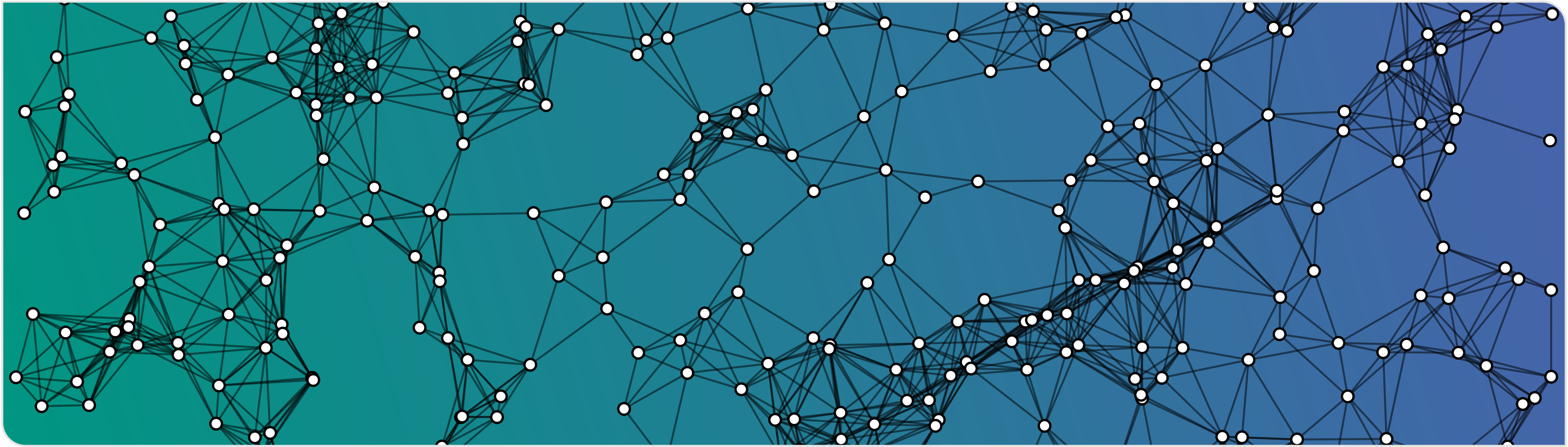
Karlsruher Institut für Technologie

Deterministic Performance Guarantees for Bidirectional BFS on Real-World Networks

Marcus Wilhelm, Thomas Bläsius

Theorietag 2022



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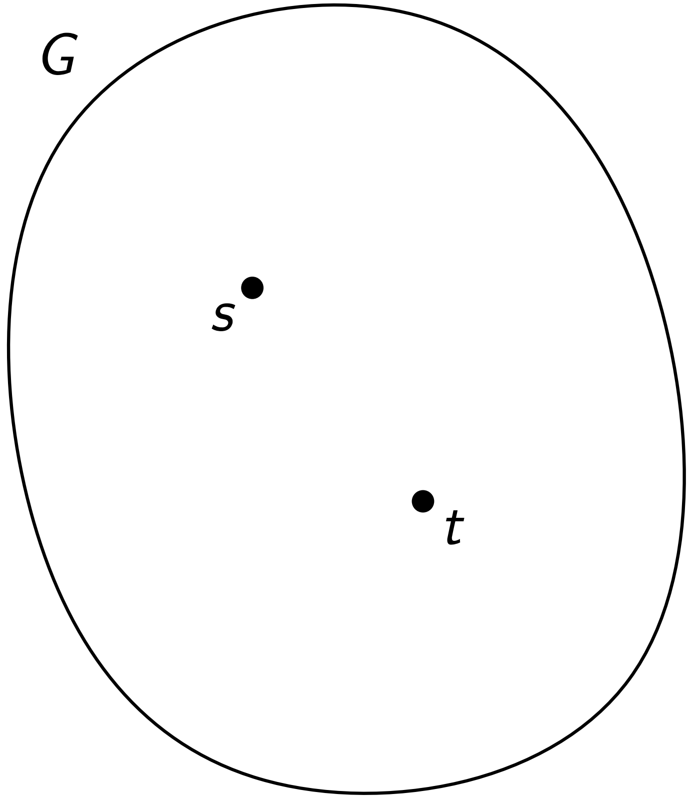

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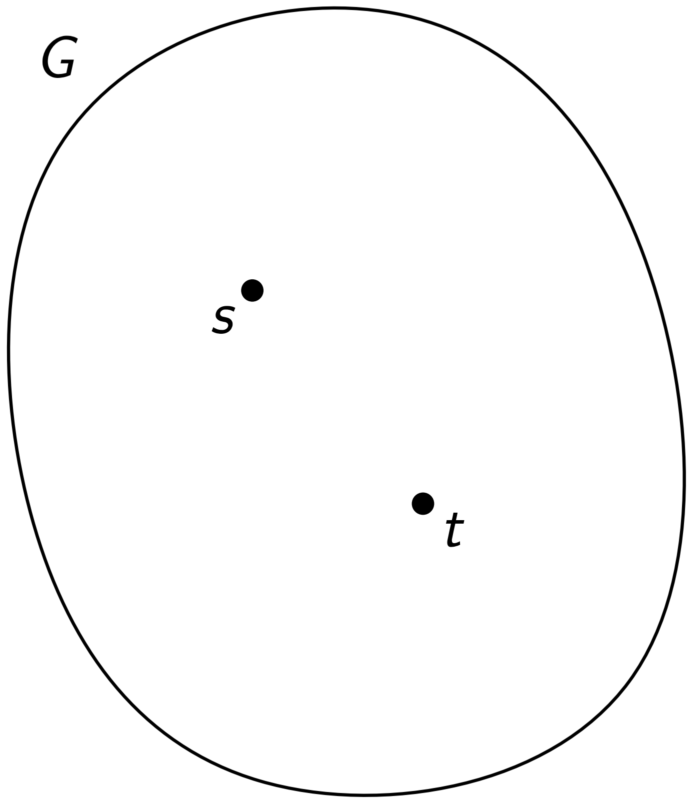

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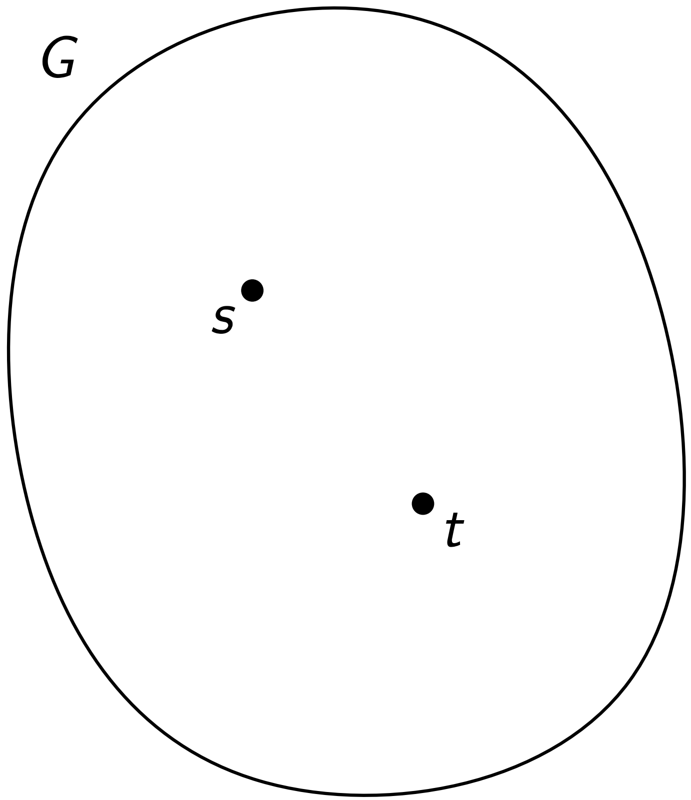

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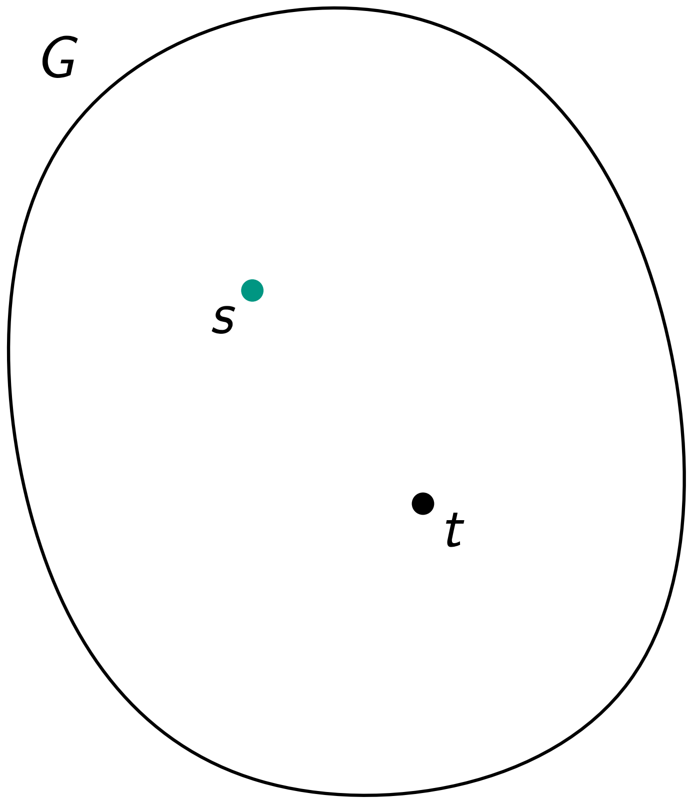

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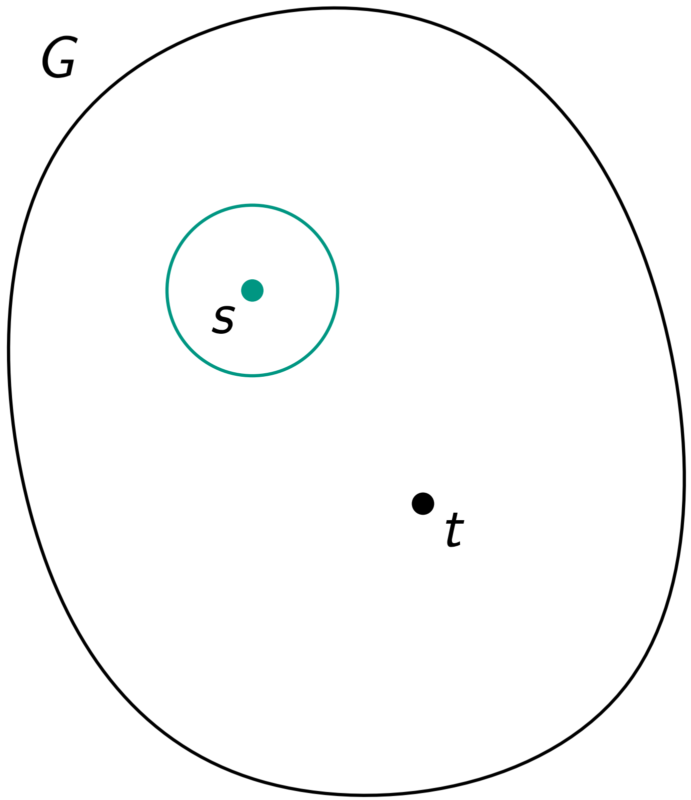

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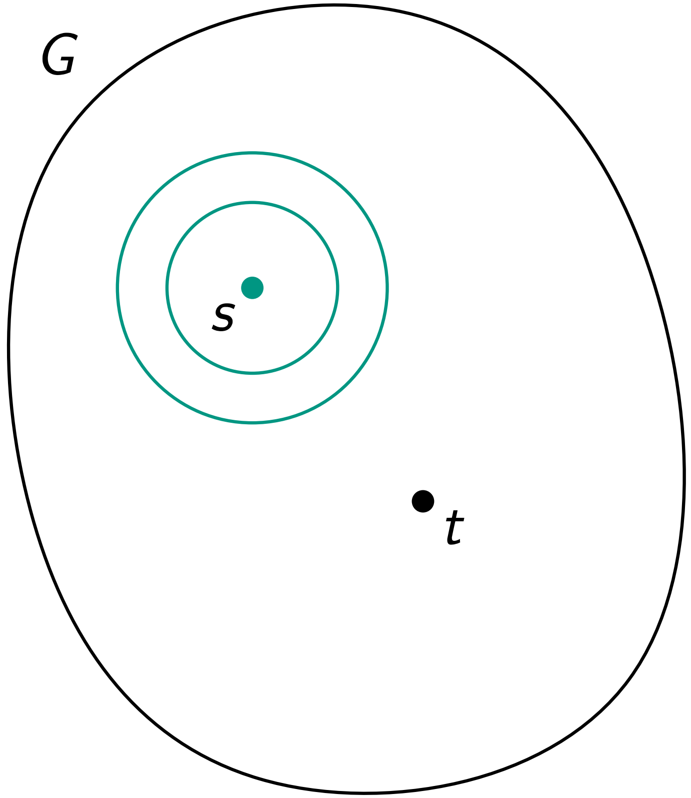

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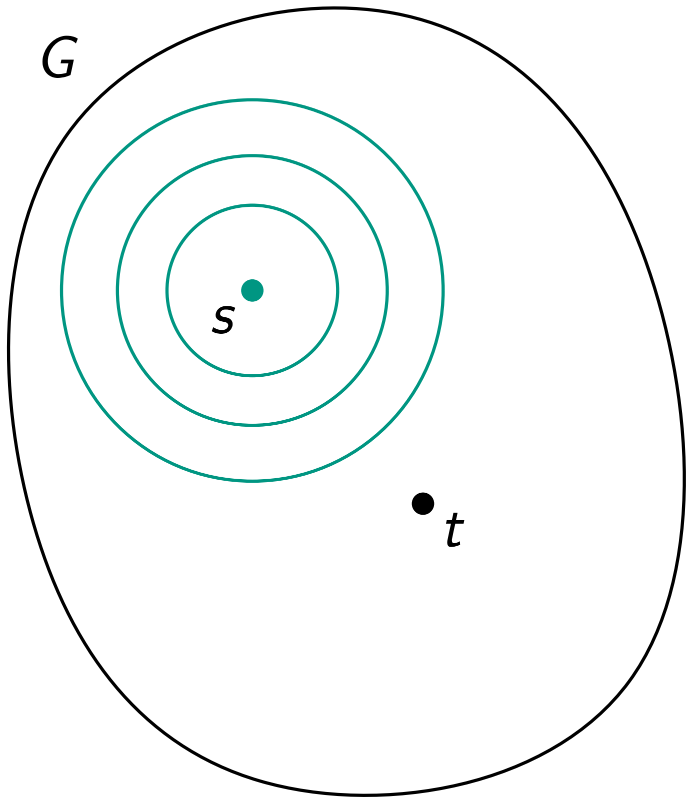

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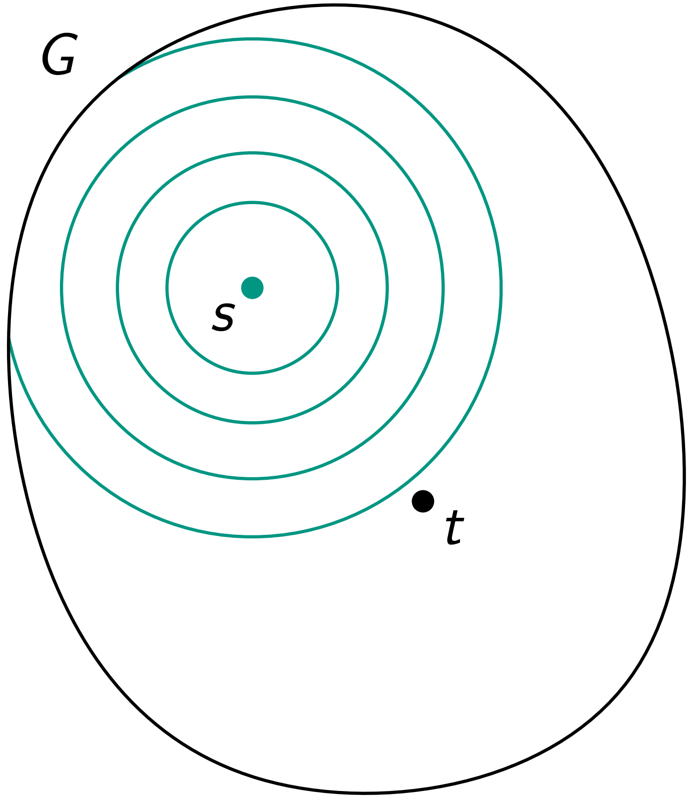

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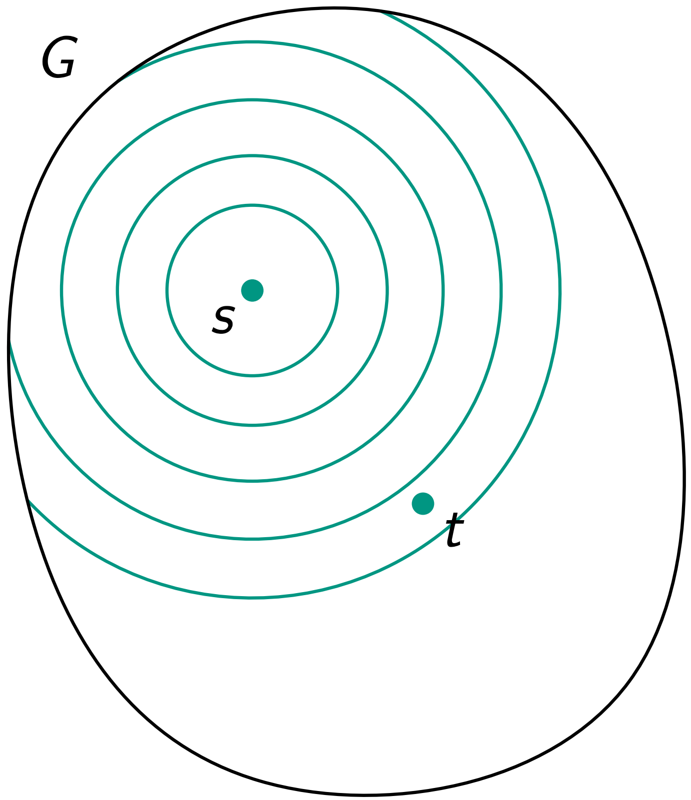

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Given: Graph G, $s, t \in V(G)$

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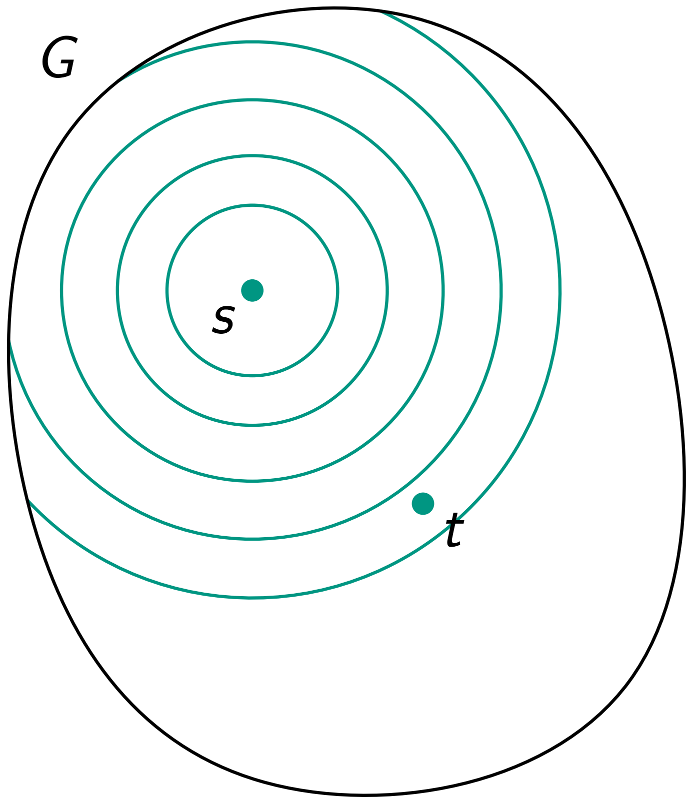

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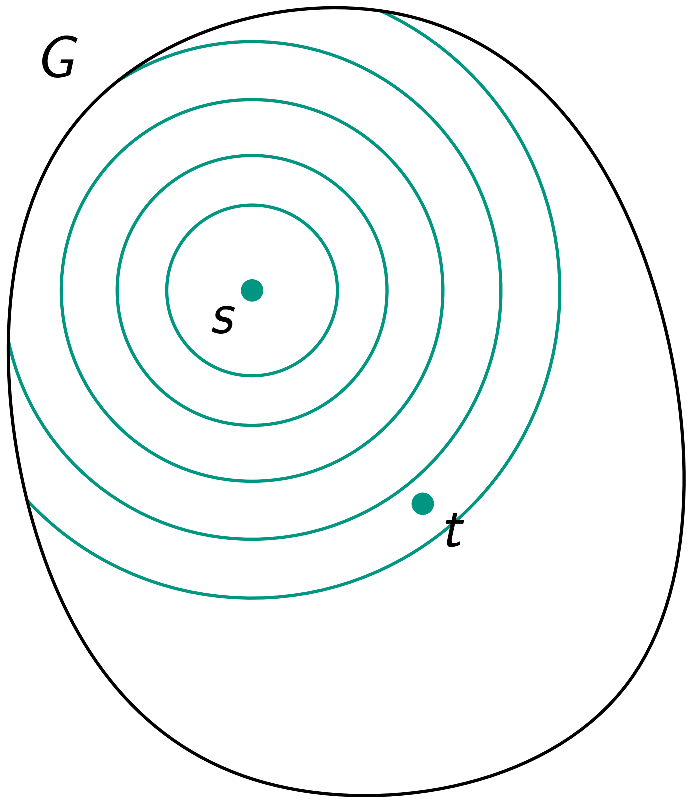

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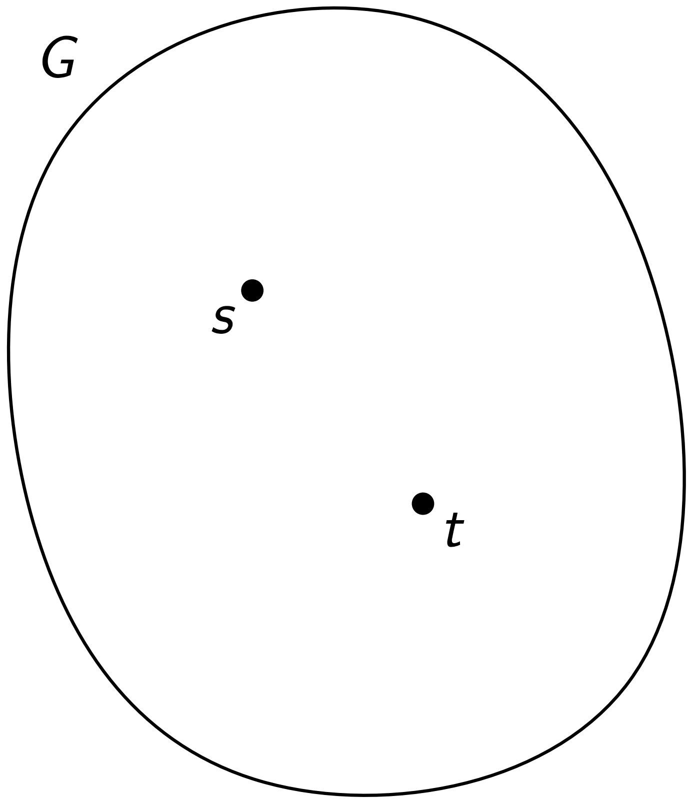

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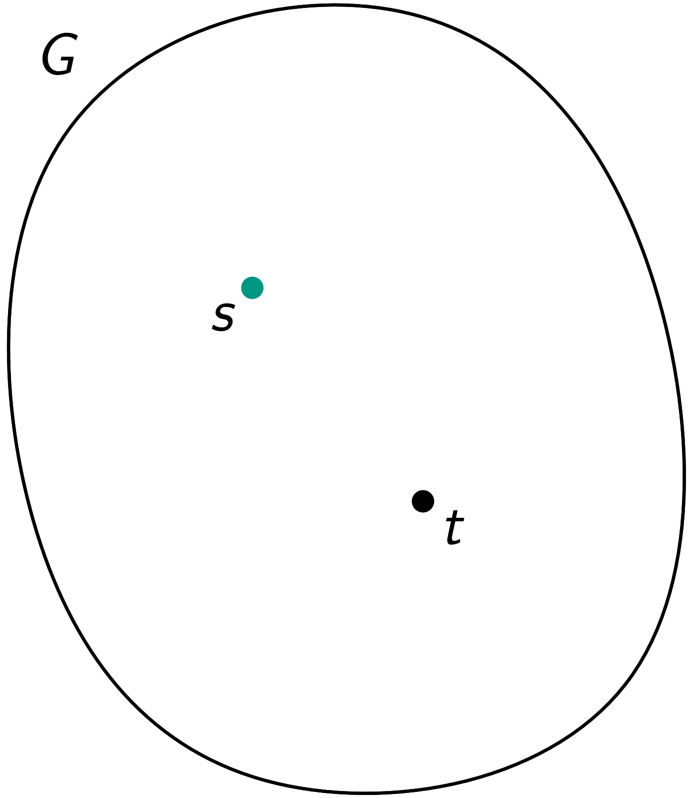

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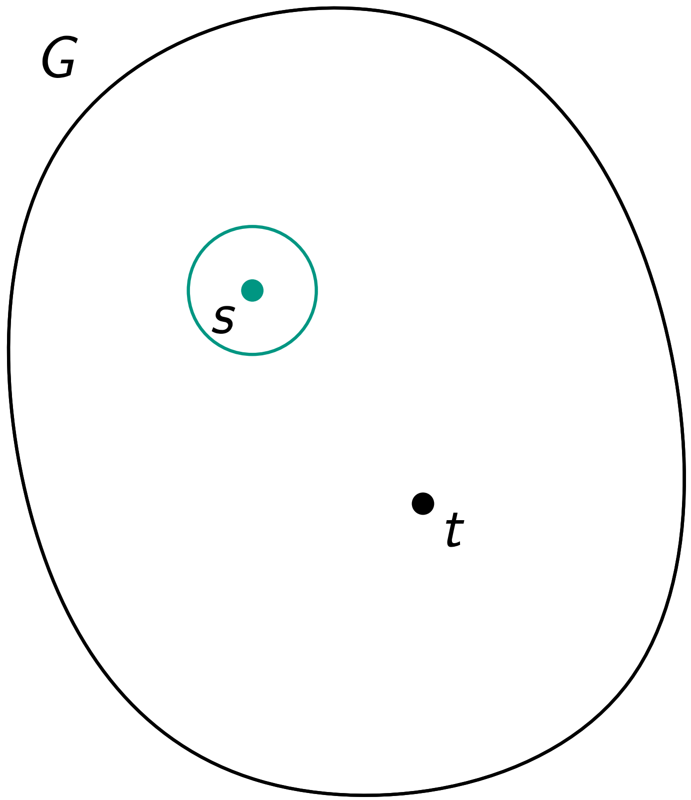

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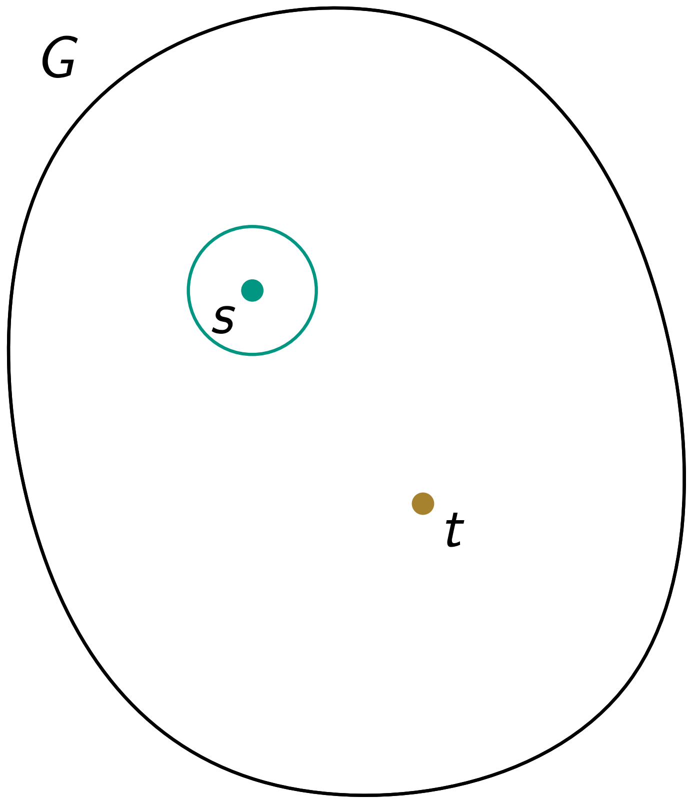

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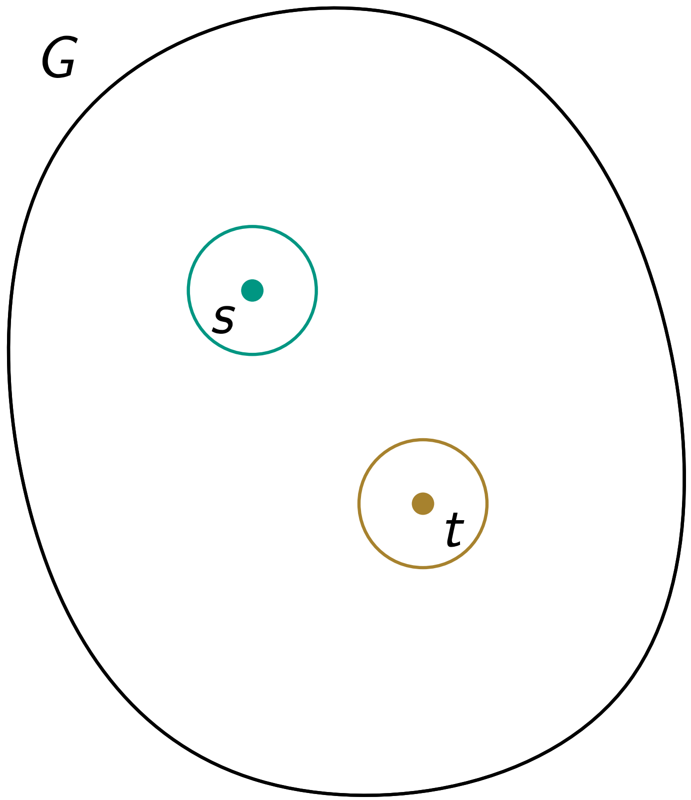

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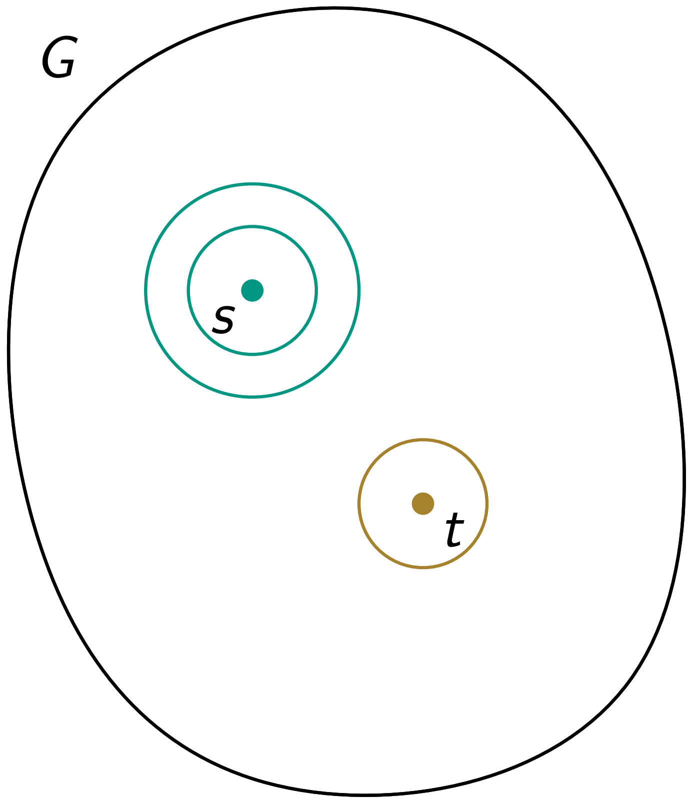

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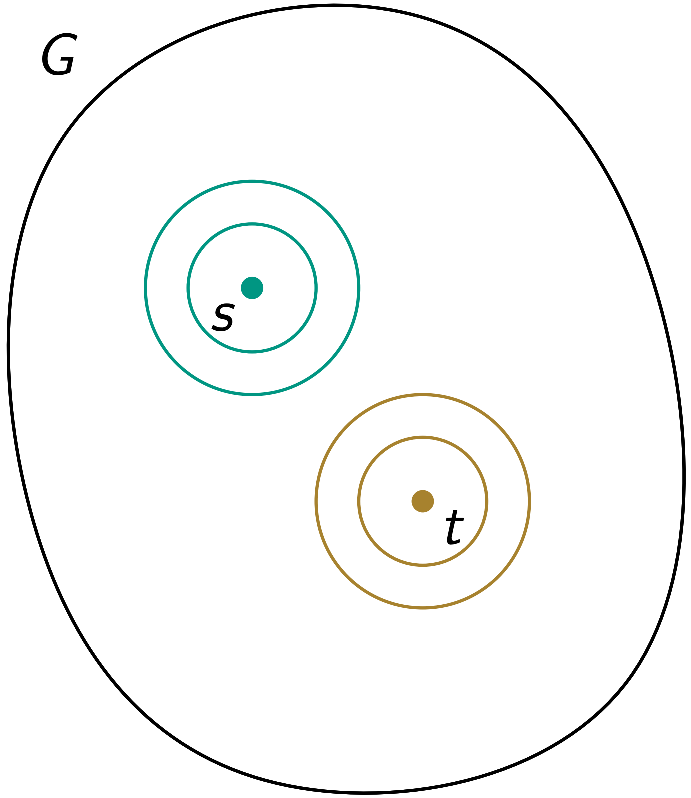

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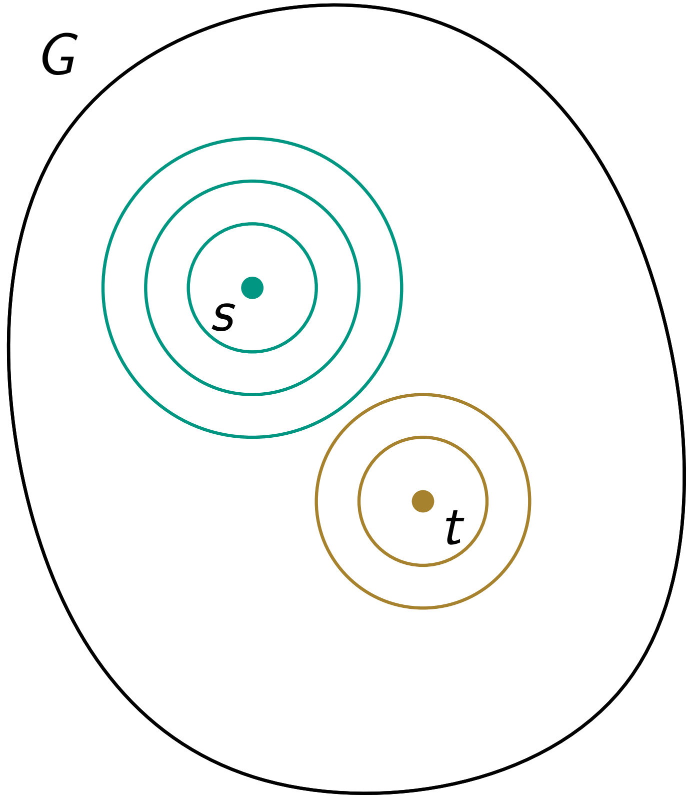

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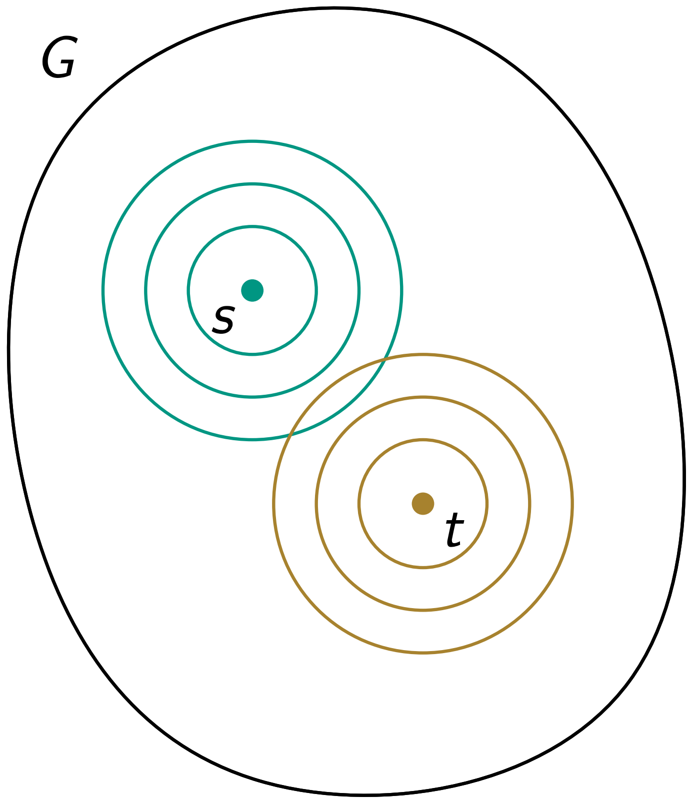

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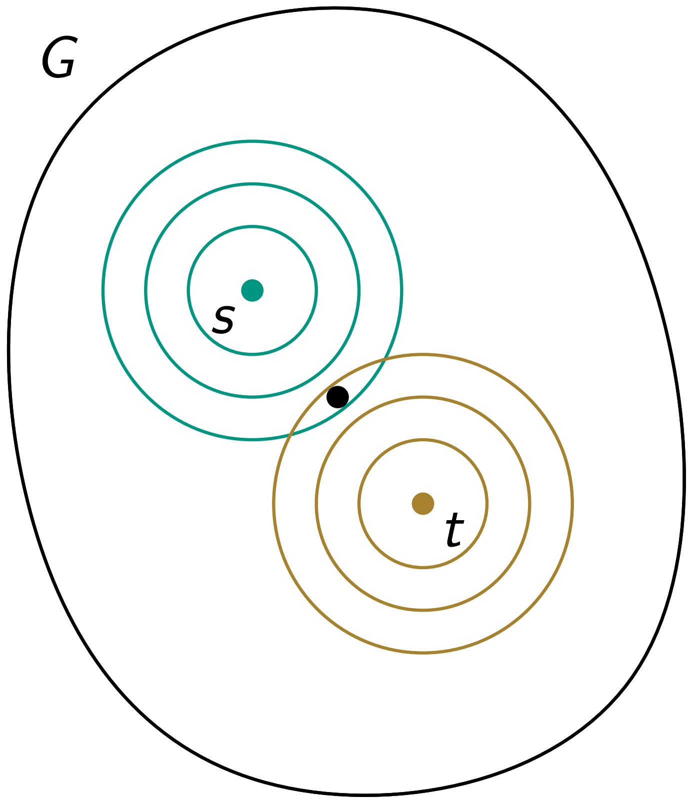

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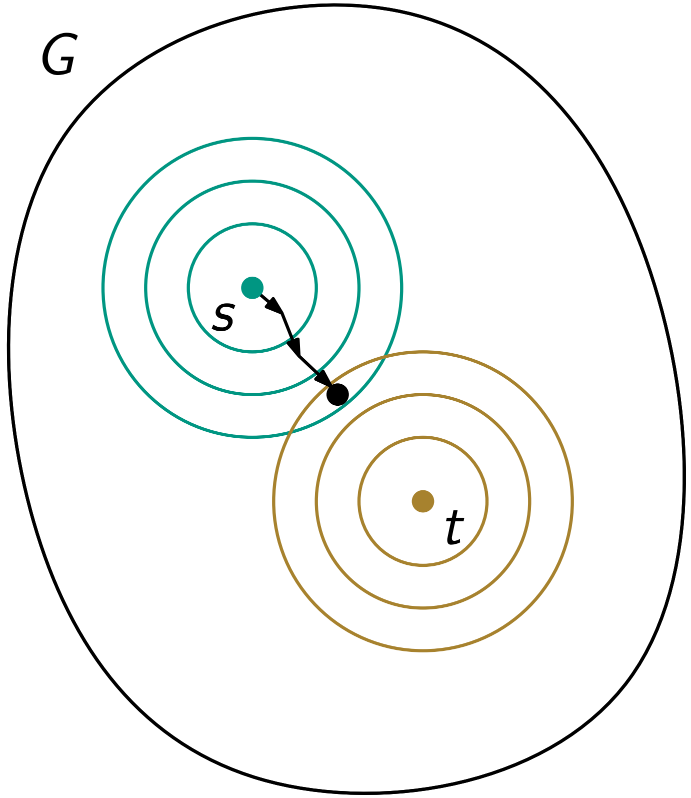

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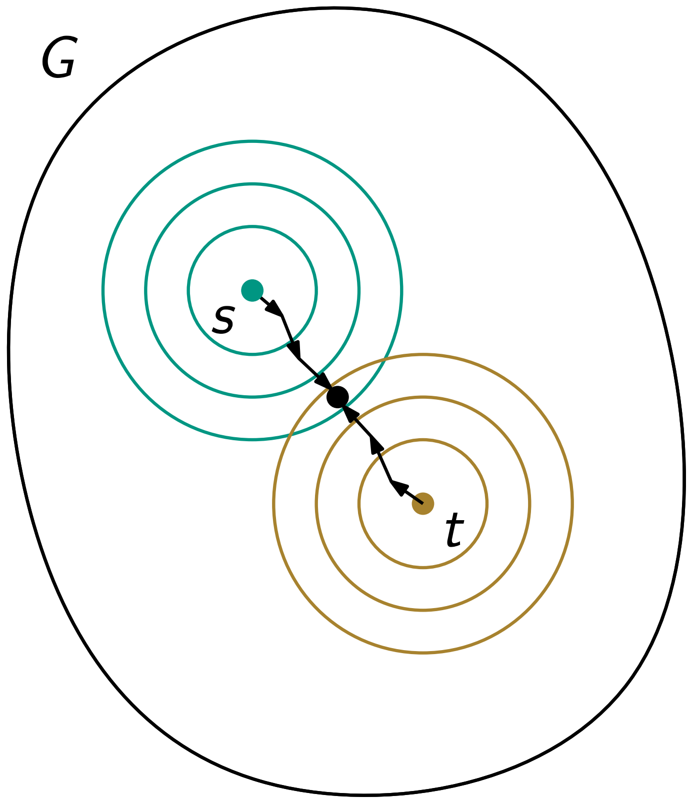

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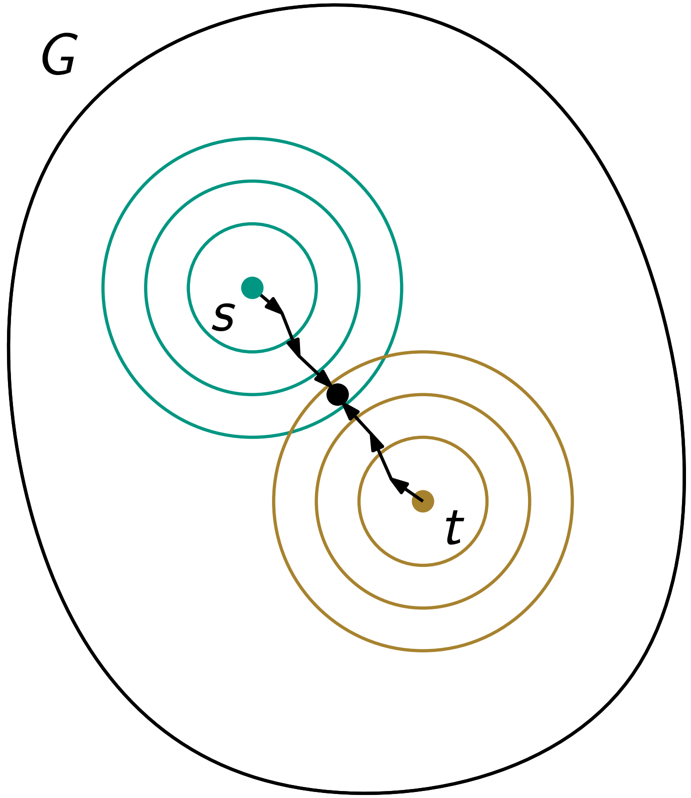

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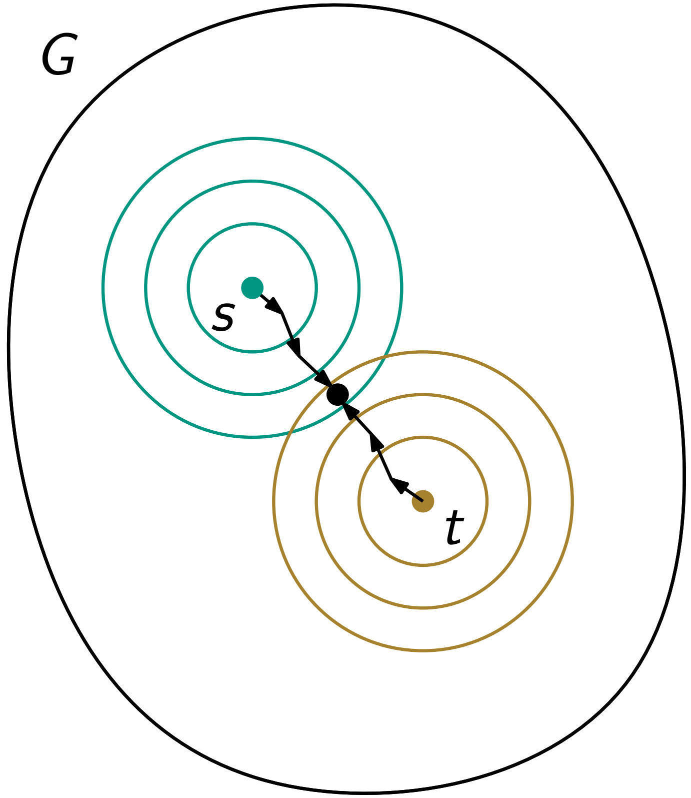
Only in the worst-case!



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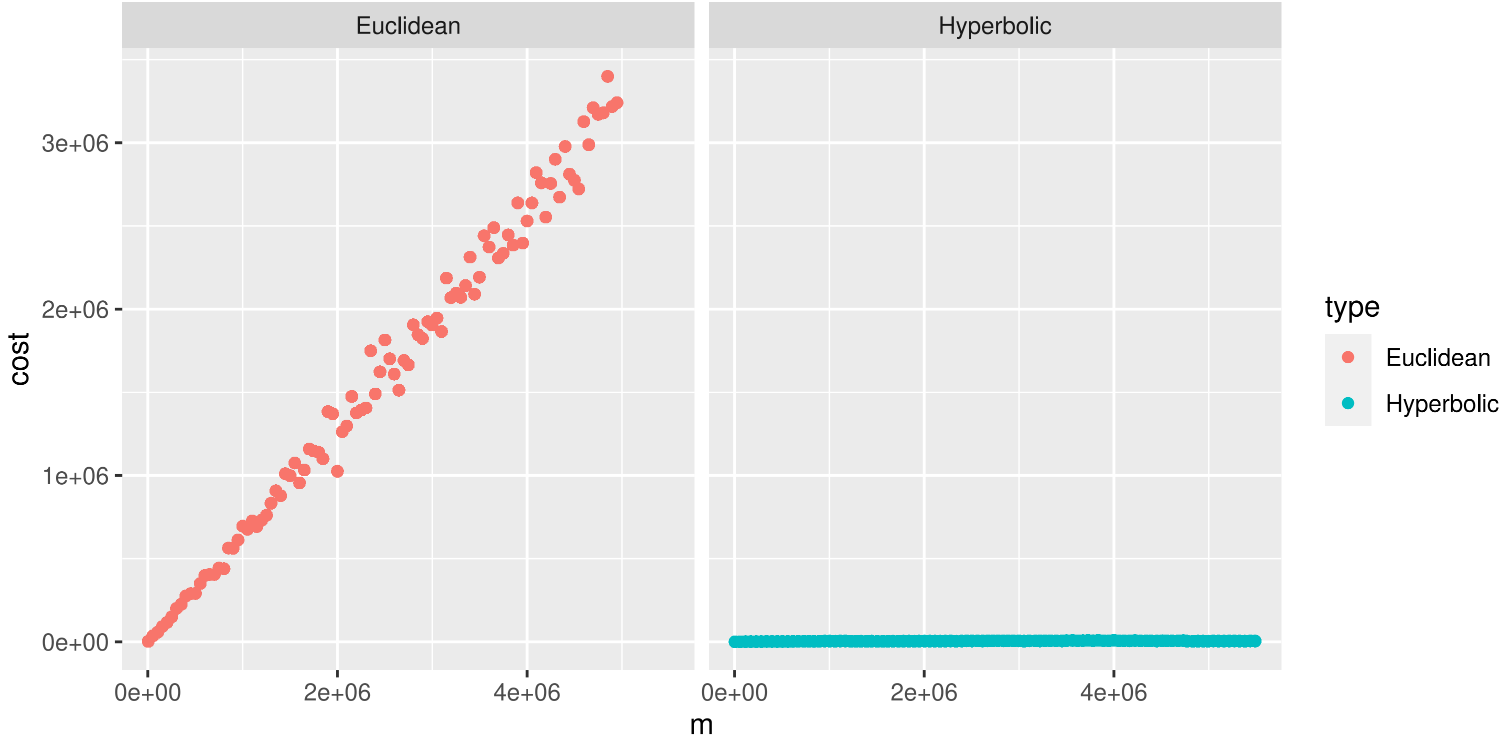

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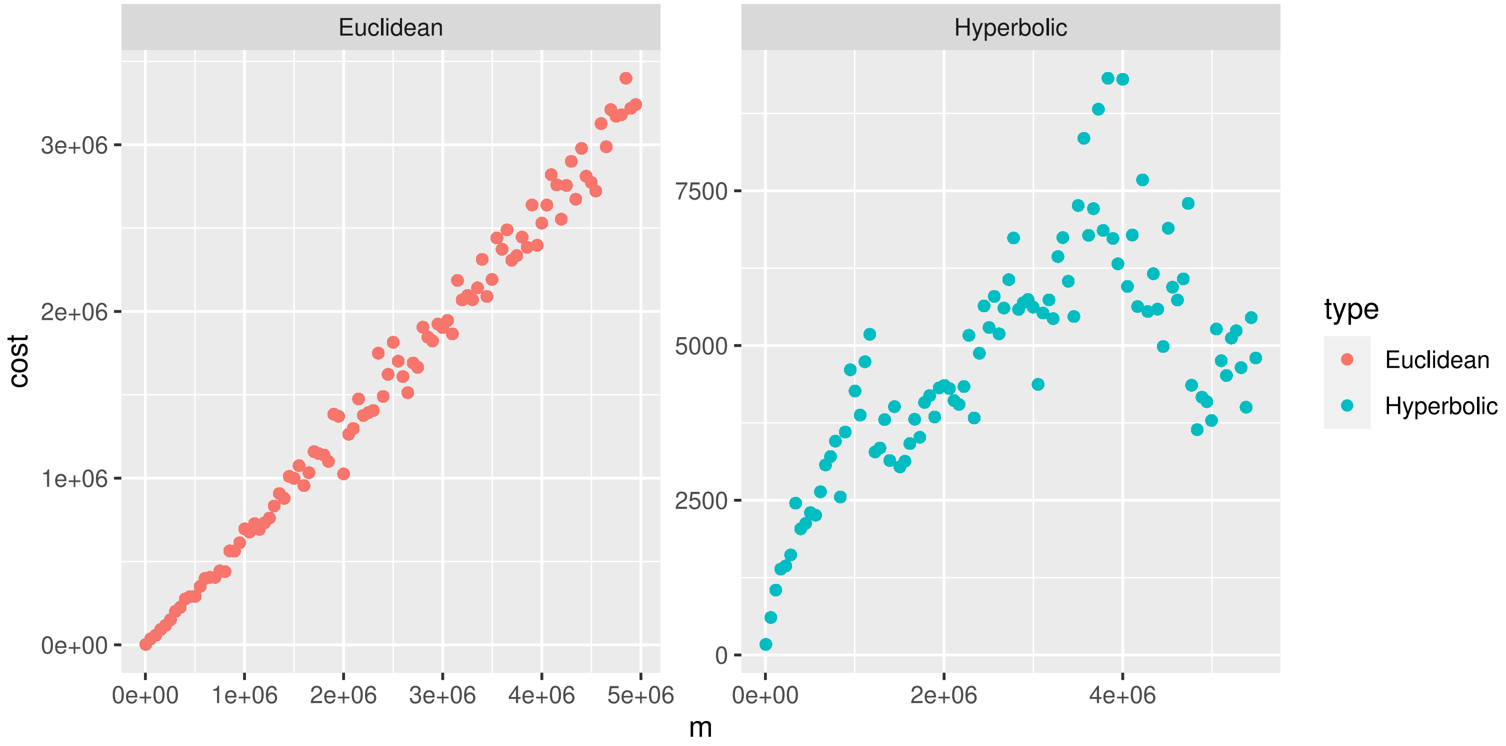

What's going on here?

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KADABRA is an ADaptive Algorithm for Betweenness via Random Approximation

MICHELE BORASSI, IMT School for Advanced Studies Lucca
EMANUELE NATALE, Max-Planck-Institut für Informatik

We present KADABRA, a new algorithm to approximate betweenness centrality in directed and undirected graphs, which significantly outperforms all previous approaches on real-world complex networks. The efficiency of the new algorithm relies on two new theoretical contributions, of independent interest.

The first contribution focuses on sampling shortest paths, a subroutine used by most algorithms that approximate betweenness centrality. We show that, on realistic random graph models, we can perform this task in time $|E|^{\frac{1}{2}+o(1)}$ with high probability, obtaining a significant speedup with respect to the $\Theta(|E|)$ worst-case performance. We experimentally show that this new technique achieves similar speedups on real-world complex networks, as well.

The second contribution is a new rigorous application of the adaptive sampling technique. This approach decreases the total number of shortest paths that need to be sampled to compute all betweenness centralities with a given absolute error, and it also handles more general problems, such as computing the k most central nodes. Furthermore, our analysis is general, and it might be extended to other settings.

CCS Concepts: • **Theory of computation** → *Graph algorithms analysis*;

Additional Key Words and Phrases: Betweenness centrality, shortest path algorithm, graph mining, sampling, network analysis

ACM Reference format:
Michele Borassi and Emanuele Natale. 2019. KADABRA is an ADaptive Algorithm for Betweenness via Random Approximation. *J. Exp. Algorithmics* 24, 1, Article 1.2 (February 2019), 35 pages.
https://doi.org/10.1145/3284359

1 INTRODUCTION

In this work, we focus on estimating the *betweenness centrality*, which is one of the most famous measures of *centrality* for nodes and edges of real-world complex networks [24, 36]. The rigorous definition of betweenness centrality has its roots in sociology, dating back to the 1970s, when Freeman formalized the informal concept discussed in the previous decades in different scientific communities [6, 17, 22, 44, 45], although the definition already appeared in [3]. Since then, this notion has been very successful in network science [28, 36, 37, 51].

This work was done while the authors were visiting the Simons Institute for the Theory of Computing.
Authors' addresses: M. Borassi, IMT School for Advanced Studies Lucca, Piazza S. Francesco 19 - 55100 Lucca (LU) - Italy; email: michele.borassi@gmail.com; E. Natale, COATI Team, I3S, 2004 route des Lucioles - B.P. 93 - F-06902 Sophia Antipolis Cedex - France; email: emanuele.natale@inria.fr.

1084-6654/2019/02-ART1.2 $15.00
https://doi.org/10.1145/3284359

Efficient Shortest Paths in Scale-Free Networks with Underlying Hyperbolic Geometry

THOMAS BLÄSIUS, Karlsruhe Institute of Technology
CEDRIC FREIBERGER, TOBIAS FRIEDRICH, MAXIMILIAN KATZMANN,
FELIX MONTENEGRO-RETANA, and MARIANNE THIEFFRY, Hasso Plattner Institute, University of Potsdam

A standard approach to accelerating shortest path algorithms on networks is the bidirectional search, which explores the graph from the start and the destination, simultaneously. In practice this strategy performs particularly well on scale-free real-world networks. Such networks typically have a heterogeneous degree distribution (e.g., a power-law distribution) and high clustering (i.e., vertices with a common neighbor are likely to be connected themselves). These two properties can be obtained by assuming an underlying hyperbolic geometry.

To explain the observed behavior of the bidirectional search, we analyze its running time on hyperbolic random graphs and prove that it is $\tilde{O}(n^{2-1/\alpha} + n^{1/(2\alpha)} + \delta_{\max})$ with high probability, where $\alpha \in (1/2, 1)$ controls the power-law exponent of the degree distribution, and $\delta_{\max}$ is the maximum degree. This bound is sublinear, improving the obvious worst-case linear bound. Although our analysis depends on the underlying geometry, the algorithm itself is oblivious to it.

CCS Concepts: • **Theory of computation** → **Random network models**; **Shortest paths**; • **Mathematics of computing** → **Random graphs**; *Paths and connectivity problems*; **Graph algorithms;**

Additional Key Words and Phrases: Random graphs, hyperbolic geometry, scale-free networks, bidirectional shortest path

ACM Reference format:
Thomas Bläsius, Cedric Freiberger, Tobias Friedrich, Maximilian Katzmann, Felix Montenegro-Retana, and Marianne Thieffry. 2022. Efficient Shortest Paths in Scale-Free Networks with Underlying Hyperbolic Geometry. *ACM Trans. Algorithms* 18, 2, Article 19 (March 2022), 32 pages.
https://doi.org/10.1145/3516483

A preliminary version of this article appeared in [4].
This research was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Grant No. 390859508.
Authors' addresses: T. Bläsius, Karlsruhe Institute of Technology, Am Fasanengarten 5, Karlsruhe, Baden-Württemberg, 76131, Germany; email: thomas.blaesius@kit.edu; C. Freiberger, T. Friedrich, M. Katzmann, F. Montenegro-Retana, and M. Thieffry, Hasso Plattner Institute, Prof.-Dr.-Helmert-Straße 2-3, Potsdam, Brandenburg, 14482, Germany; emails: cedric.freiberger@student.hpi.de, {tobias.friedrich, maximilian.katzmann}@hpi.de, {felix.montenegro-retana, marianne.thieffry}@student.hpi.de.

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Expansion

How many vertices have distance k from v?

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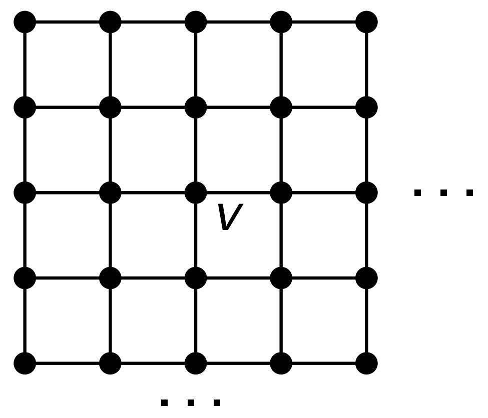

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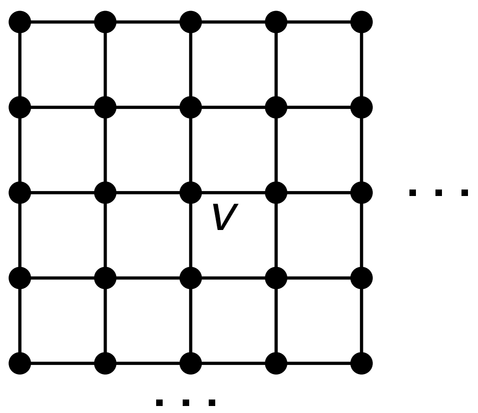

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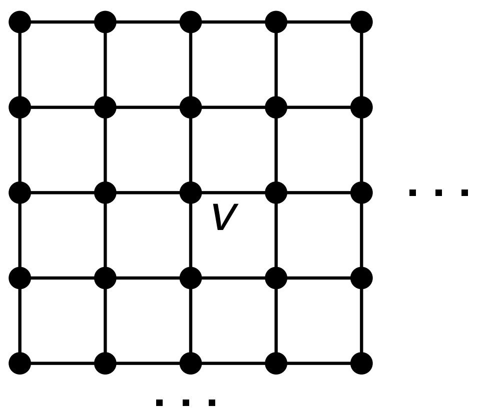

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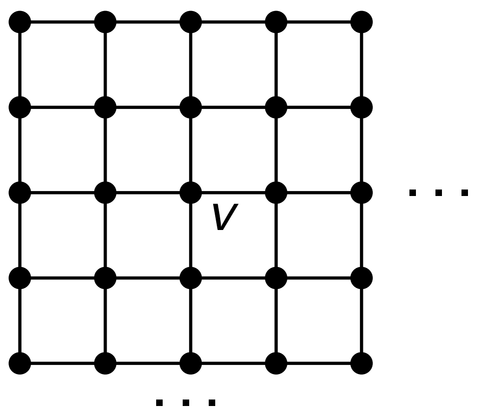

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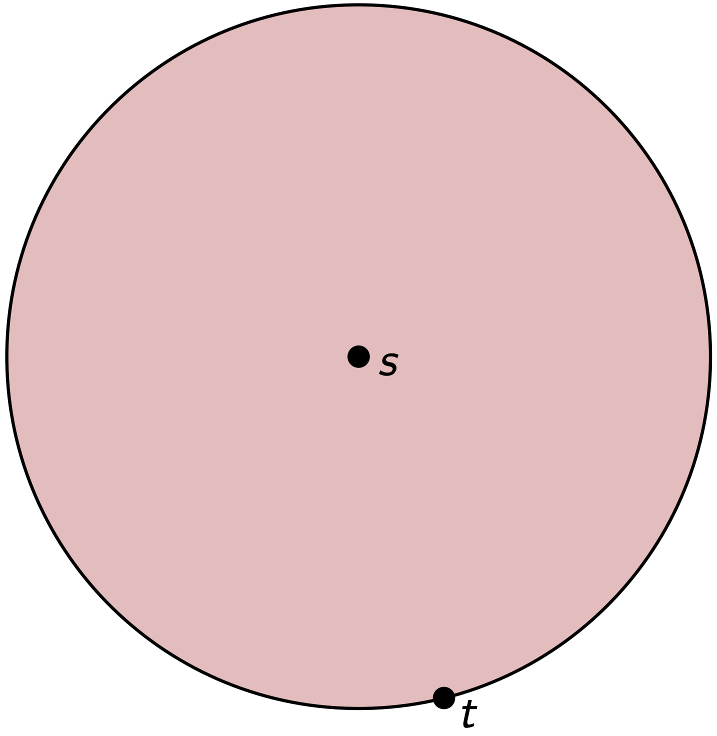

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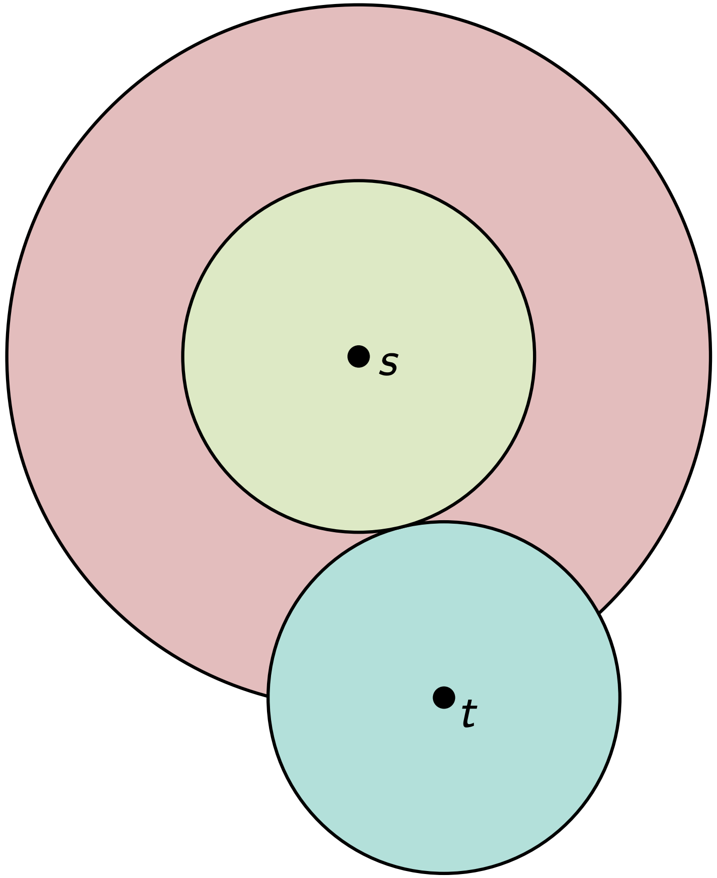

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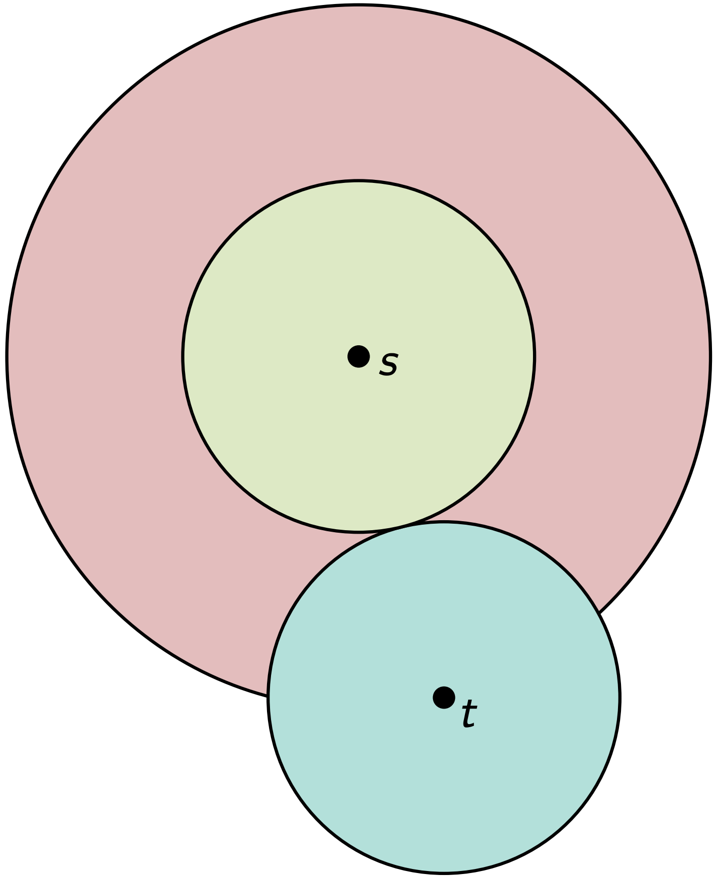

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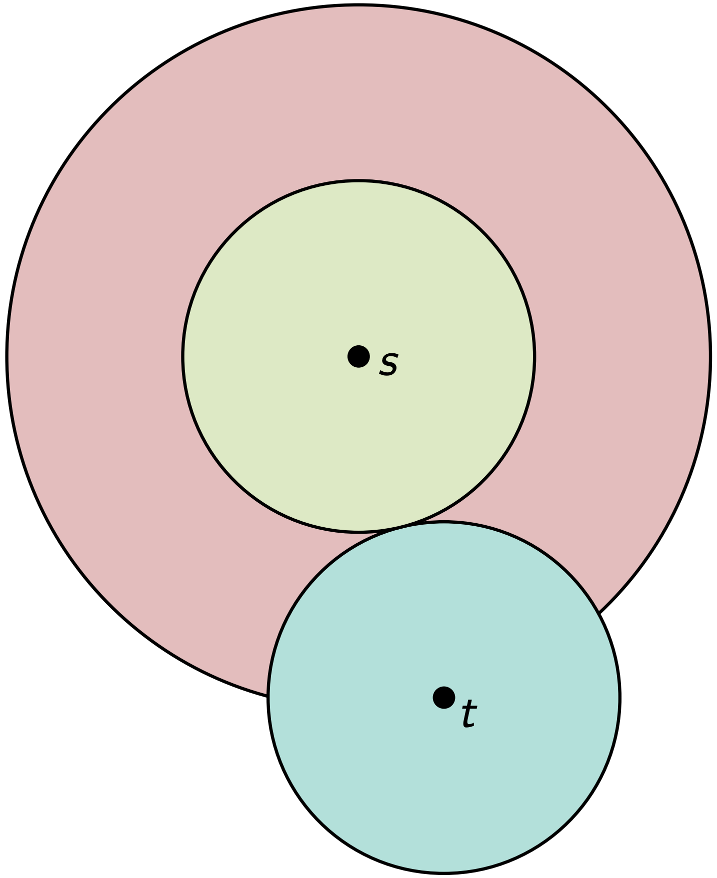

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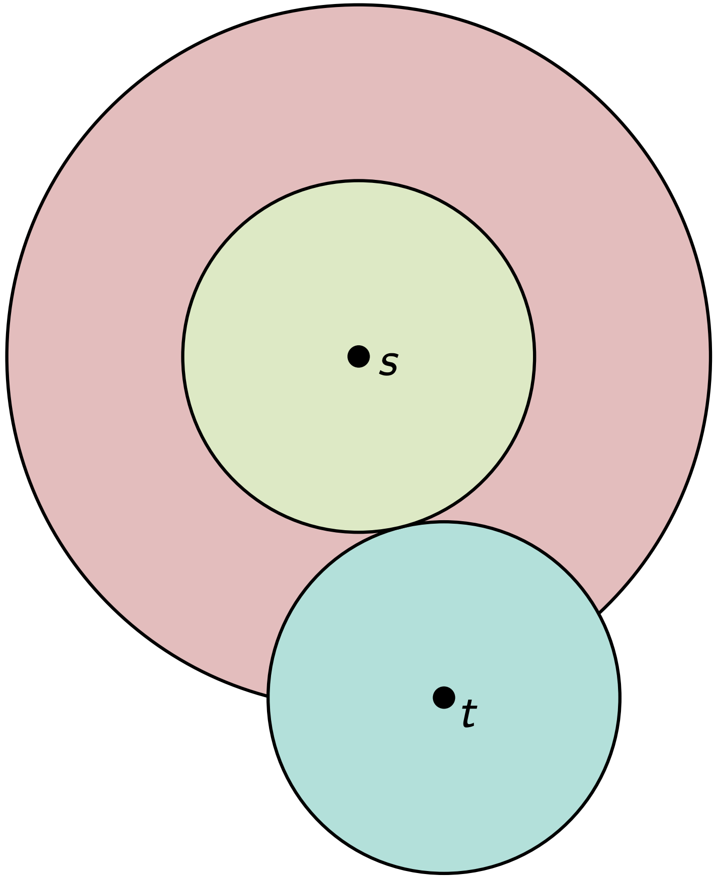
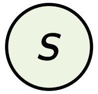


Goal: find definition that works on real graphs *and* in proofs

Definitions



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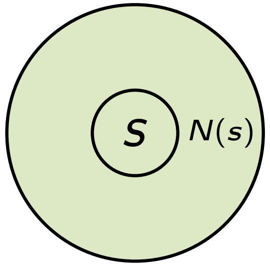

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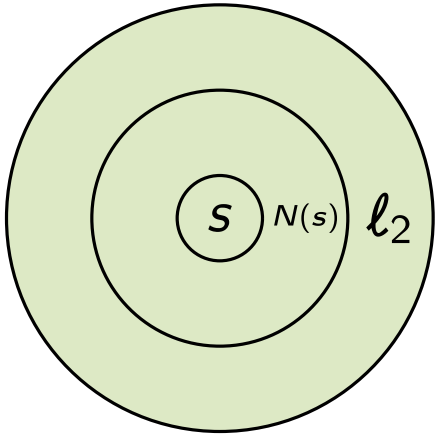

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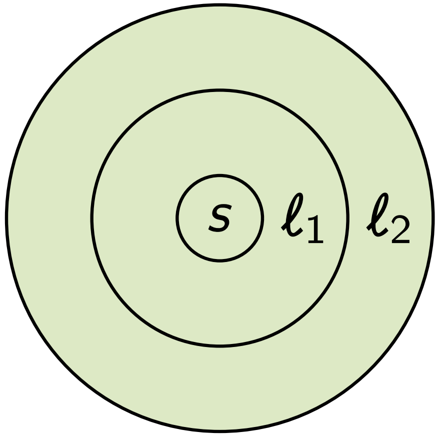

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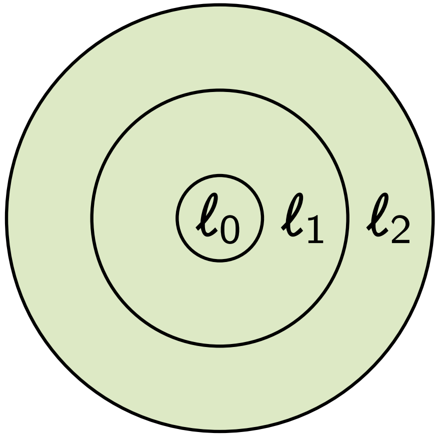

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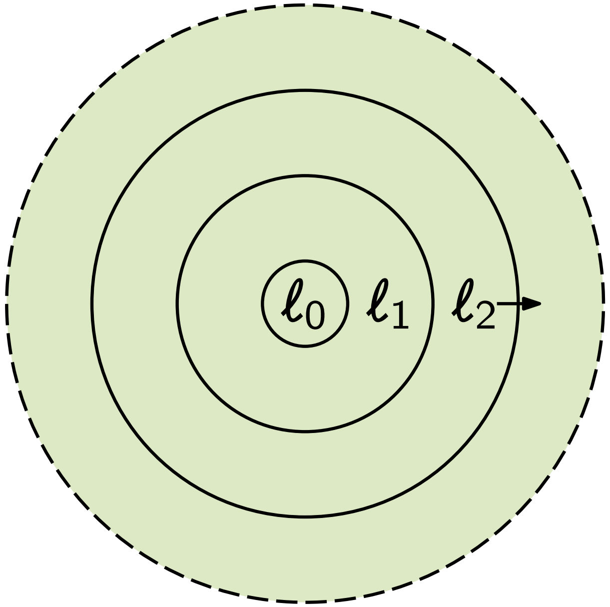

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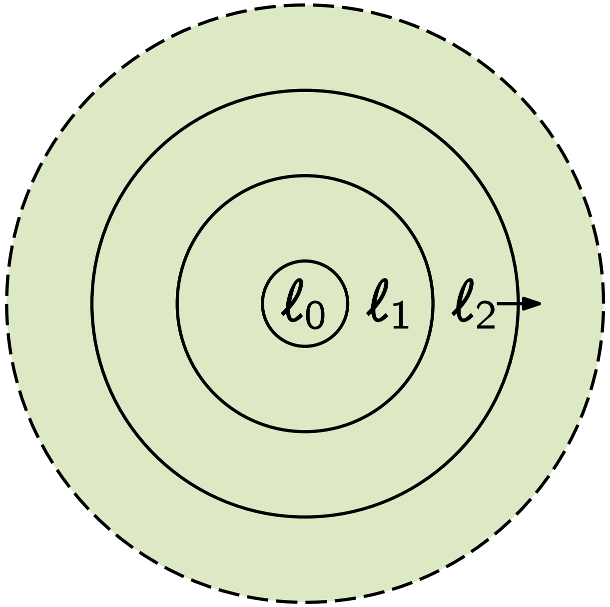
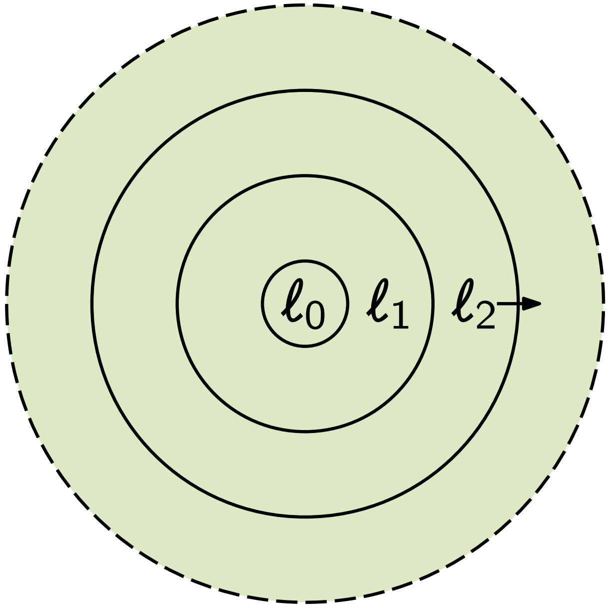


Cost of exploration step:

Definitions



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$$c(i) = \sum_{v \in \ell_{i-1}} \deg(v)$$

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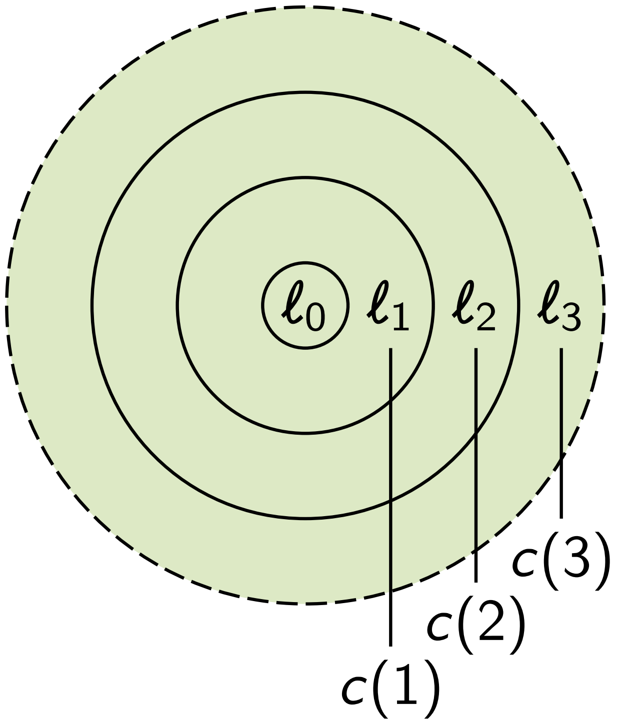

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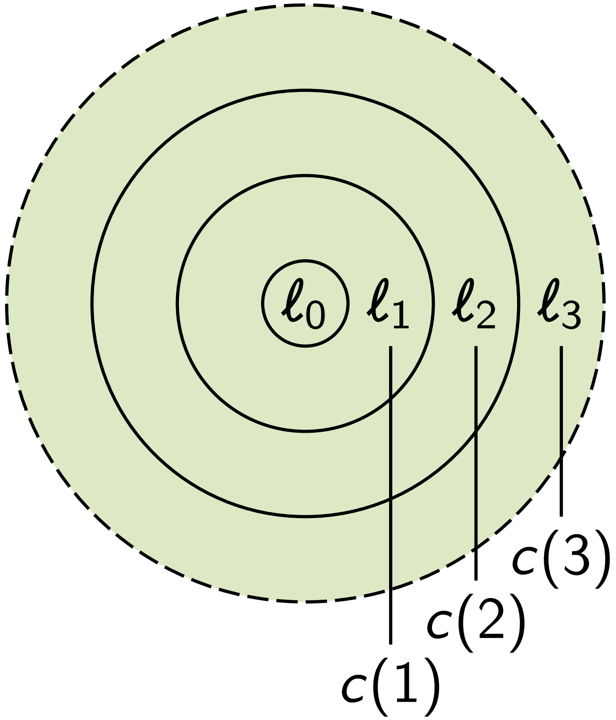

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- $e(i) = \frac{c(i+1)}{c(i)}$

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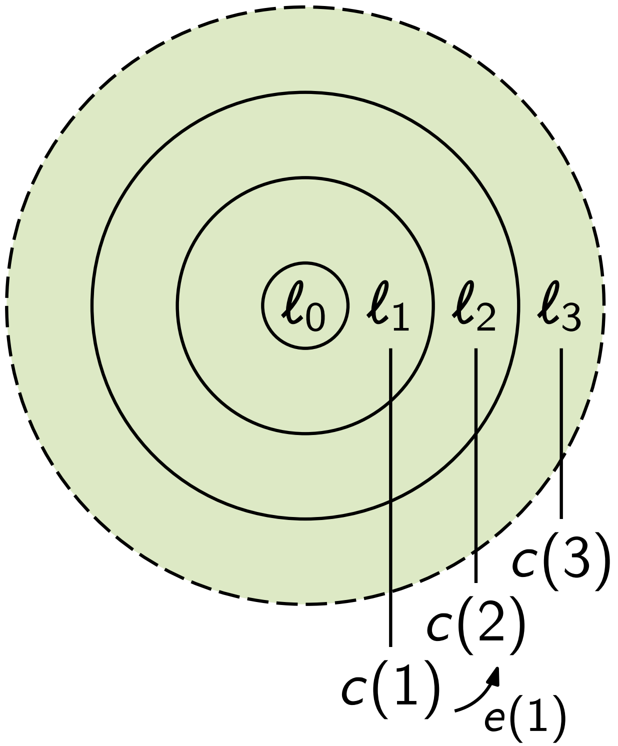

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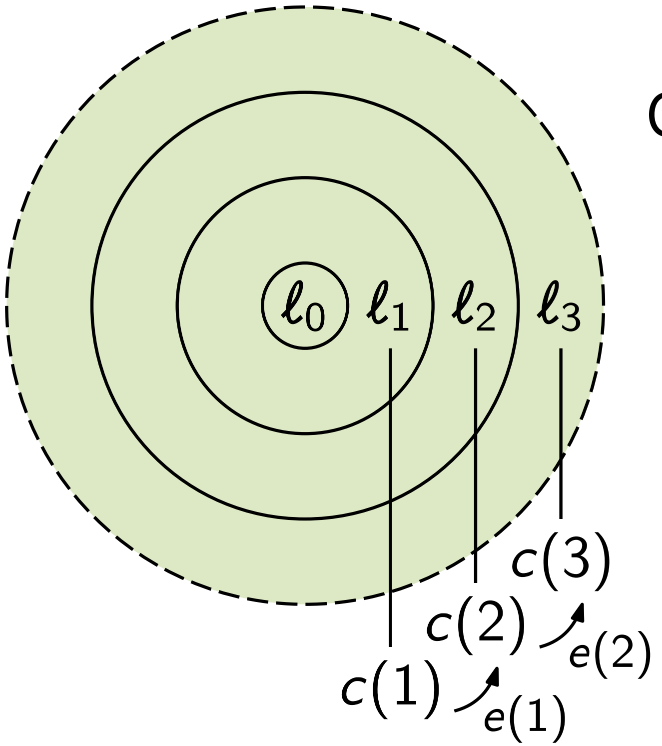

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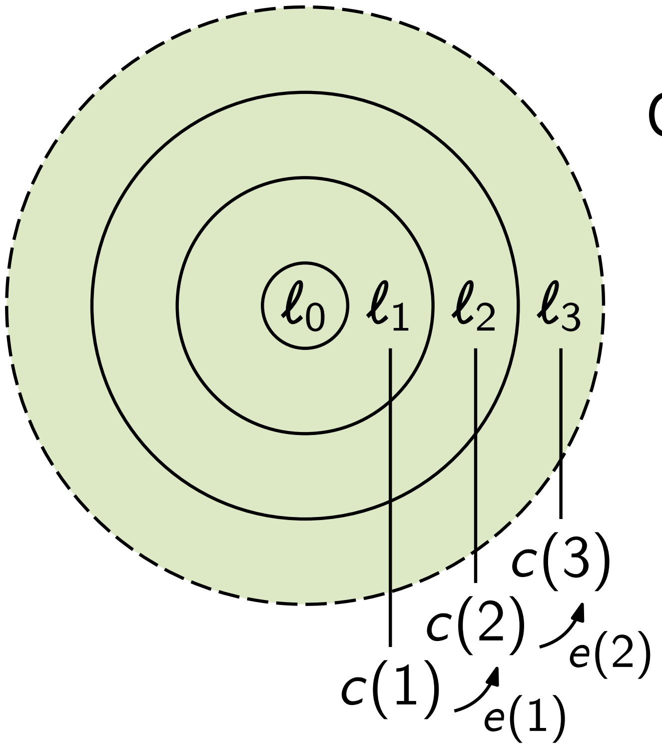

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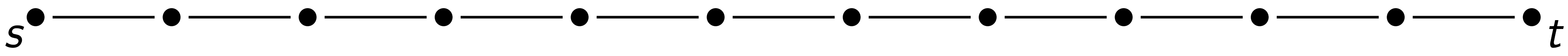
$c(i) = \sum_{v \in \ell_{i-1}} \deg(v)$

Expansion:

- $e(i) = \frac{c(i+1)}{c(i)}$
- steps $i, \ldots, j$ are b-expanding: $e(k) \geq b$ for all $k \in [i, j)$

Expansion Overlap



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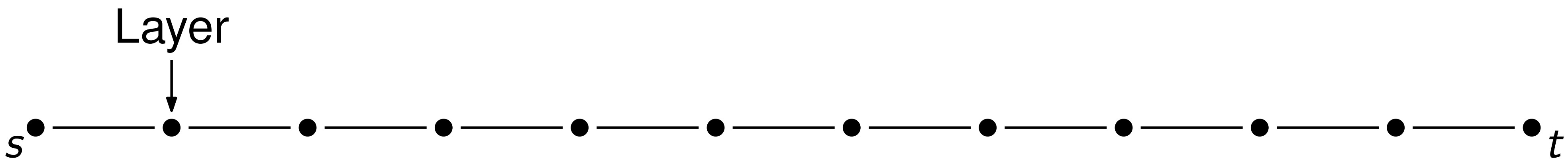

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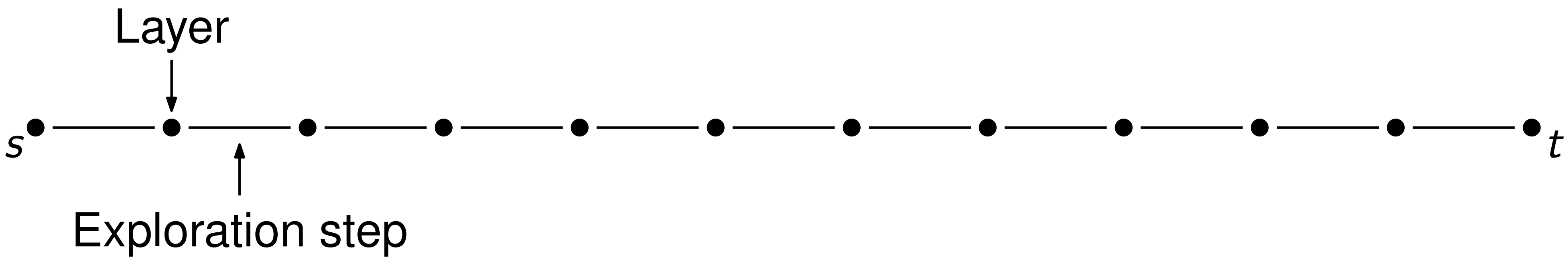

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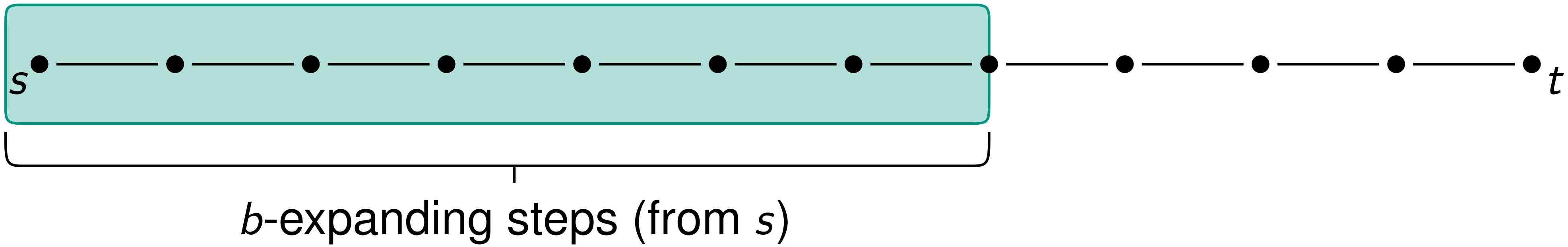

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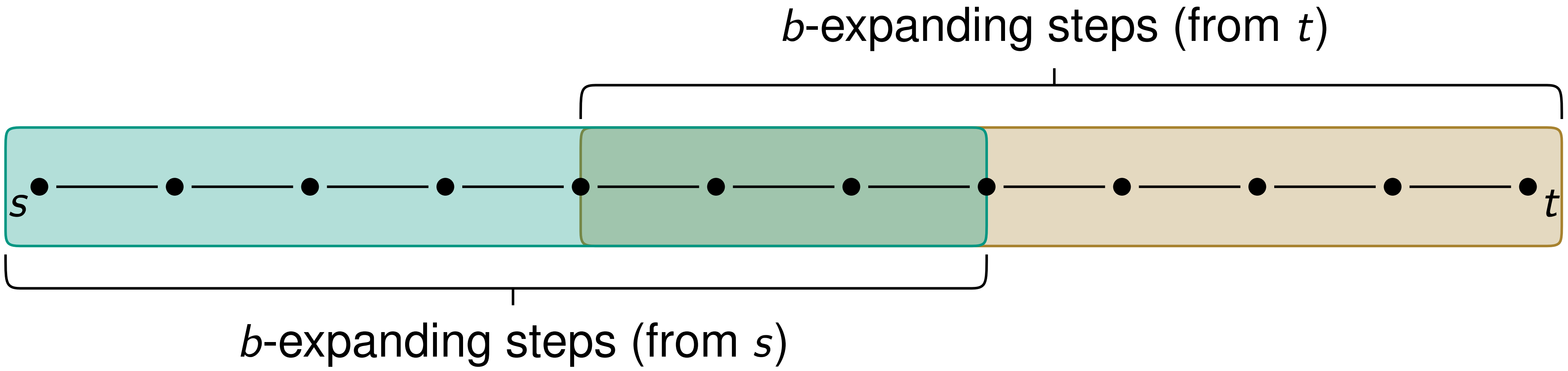

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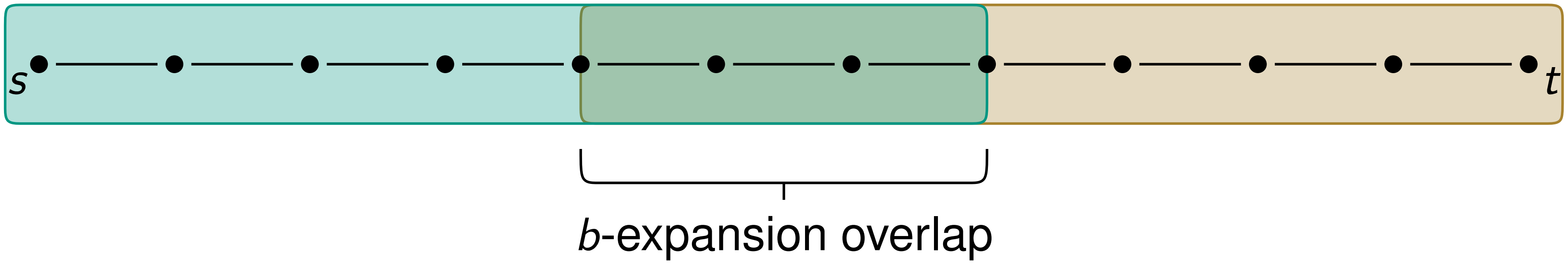

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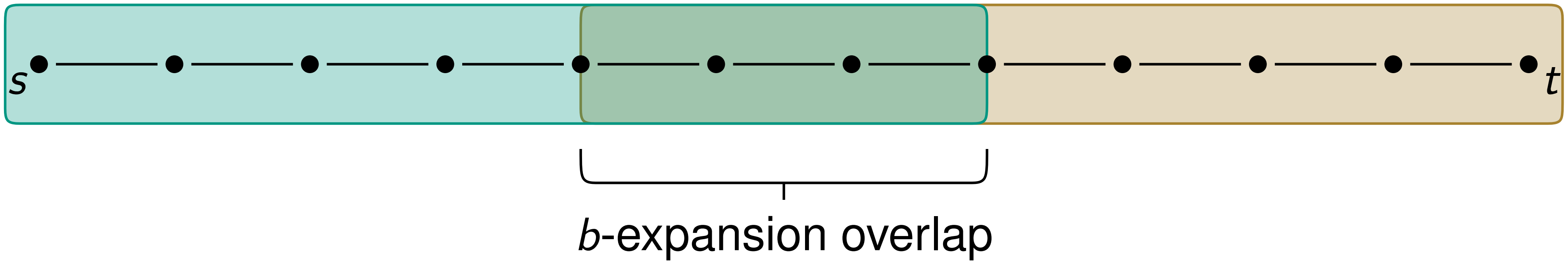

Theorem 1
We have $c_{\text{bi}}(s,t) \in \tilde{\mathcal{O}}\left(m^{1-c/2}\right)$ for $s, t \in V$ with b-expansion overlap of length $c \cdot \log_b m$.

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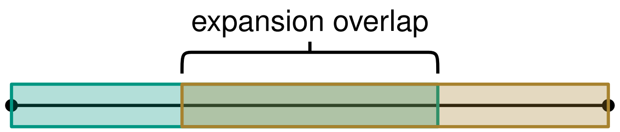

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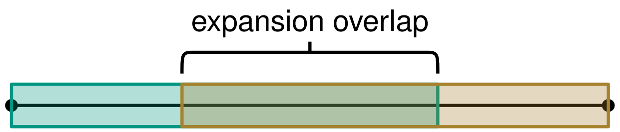

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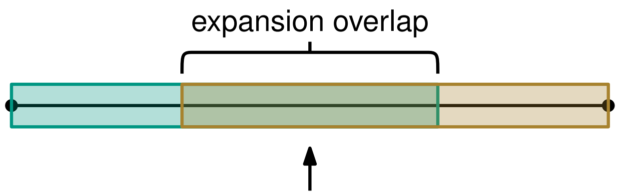

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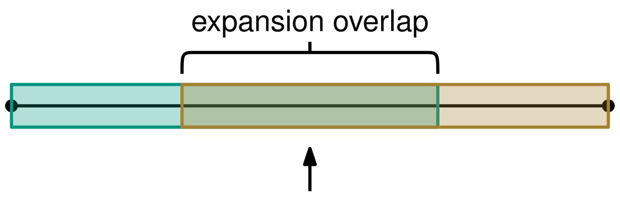

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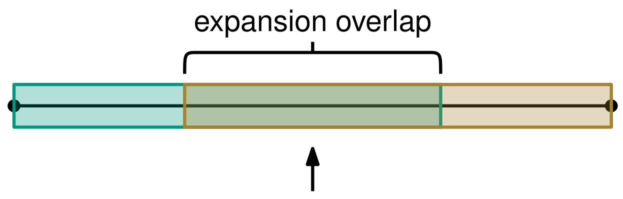

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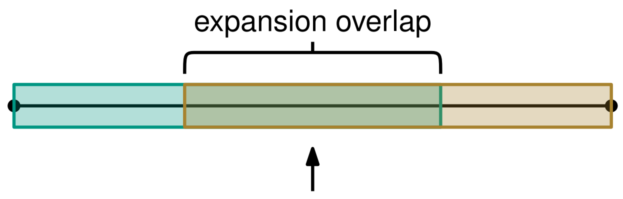

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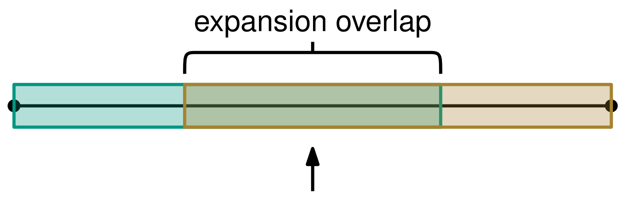

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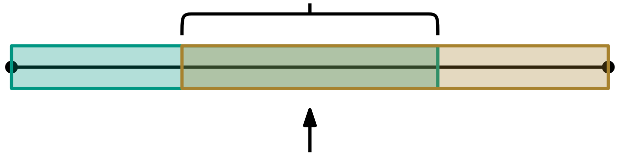

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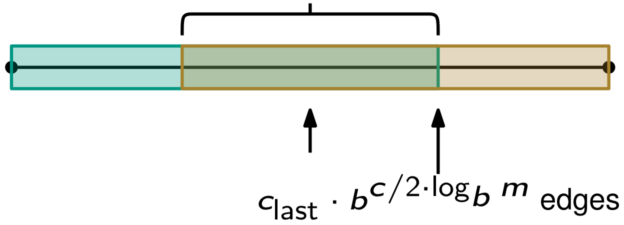

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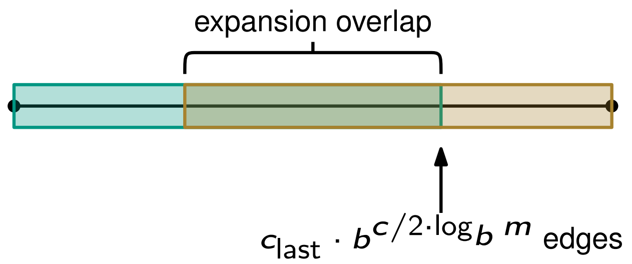

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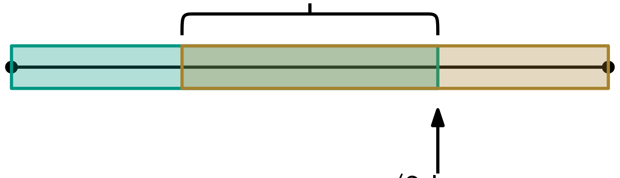

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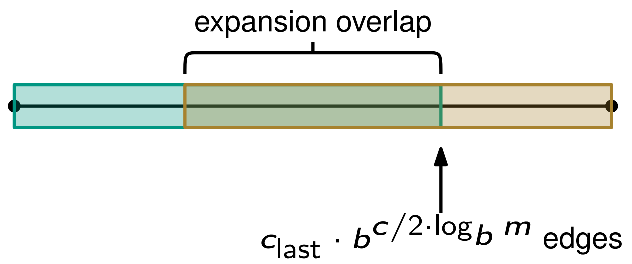

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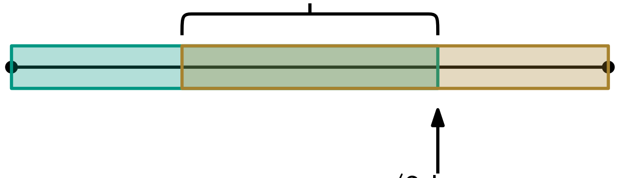

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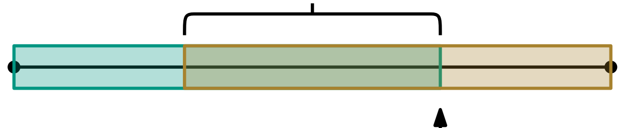

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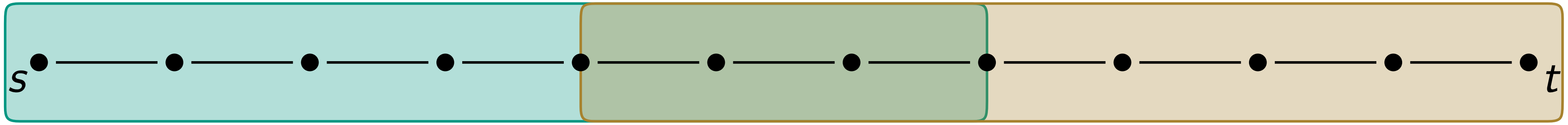
Relative expansion overlap



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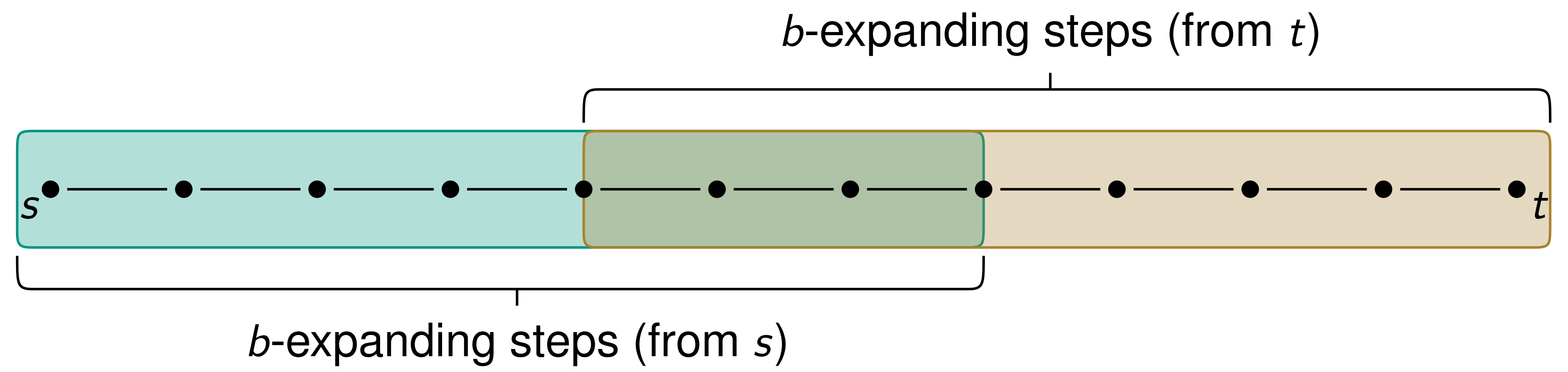

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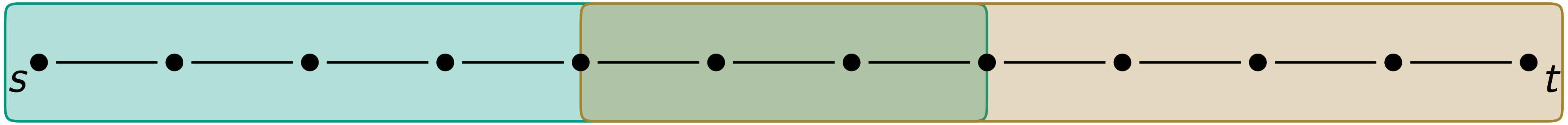

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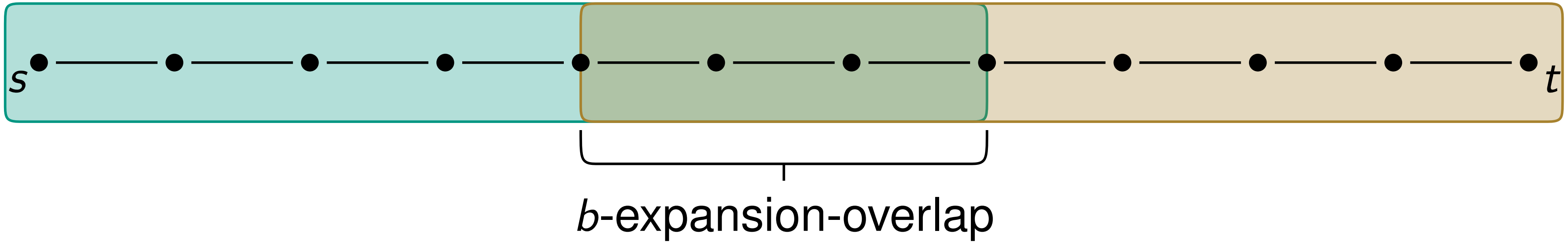

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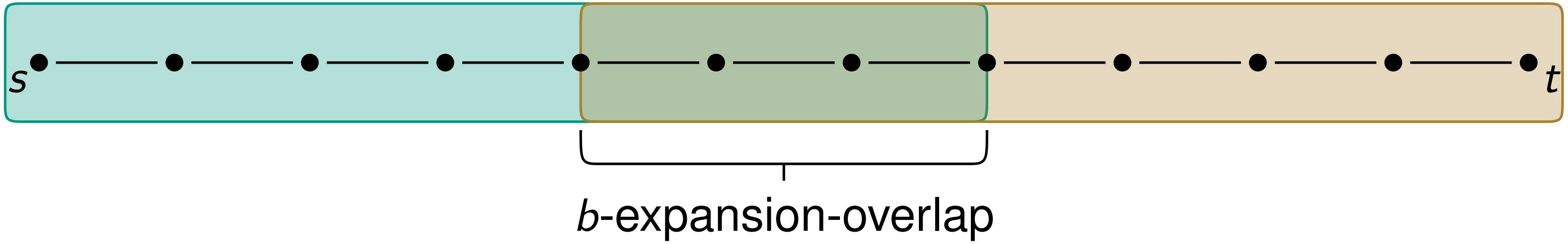

- idea: sublinear if $\frac{\text{expansion-overlap}}{d(s,t)} > \text{const.}$

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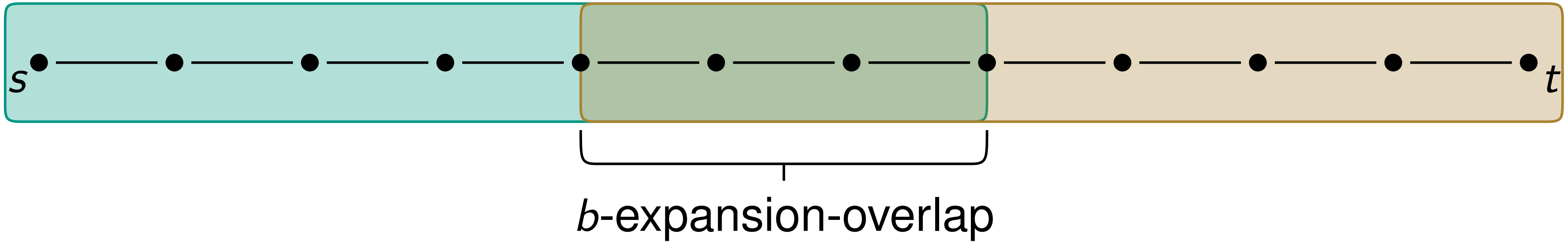

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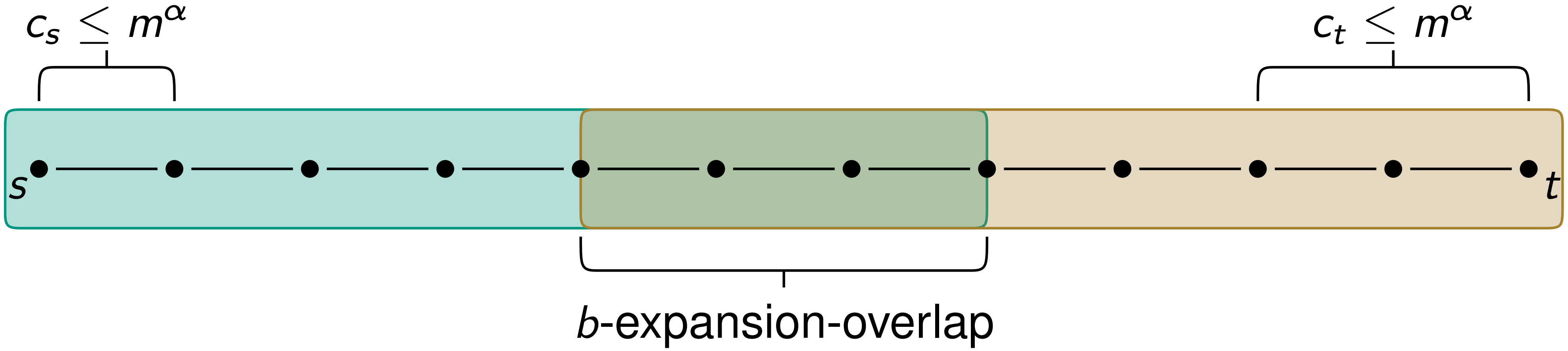

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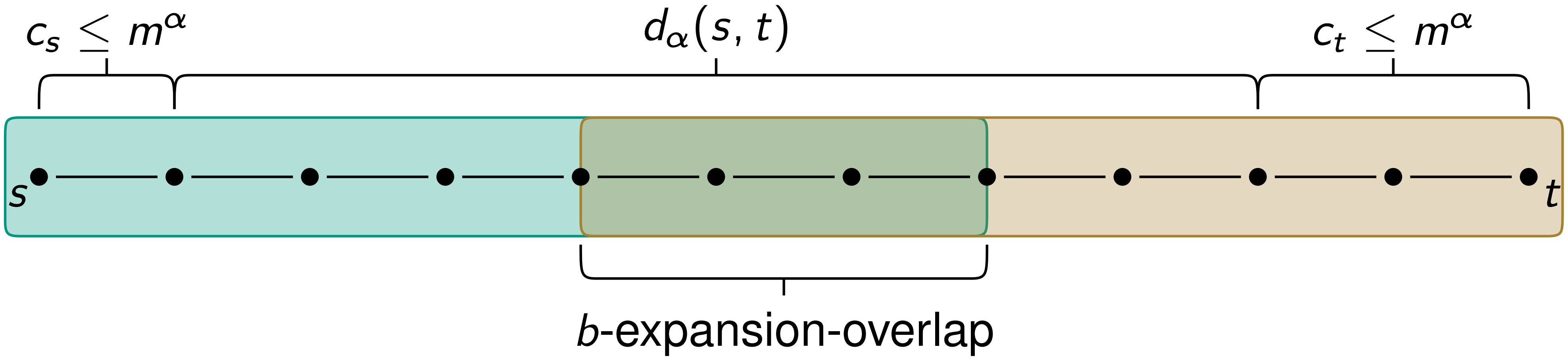

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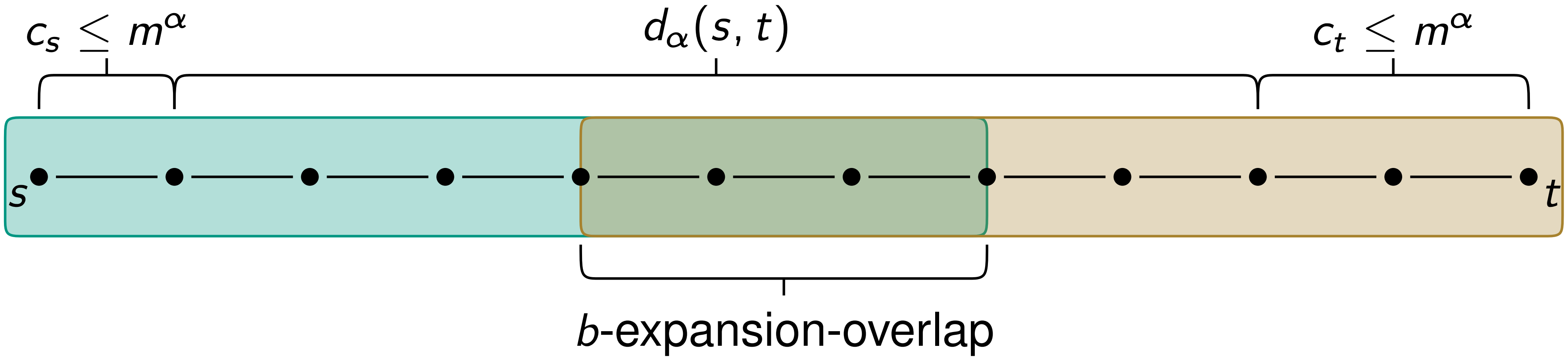

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Theorem 2
We have $c_{\text{bi}}(s, t) \in \tilde{\mathcal{O}}\left(m^{1-\varepsilon}\right)$ for $s, t \in V$ with b-expansion overlap of length at least $c \cdot d_\alpha(s, t)$ for constant c and $\varepsilon = \frac{c(1-\alpha)}{2(\log_b(b^+)+c)} > 0$, for maximum expansion b^+.

Relative expansion overlap

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- Case distinction on $d_\alpha(s,t)$

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- Case distinction on $d_\alpha(s,t)$
  - $d_\alpha(s,t) \geq a \log_b m$ for any const. a: Theorem 1 applies

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  - otherwise:

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We have $c_{\text{bi}}(s,t) \in \tilde{O}\left(m^{1-\varepsilon}\right)$ for $s,t \in V$ with b-expansion overlap of length at least $c \cdot d_\alpha(s,t)$ for constant c and $\varepsilon = \frac{c(1-\alpha)}{2(\log_b(b^+)+c)} > 0$, for maximum expansion b^+.

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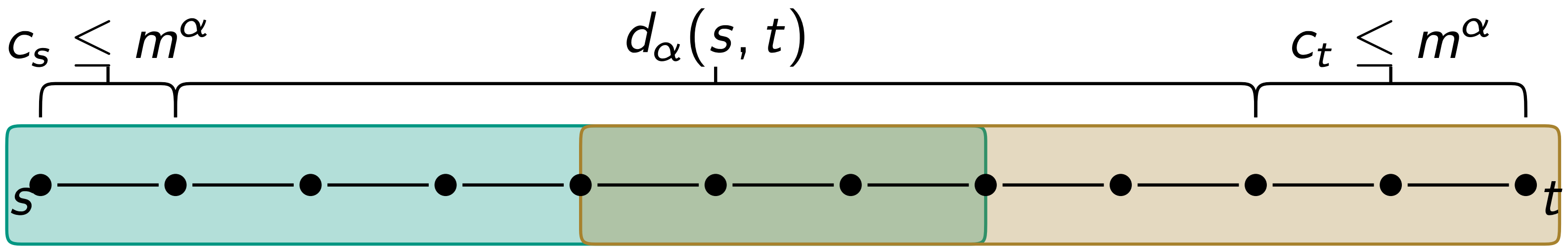

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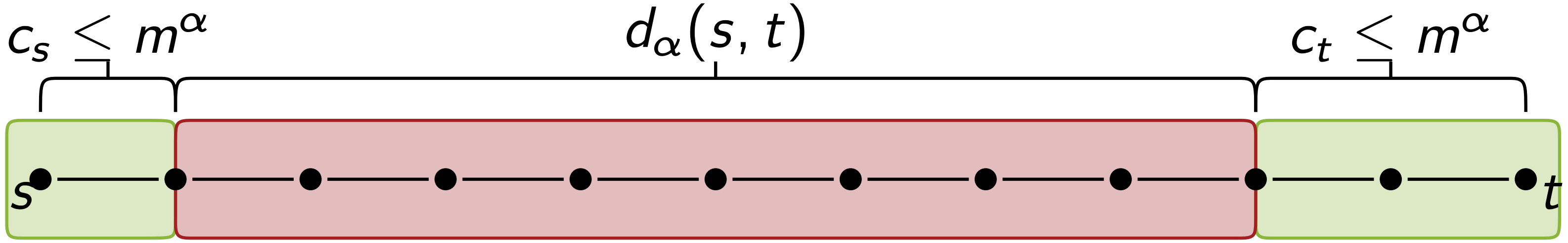

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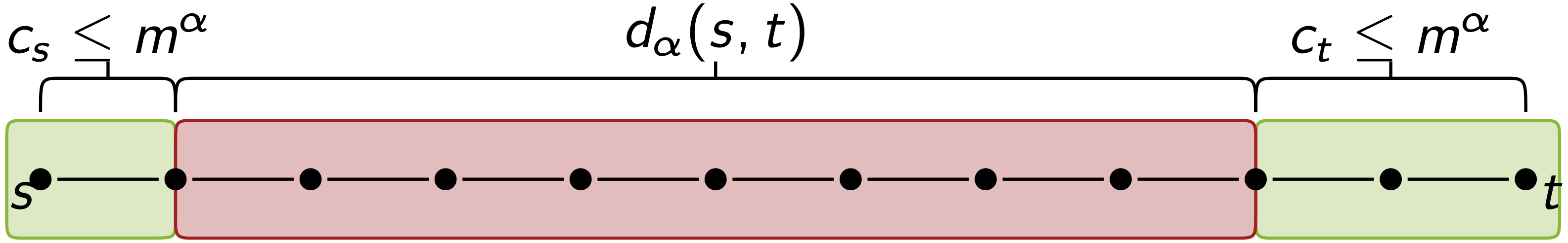

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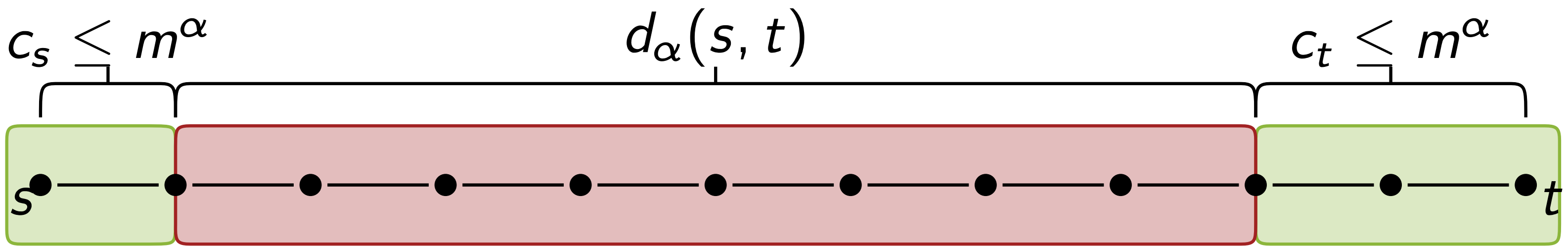
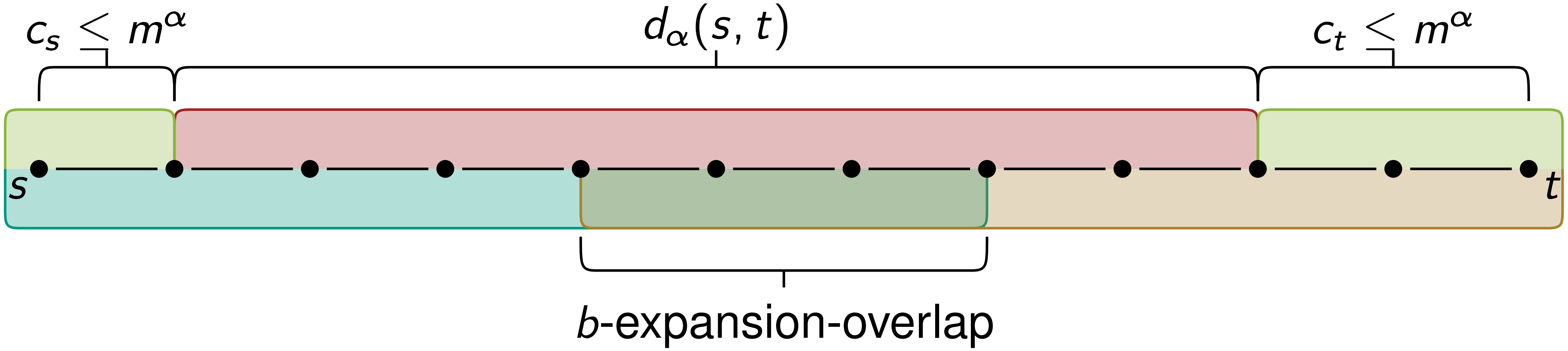


    - $c_{\mathrm{bi}}(s,t) \approx m^\alpha \cdot {b^+}^{d_\alpha(s,t)} \leq m^\alpha \cdot m^{a \log_b(b^+)}$
    - works out for suitable choice of a $\square$

A tight characterization



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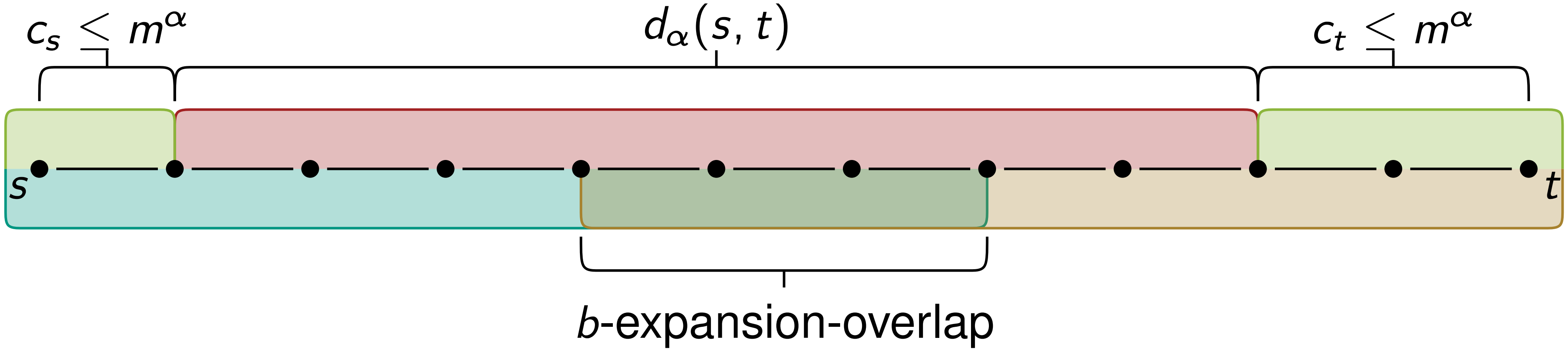

Intuition:

A tight characterization

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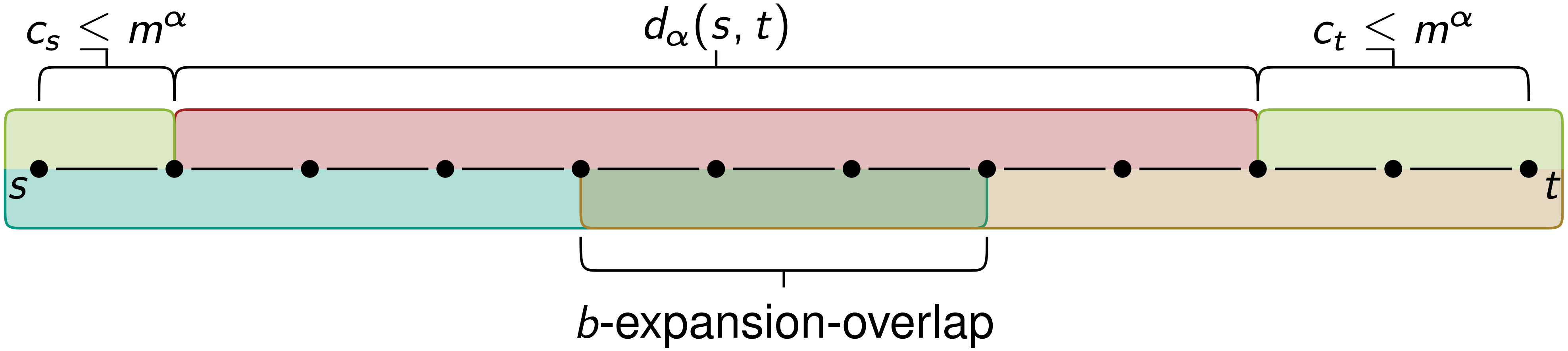

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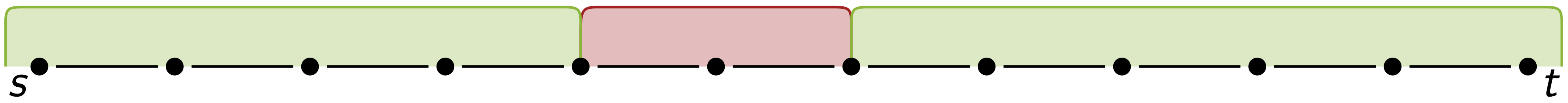

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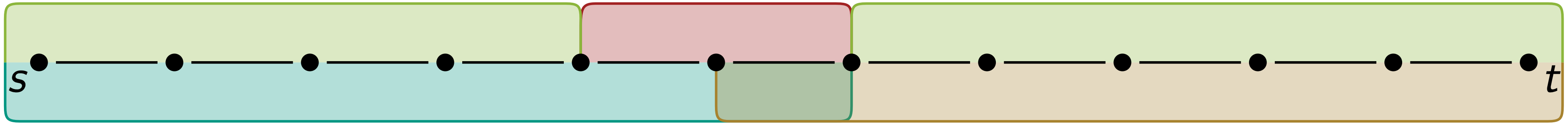

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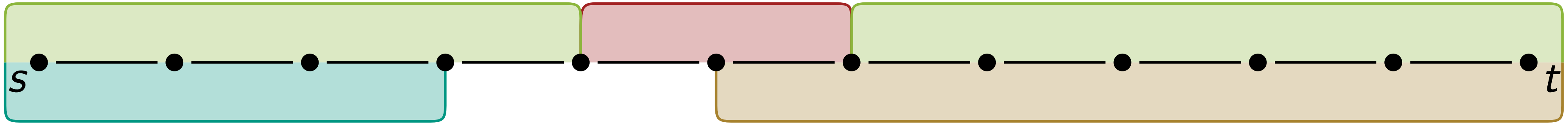

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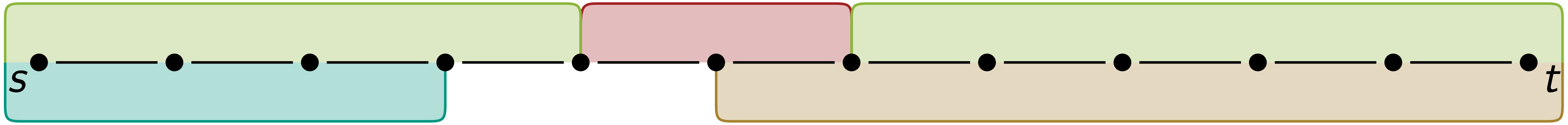

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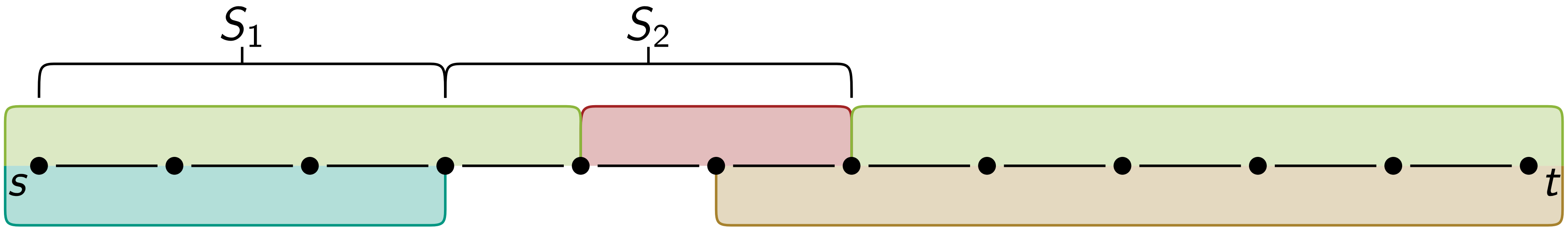

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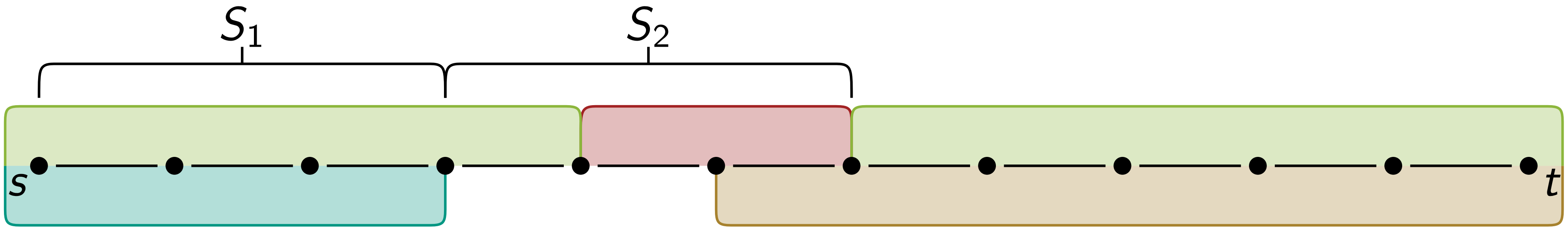

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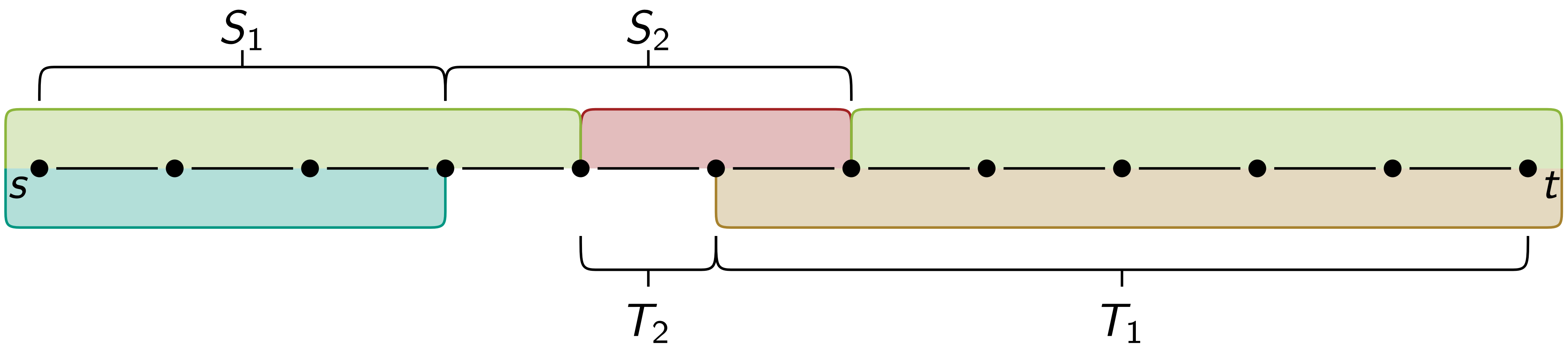

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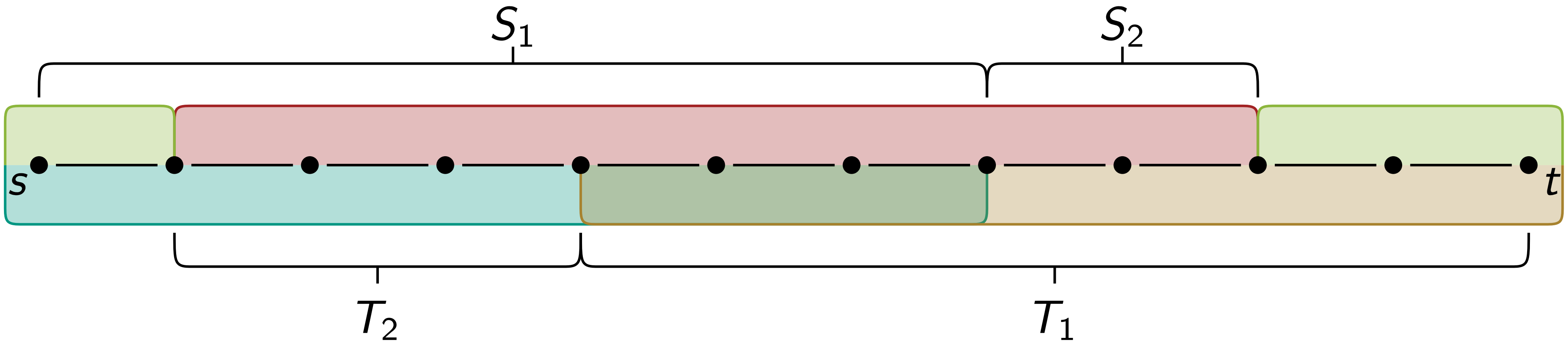

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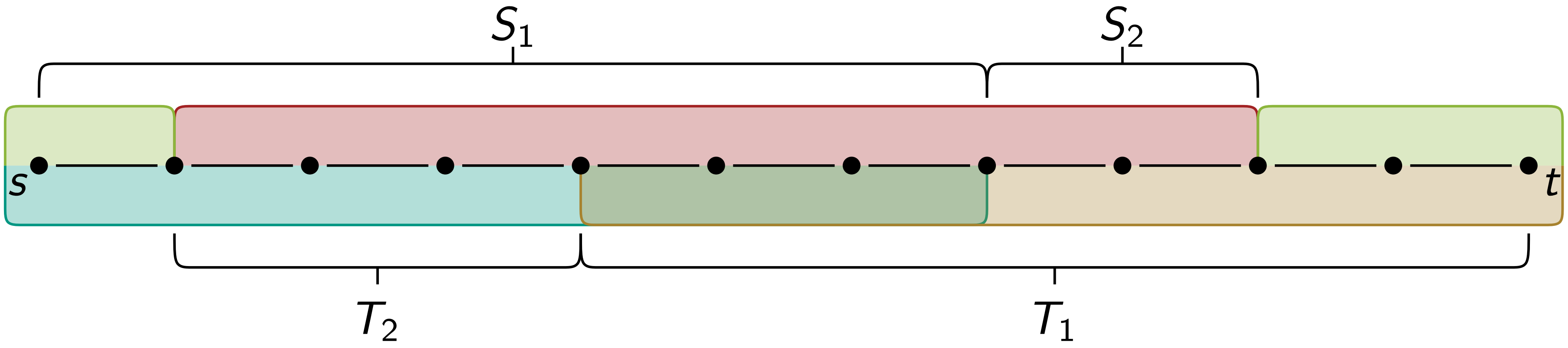

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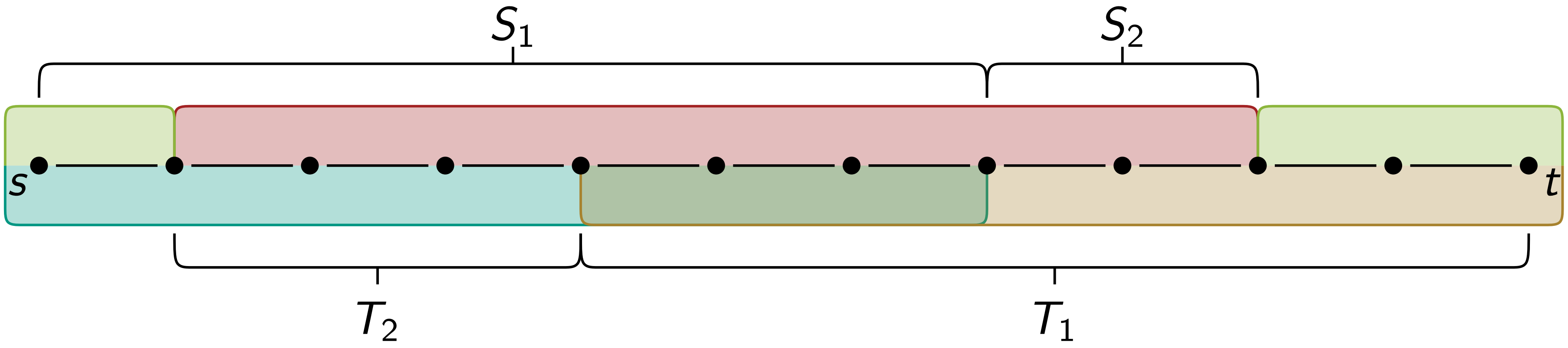

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> **Theorem 3**
> Bi-diretional BFS runs in $\tilde{\mathcal{O}}\left(m^{1-\varepsilon}\right)$ time with $\varepsilon > 0$ if $\rho < \frac{1-\alpha}{1-\alpha+\alpha \log_b(b^+)}$. Otherwise there are instances with running time in $\Theta(m)$.

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- Case distinction on length of expansion overlap

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Proof sketch 1:

- Case distinction on length of expansion overlap
  - length at least $c \cdot \log_b(m)$: Theorem 1

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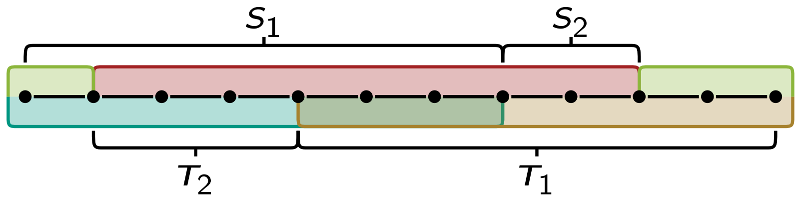

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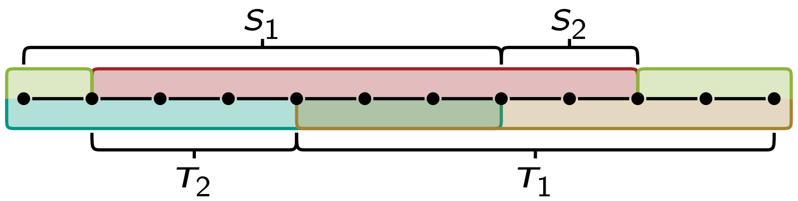

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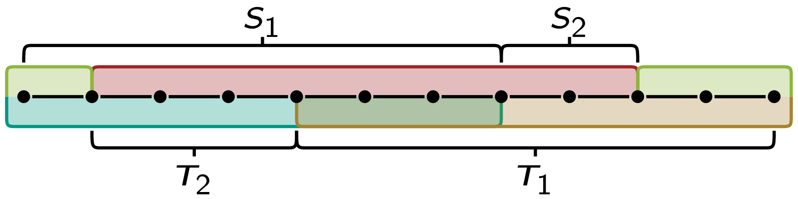

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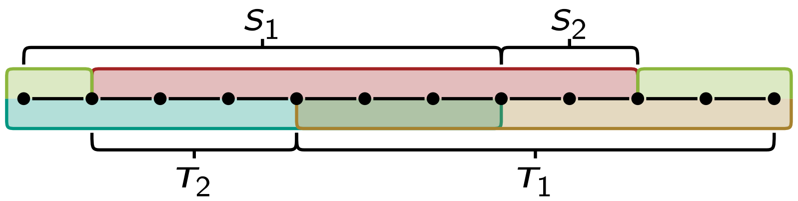

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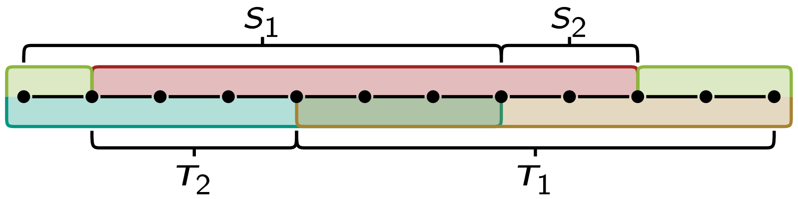

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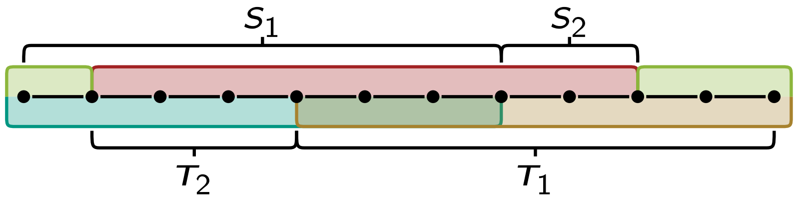

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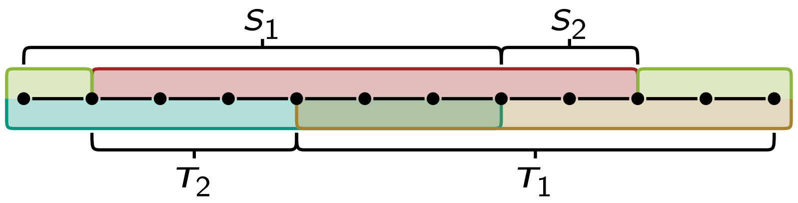

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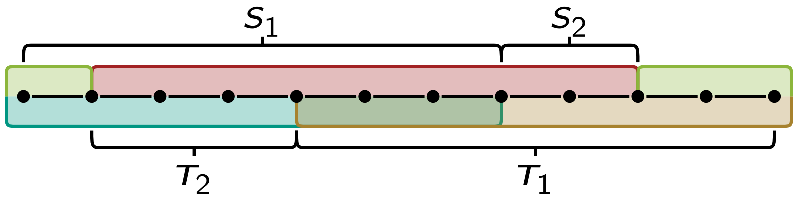

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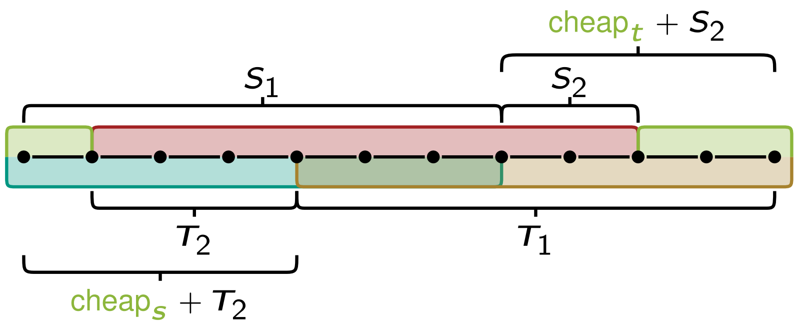

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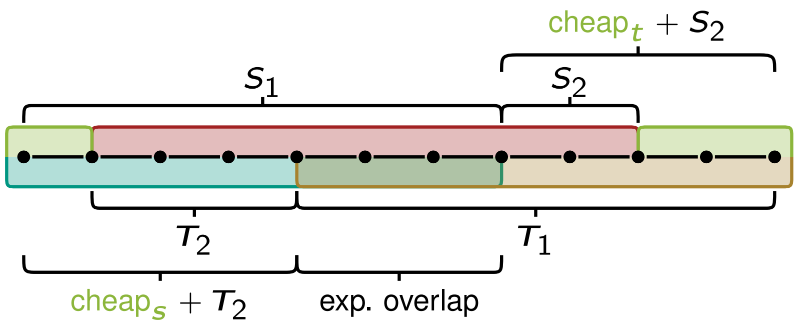

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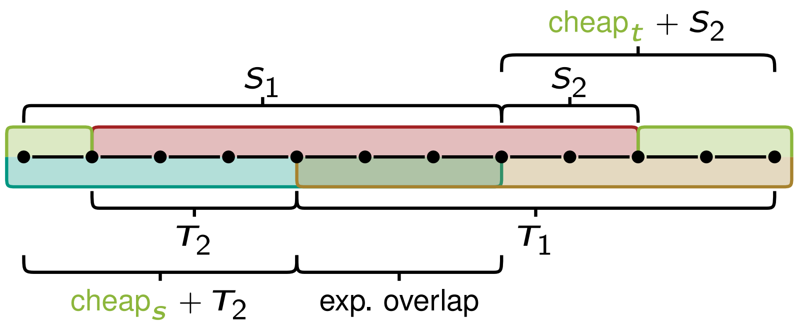

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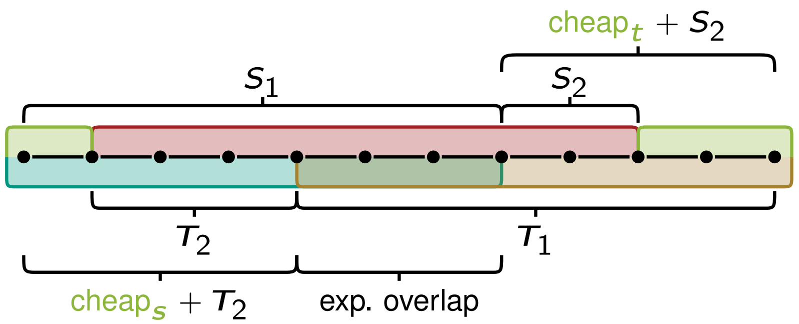

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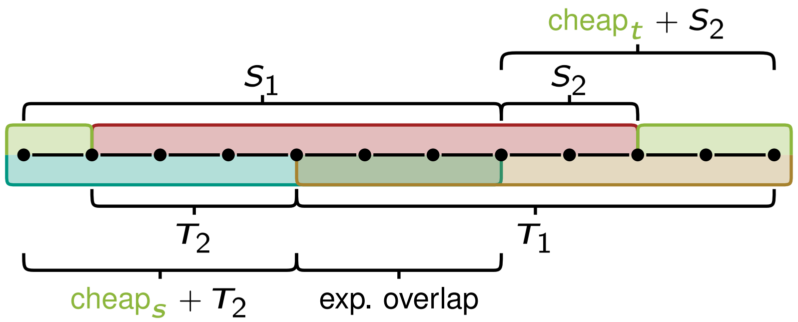

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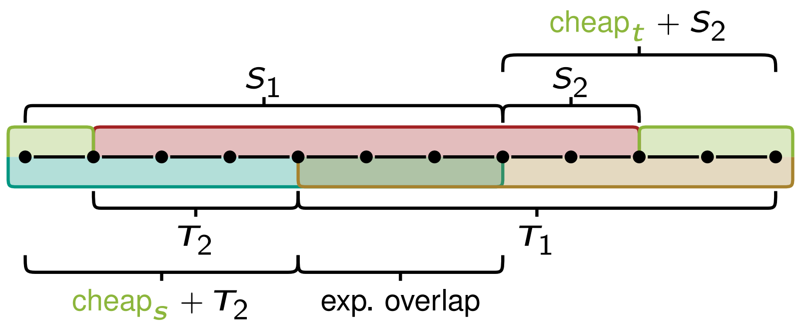

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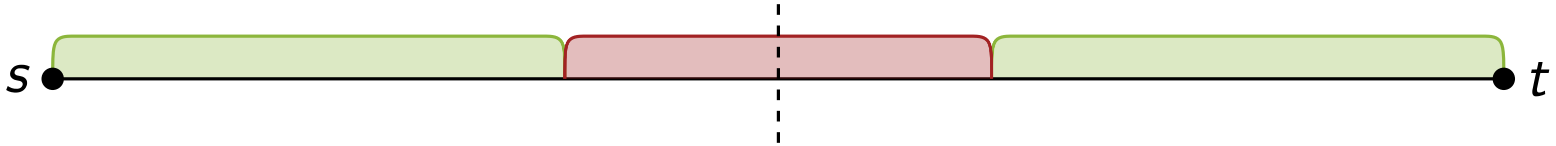

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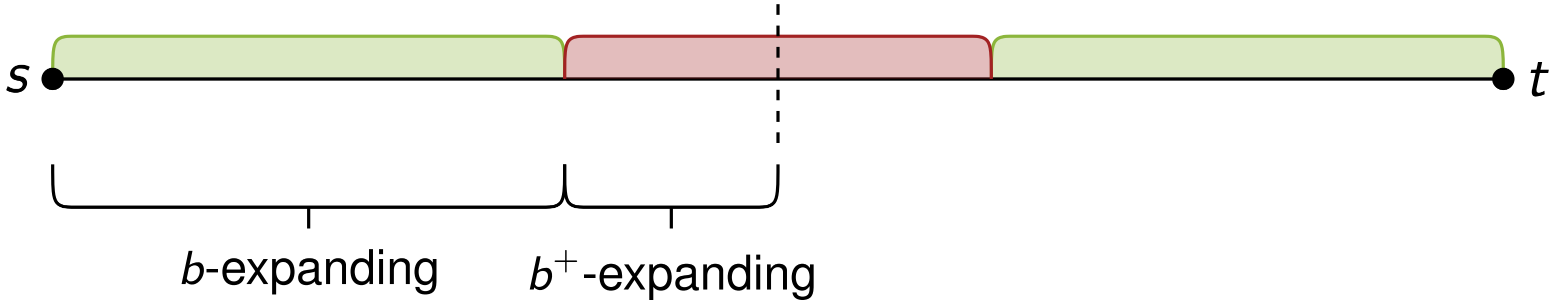

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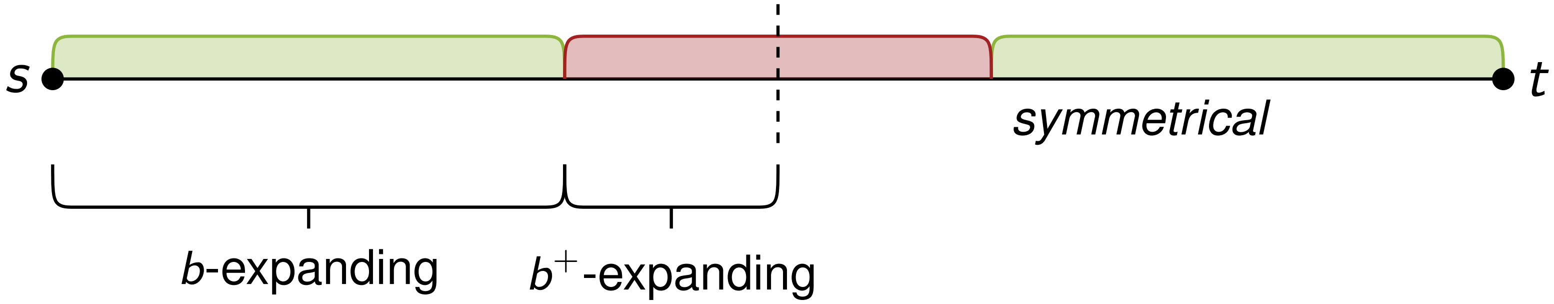

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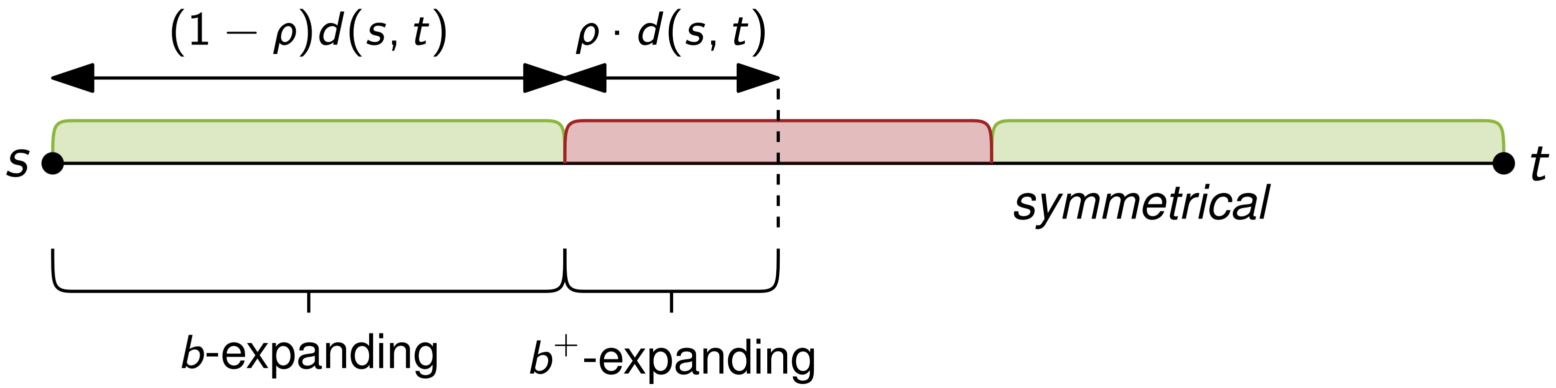

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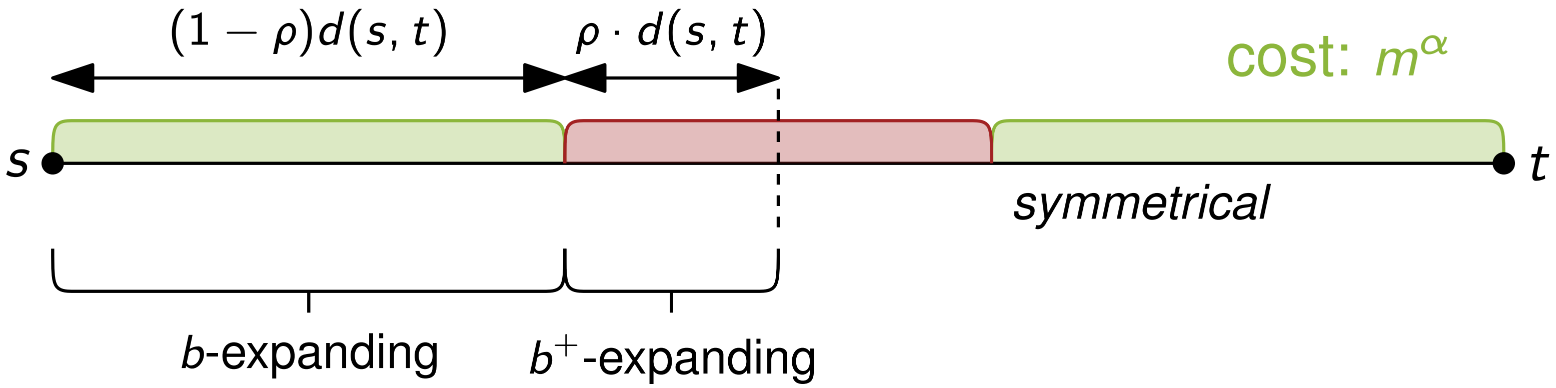

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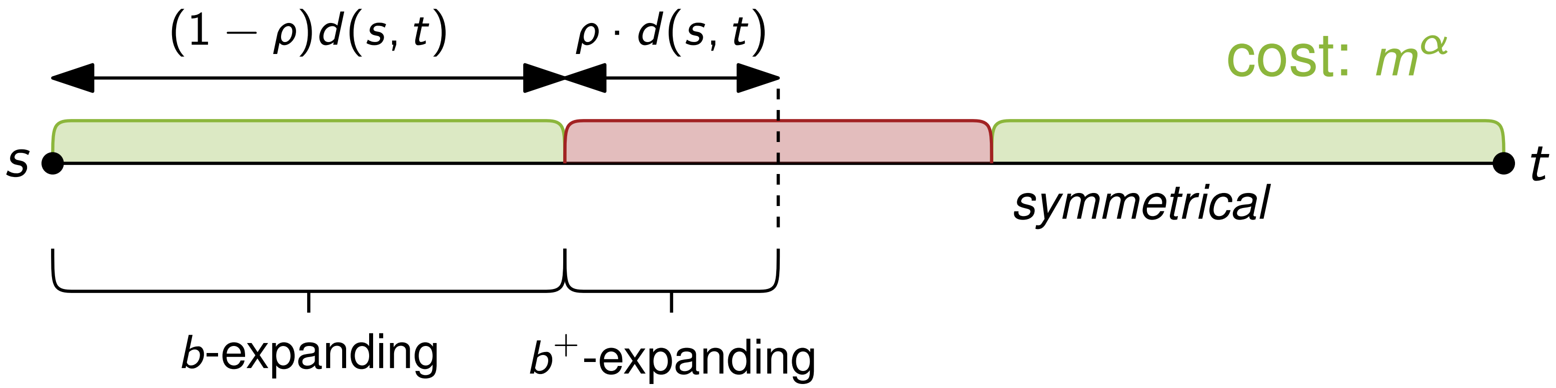

$\square$

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We have $c_{\mathrm{bi}}(s, t) \in \tilde{\mathcal{O}}\left(m^{1-c/2}\right)$ for $s, t \in V$ with b-expansion overlap of length $c \cdot \log_b m$.

Theorem 2
We have $c_{\mathrm{bi}}(s, t) \in \tilde{\mathcal{O}}\left(m^{1-\varepsilon}\right)$ for $s, t \in V$ with b-expansion overlap of length at least $c \cdot d_\alpha(s, t)$ for constant c and $\varepsilon = \frac{c(1-\alpha)}{2(\log_b(b^+)+c)} > 0$, for maximum expansion b^+.

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  $\Rightarrow \quad x = \log_m \hat{c}$

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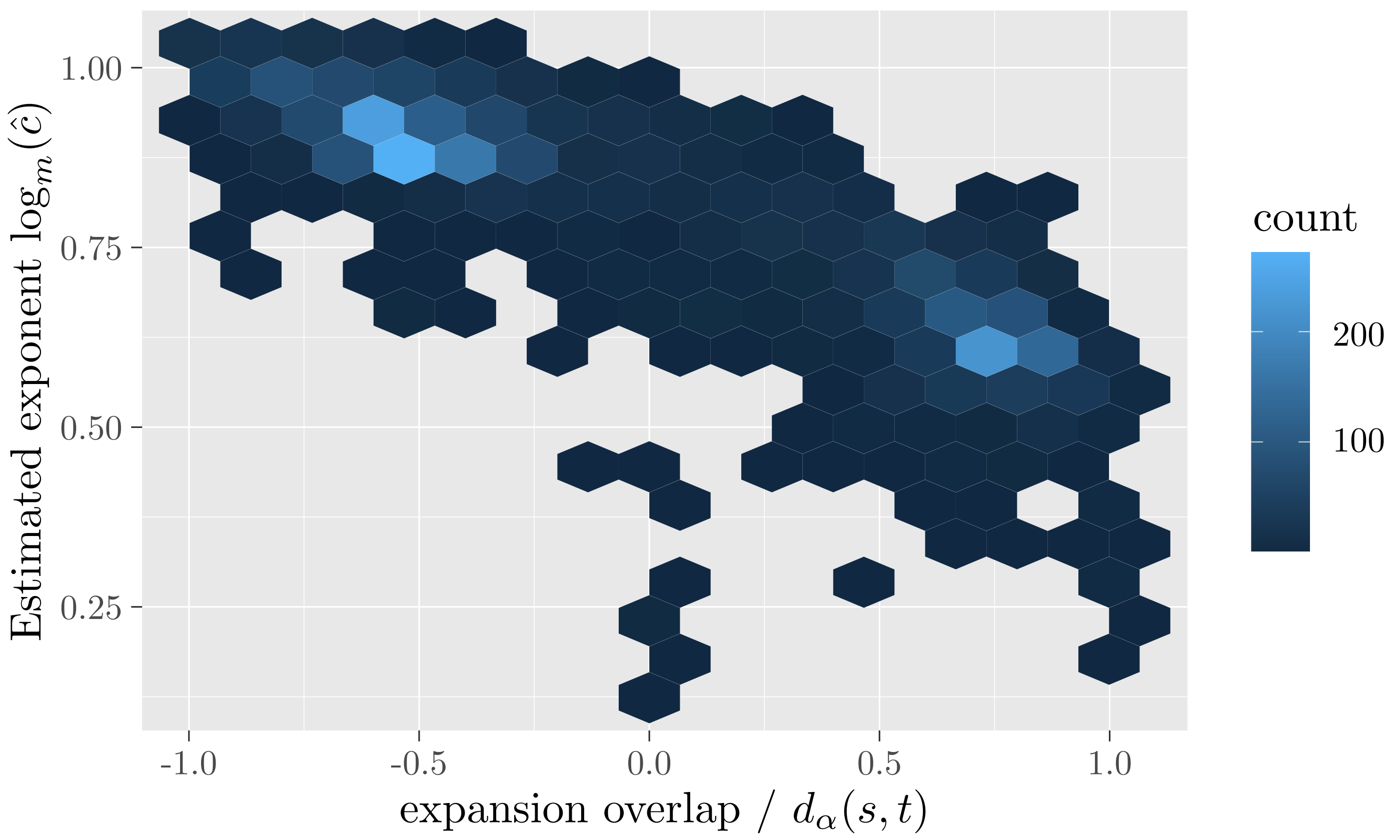

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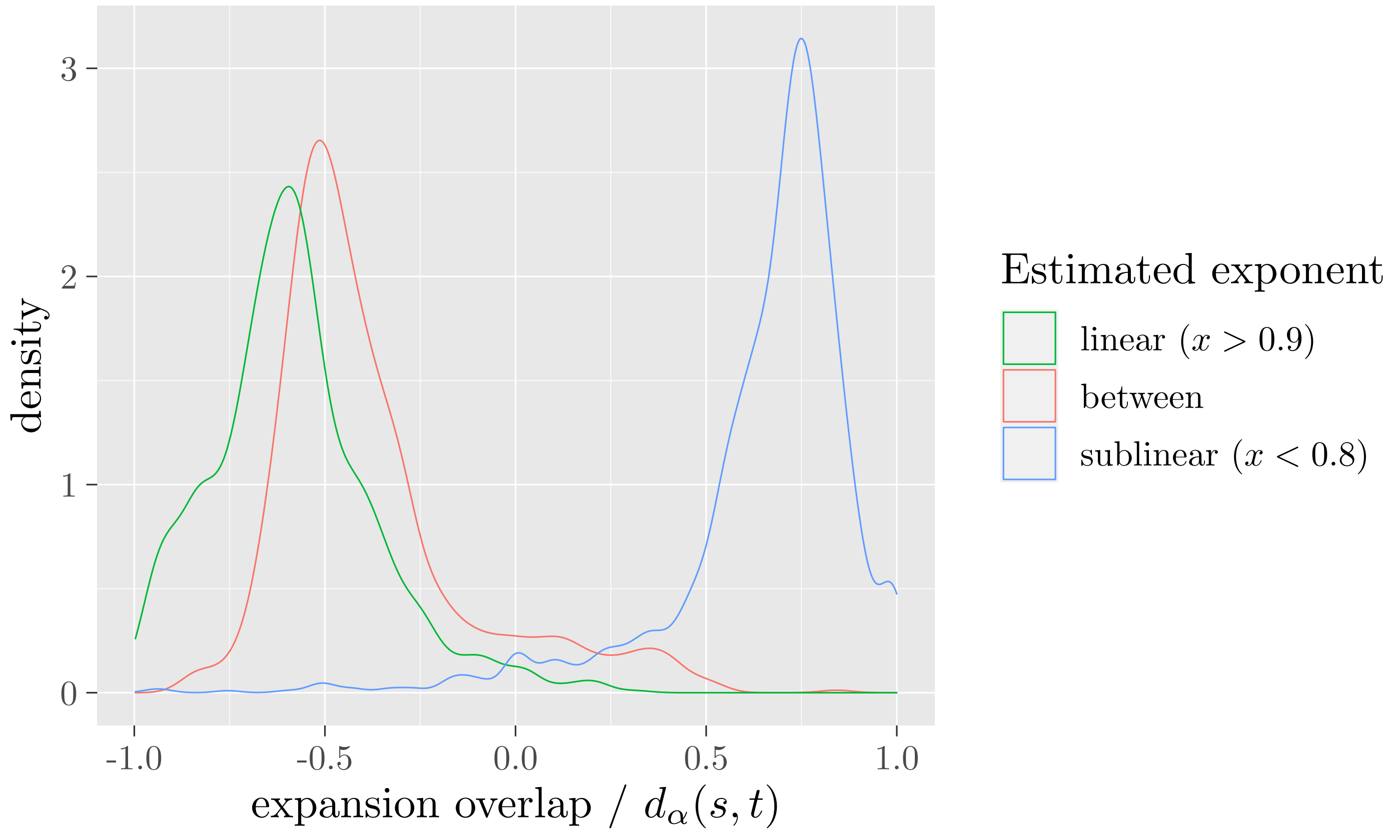

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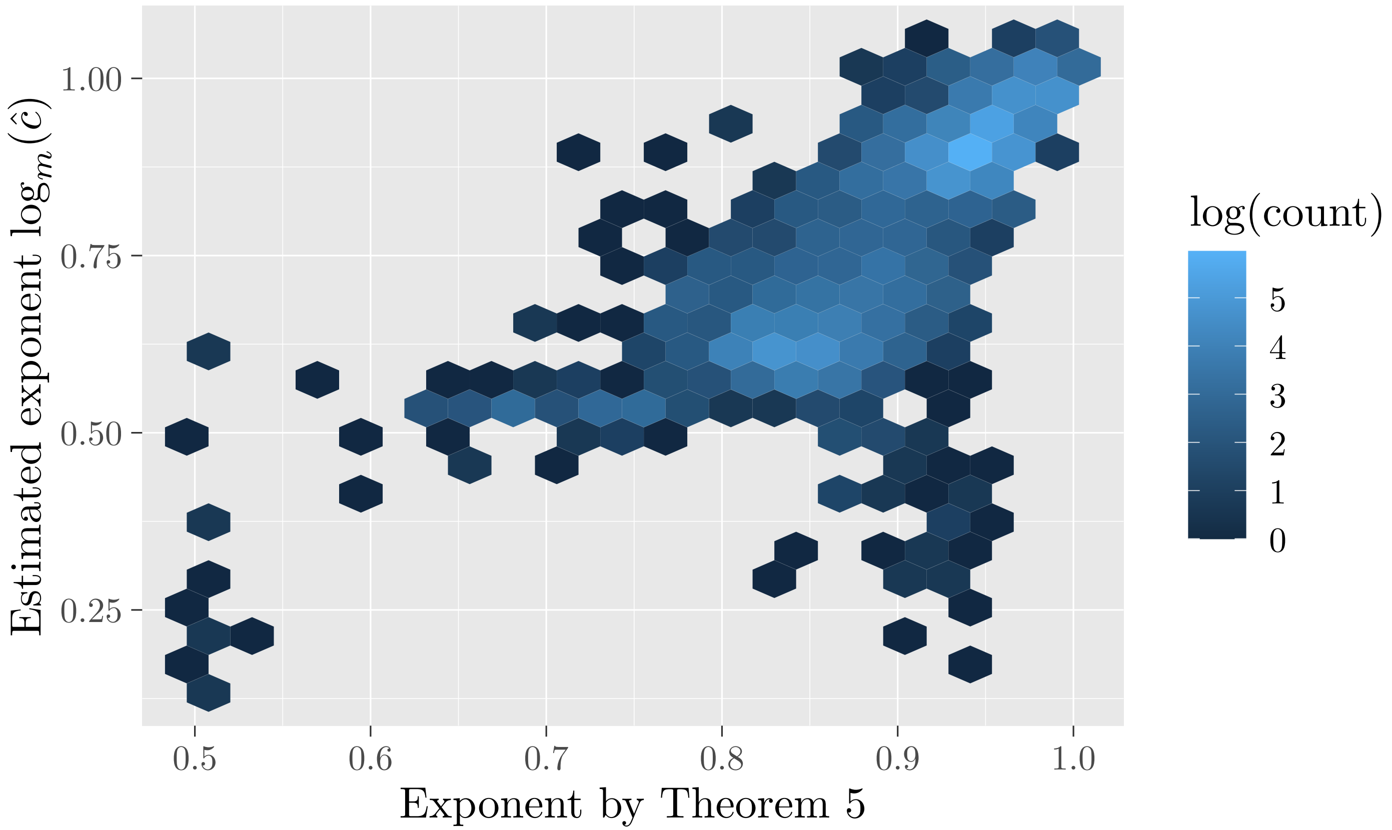

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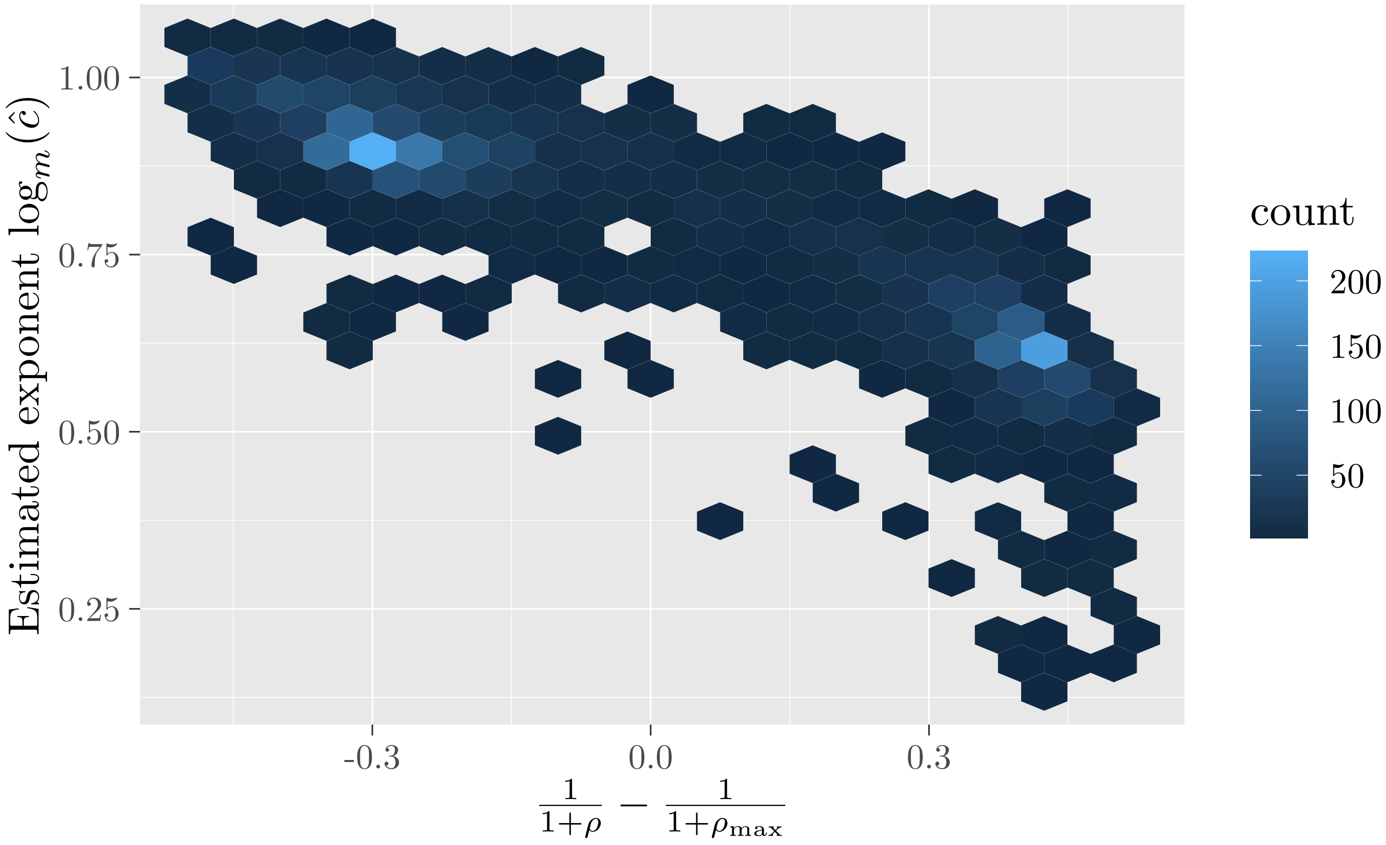

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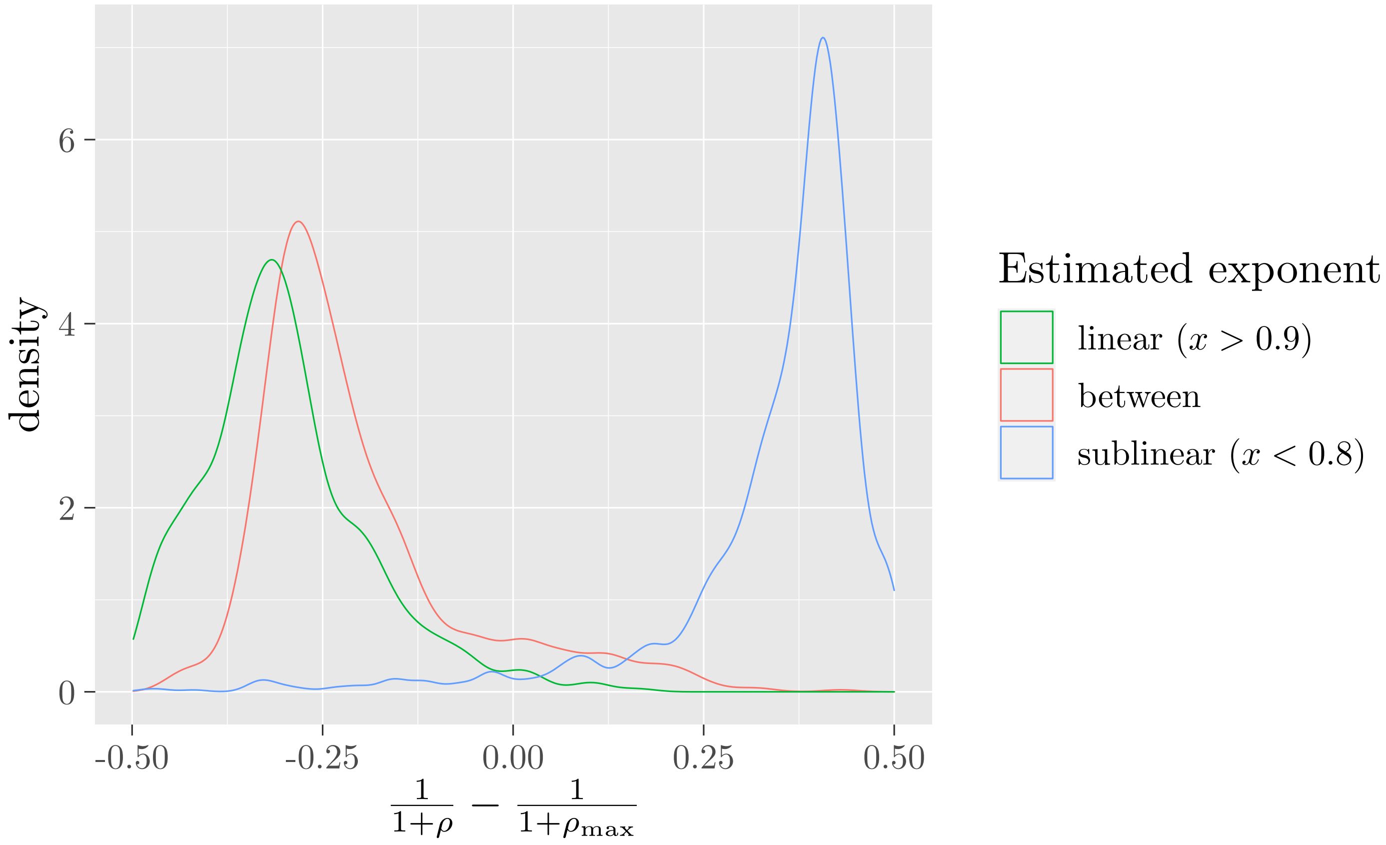

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More info:
`scale.iti.kit.edu/resources/supplemental/`
On ArXiv soon

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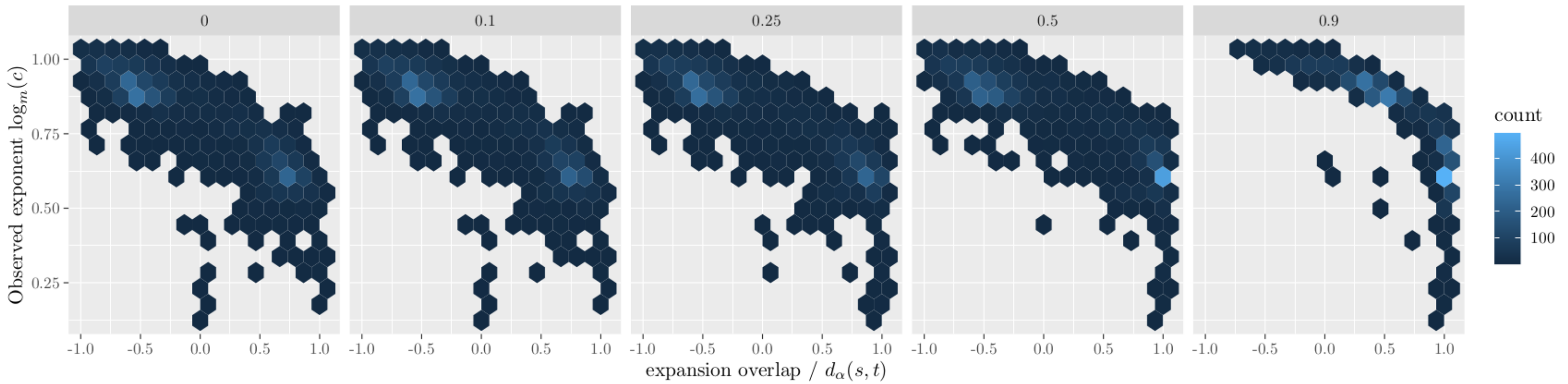

Figure 5 Estimated exponent for parameter c of Theorem 5 under $b = 2$ and different values of α

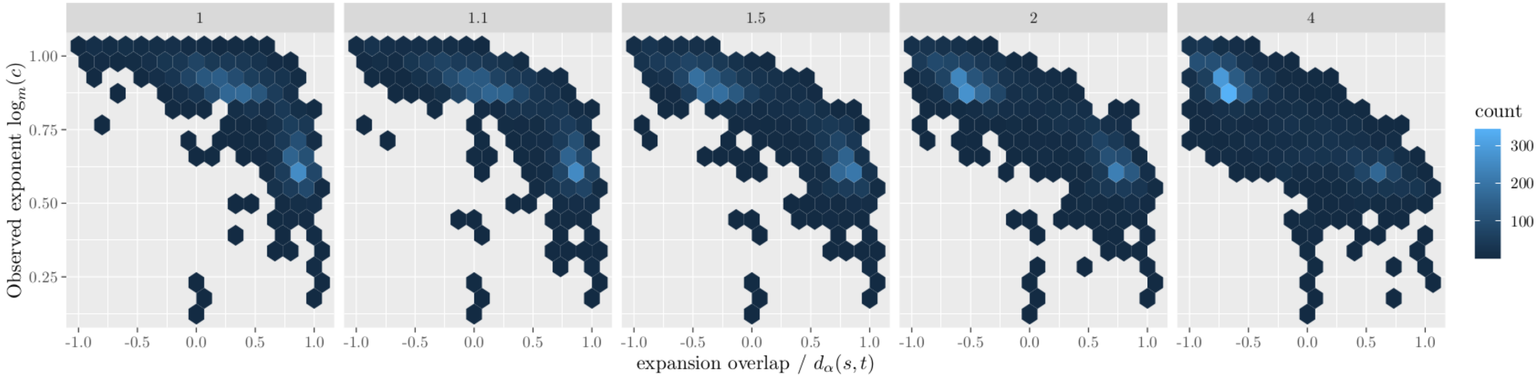

Figure 6 Estimated exponent for parameter c of Theorem 5 under $\alpha = 0.1$ and different values of b