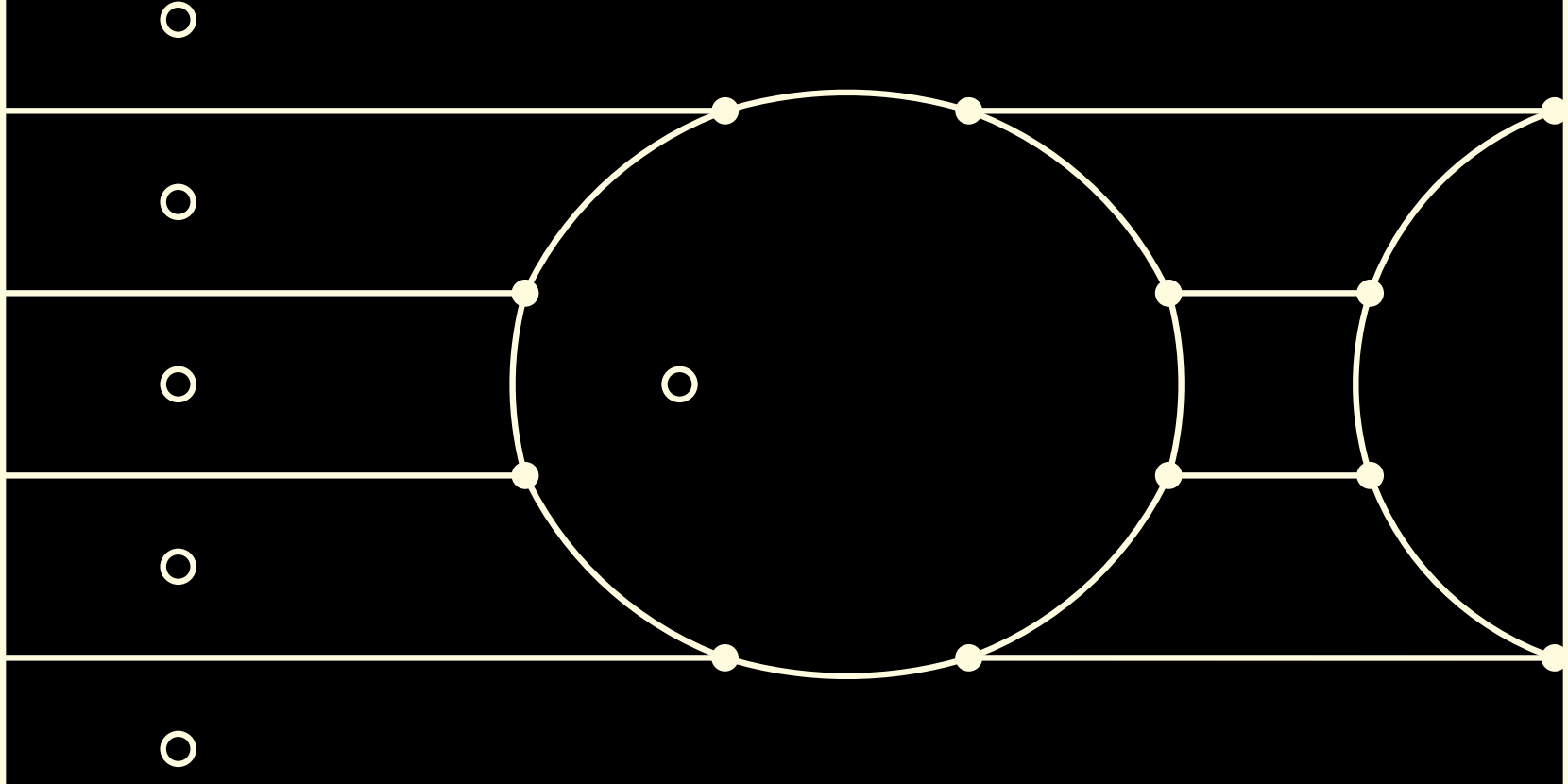

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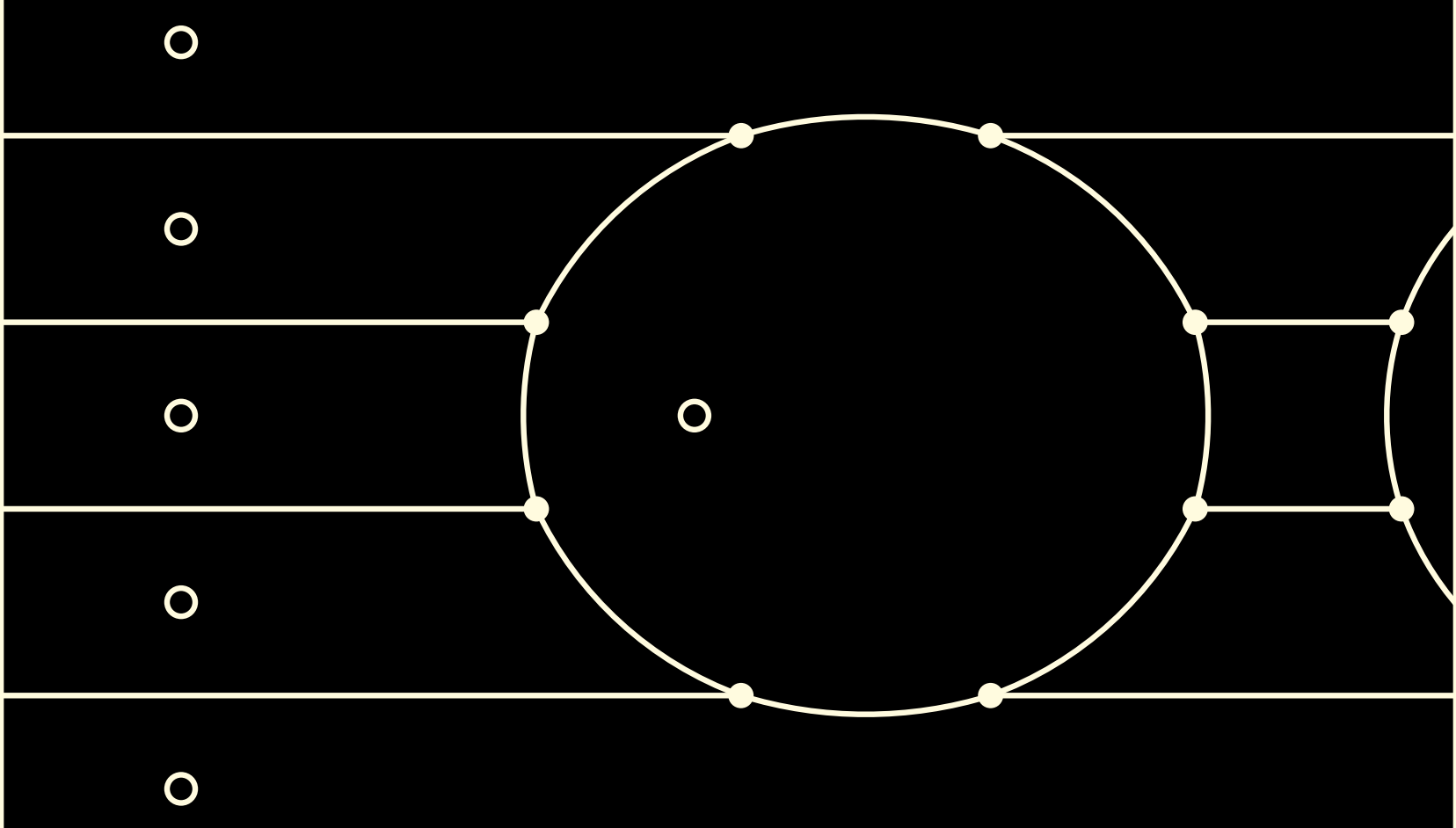

Theorem

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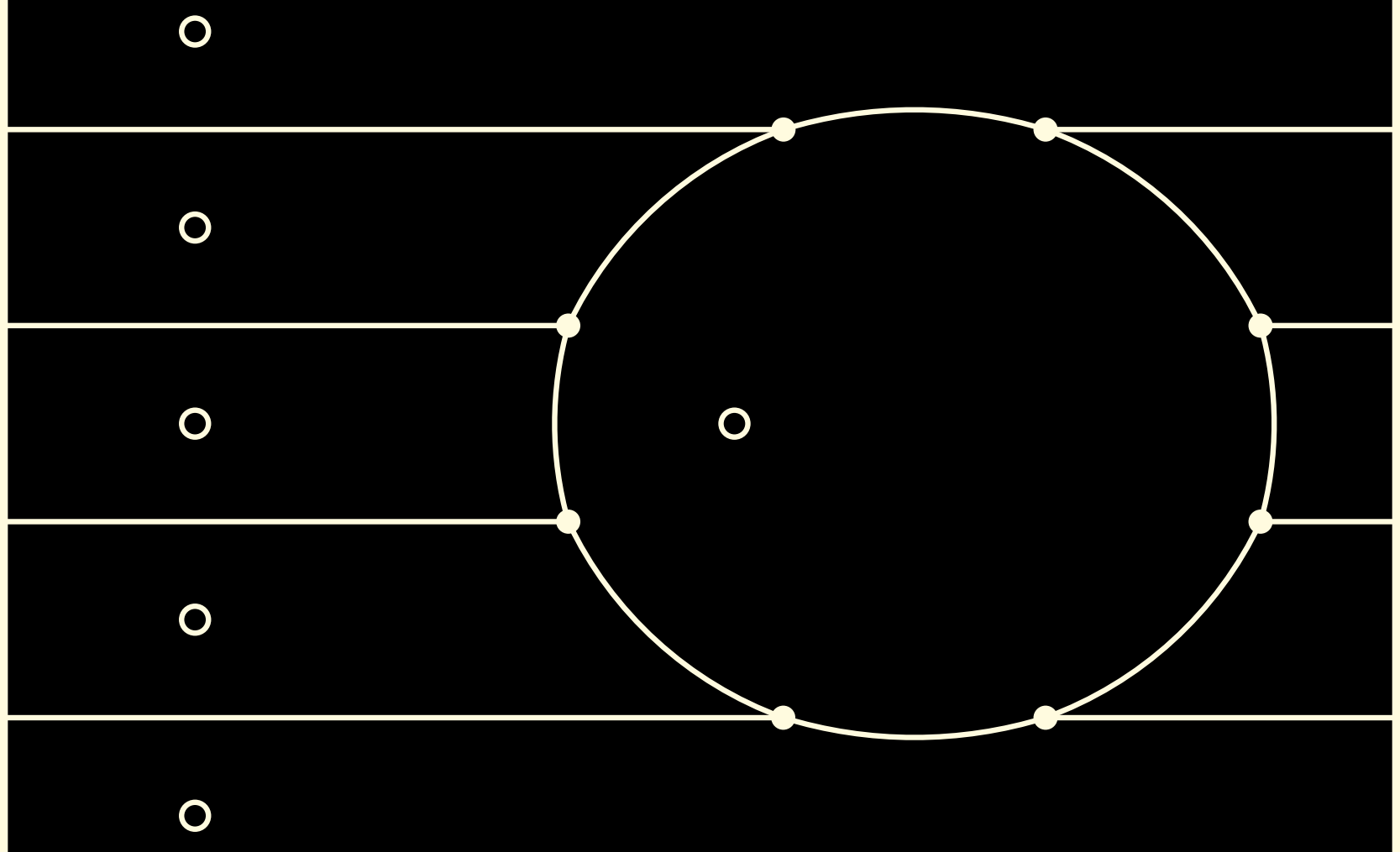

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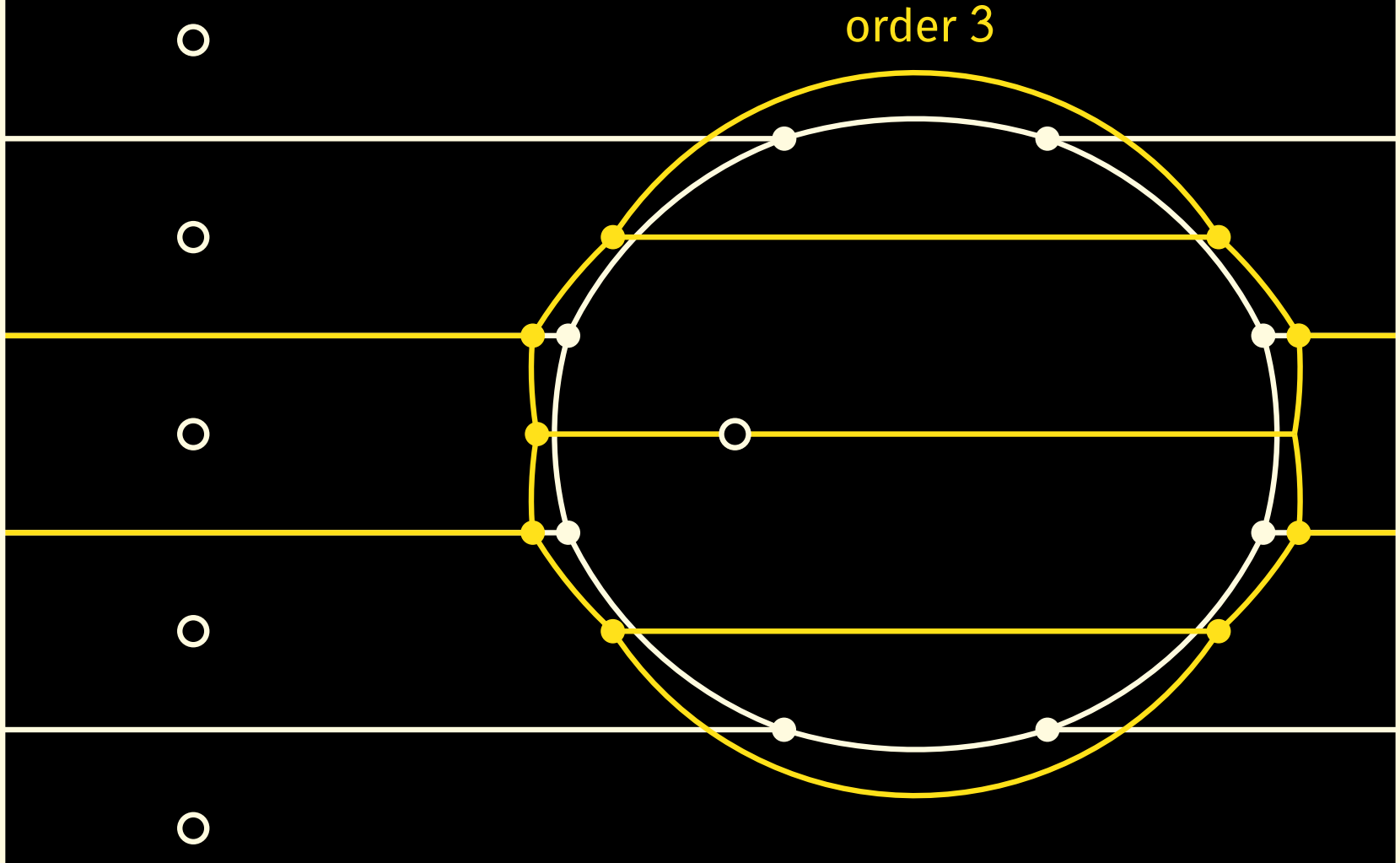

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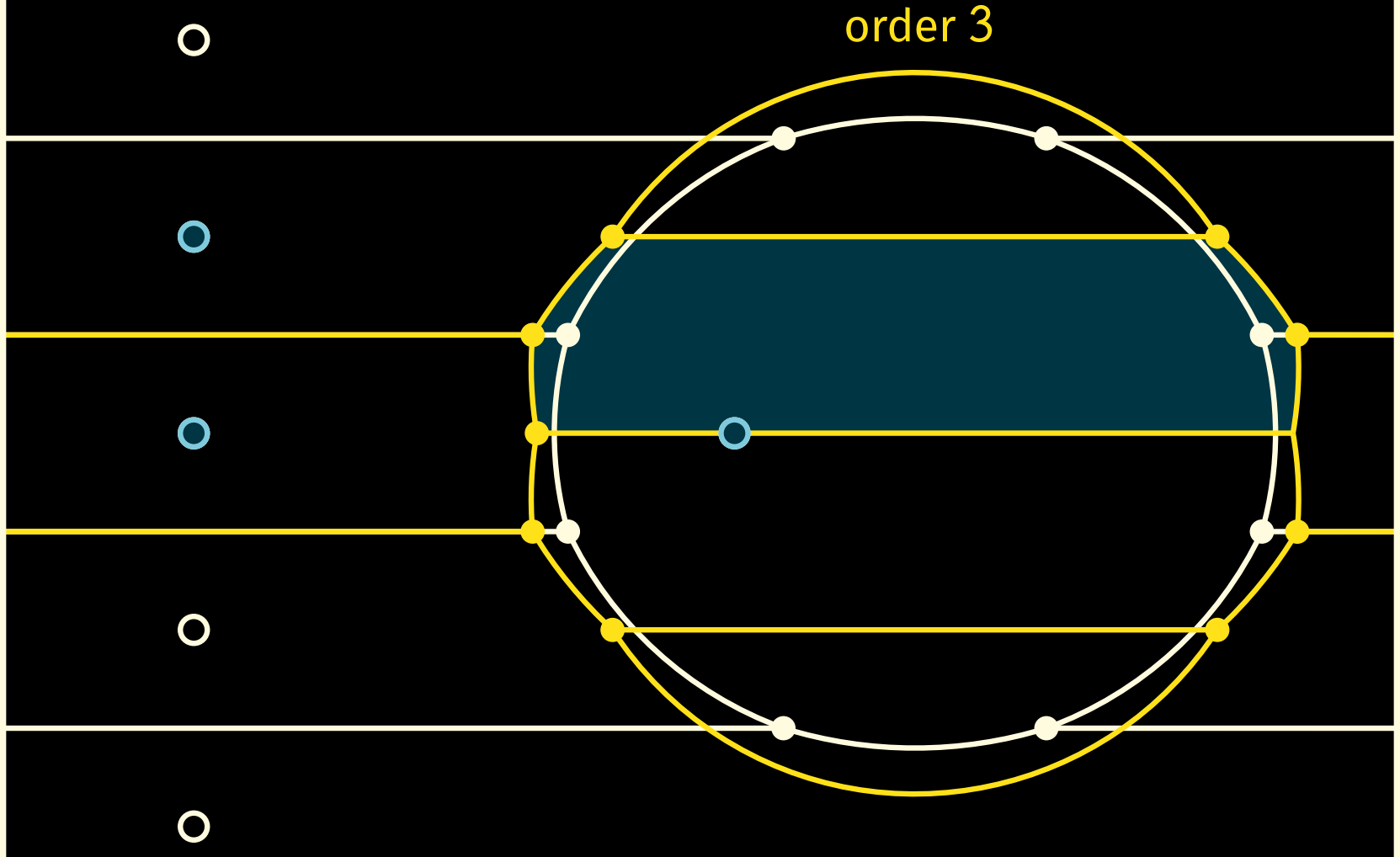

Theorem

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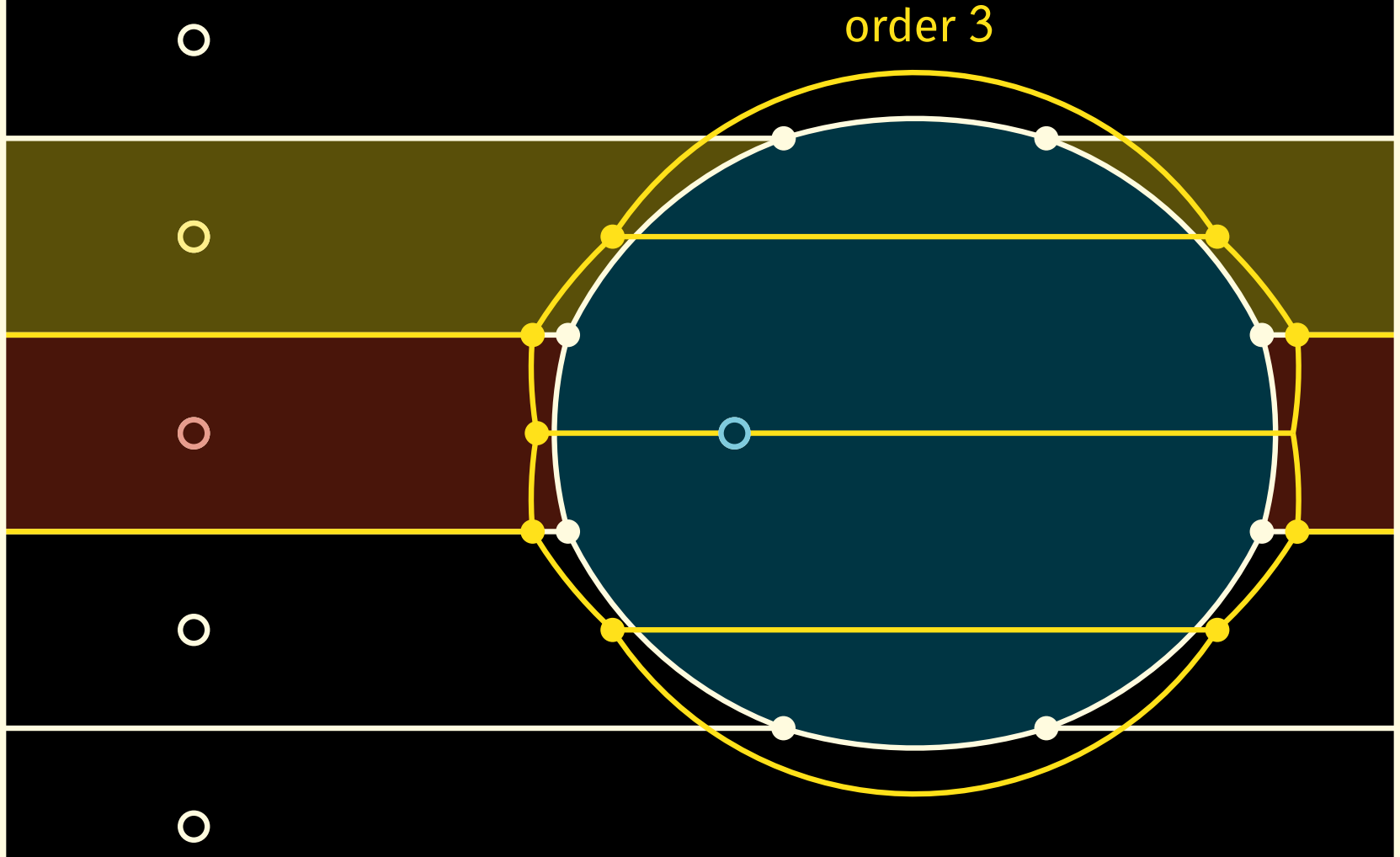
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