From Symmetry to Asymmetry: Generalizing TSP Approximations by Parametrization

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Definition: The Traveling Salesman Problem
Given a complete undirected graph with non-negative edge weights, find a cheapest Hamiltonian cycle.

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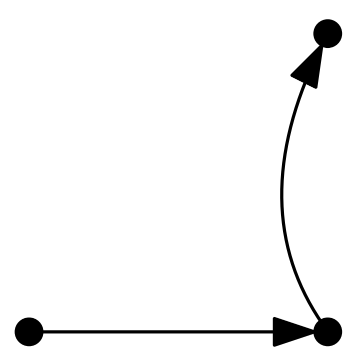

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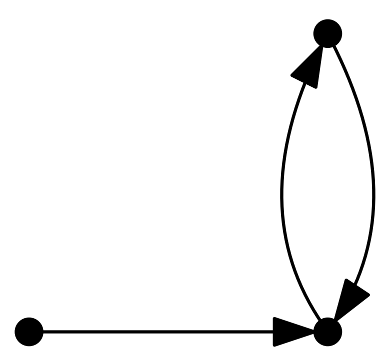

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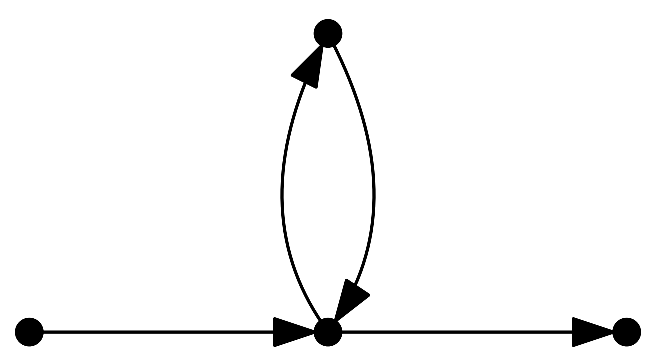

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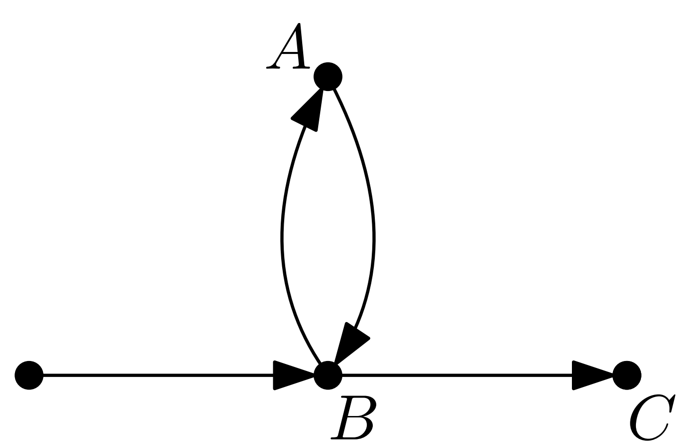

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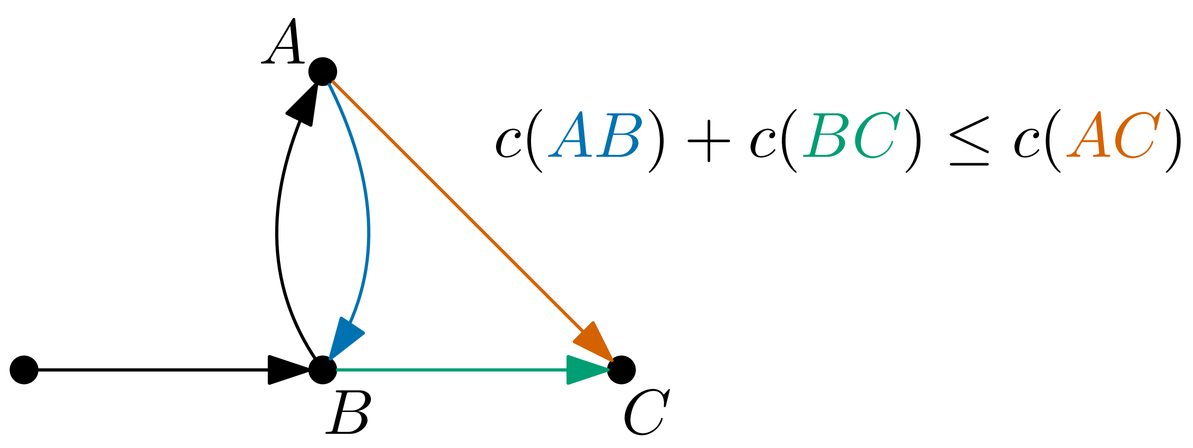

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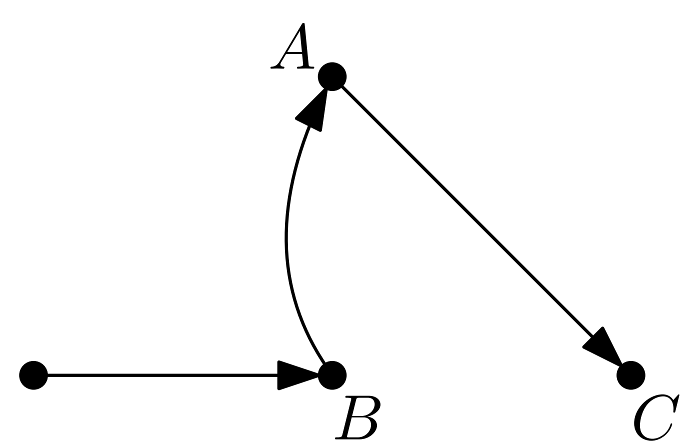

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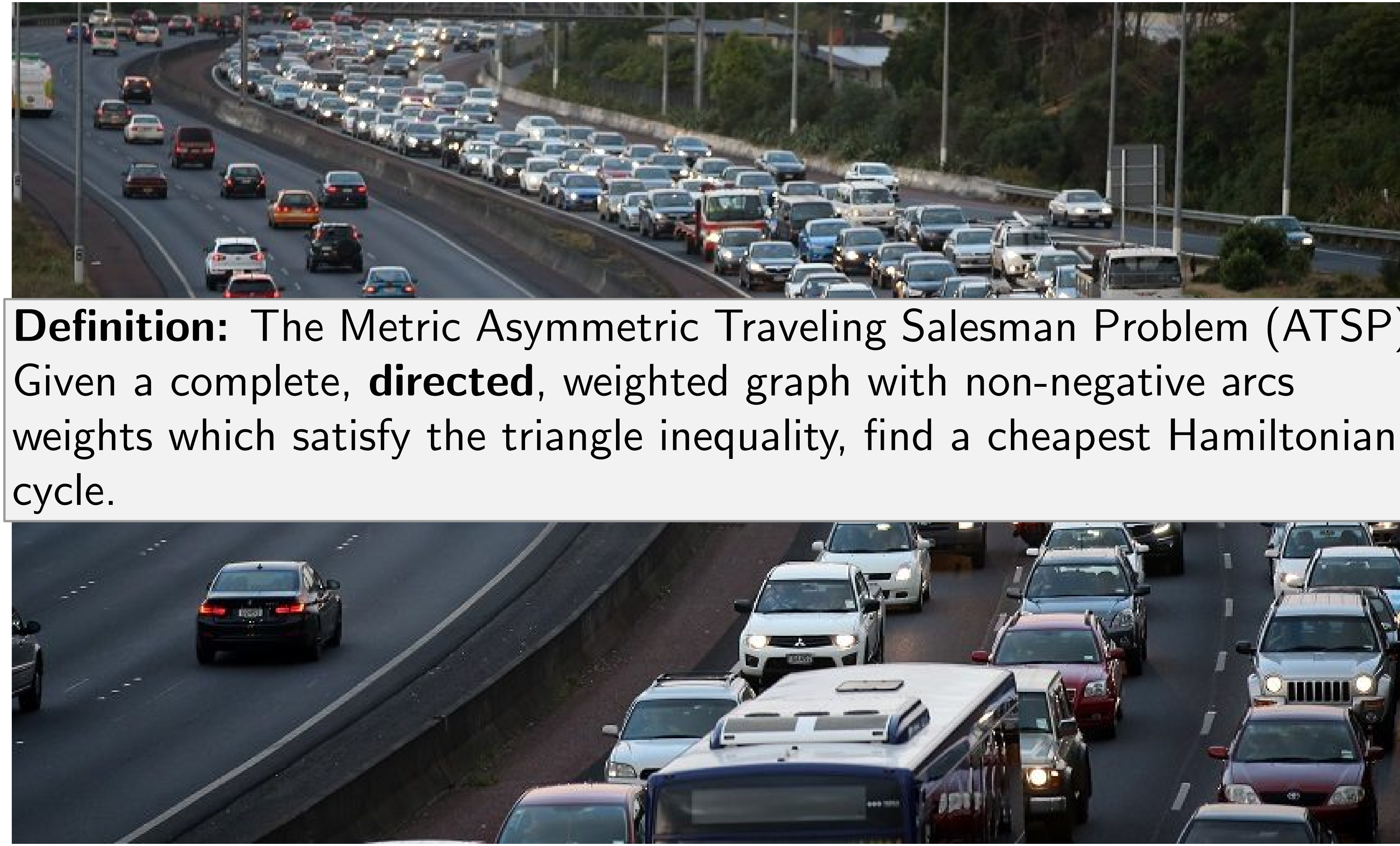

Definition: The Metric Asymmetric Traveling Salesman Problem (ATSP)
Given a complete, **directed**, weighted graph with non-negative arcs weights which satisfy the triangle inequality, find a cheapest Hamiltonian cycle.

State of the Art

Exact Solutions

- $\mathcal{O}(2^n n^2)$ by Held and Karp

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- practical solvers without worst case guarantees

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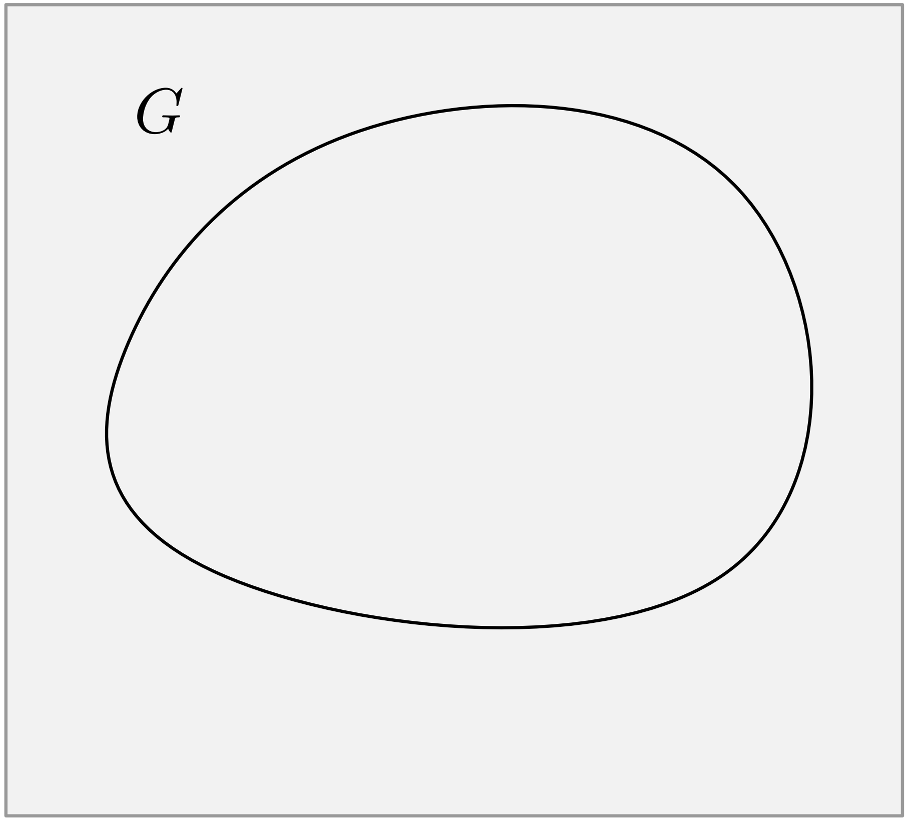

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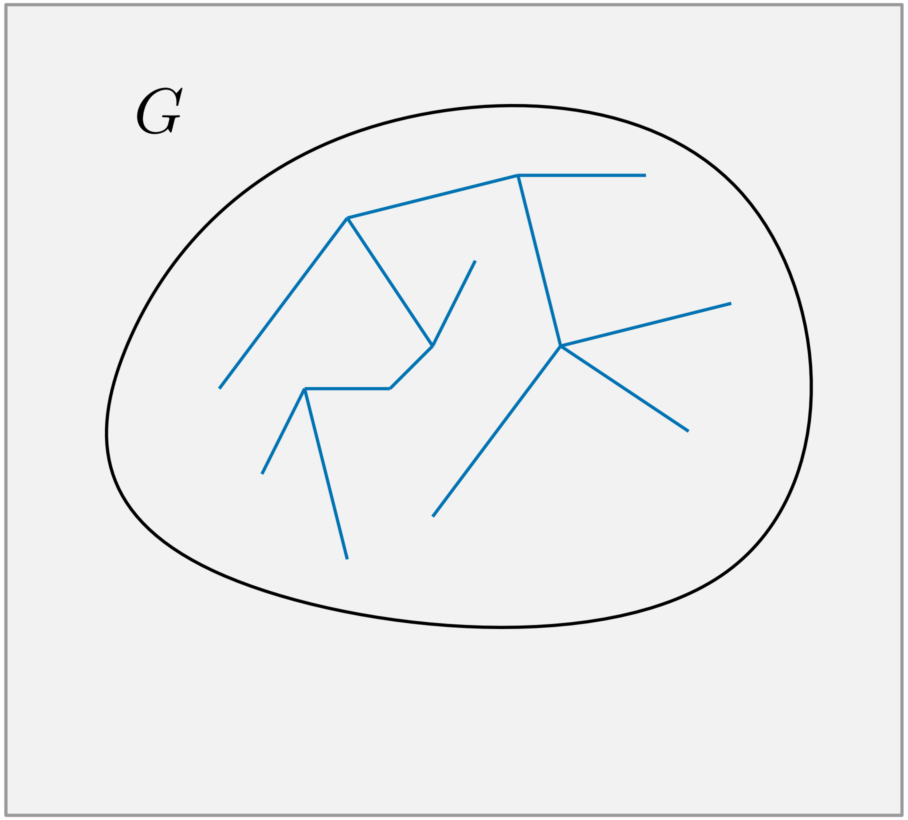

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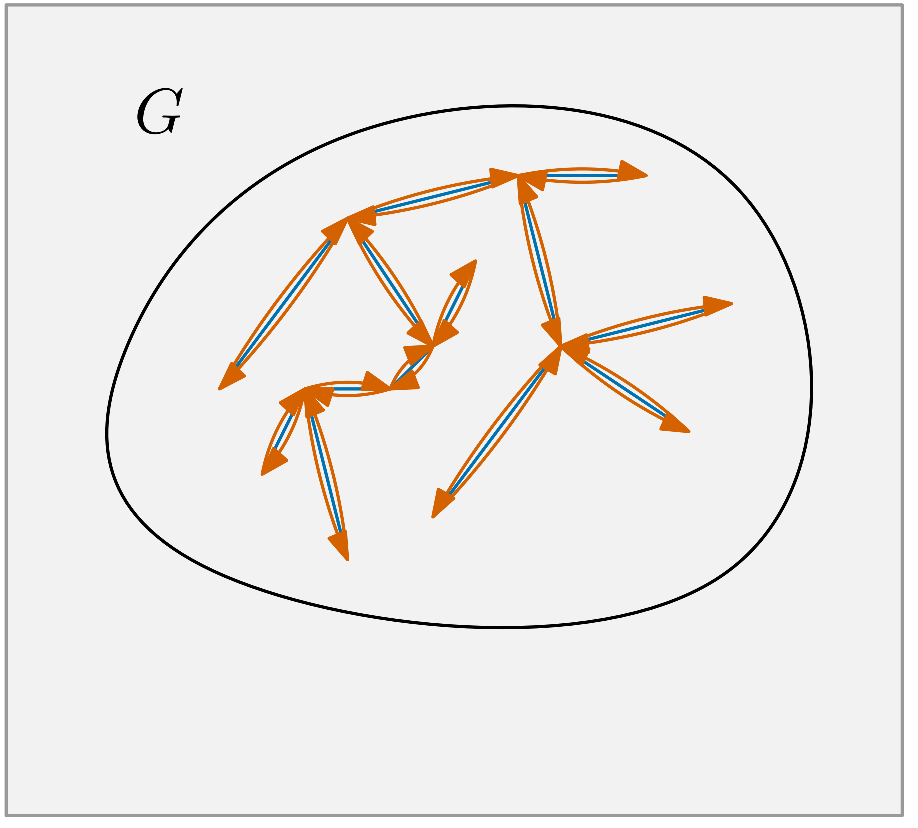

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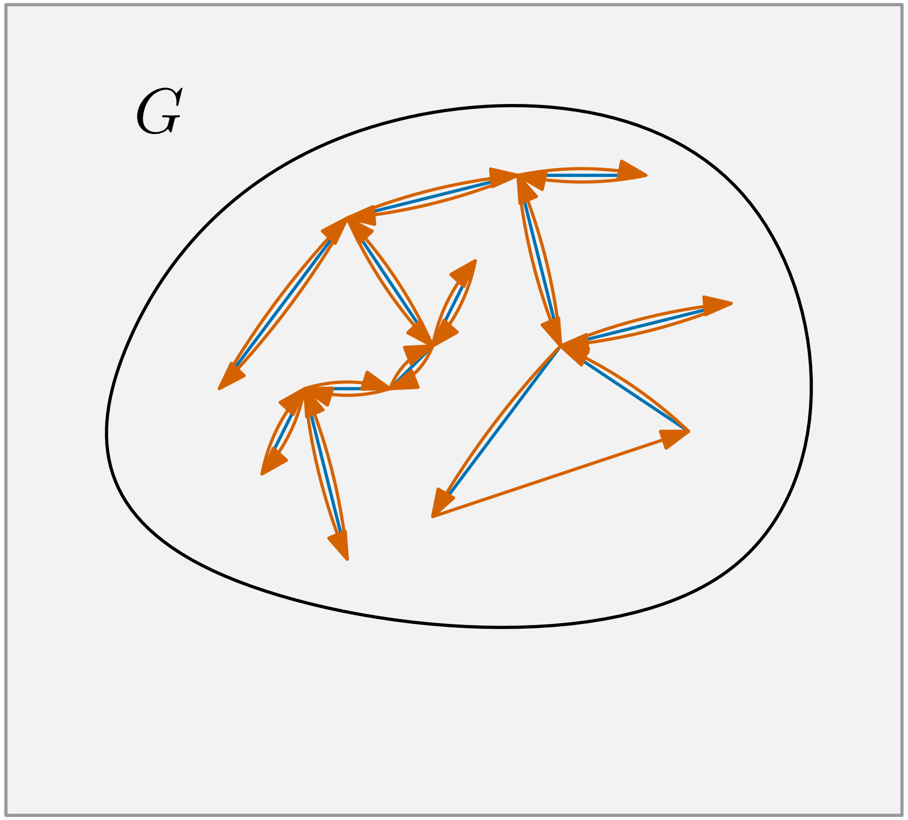

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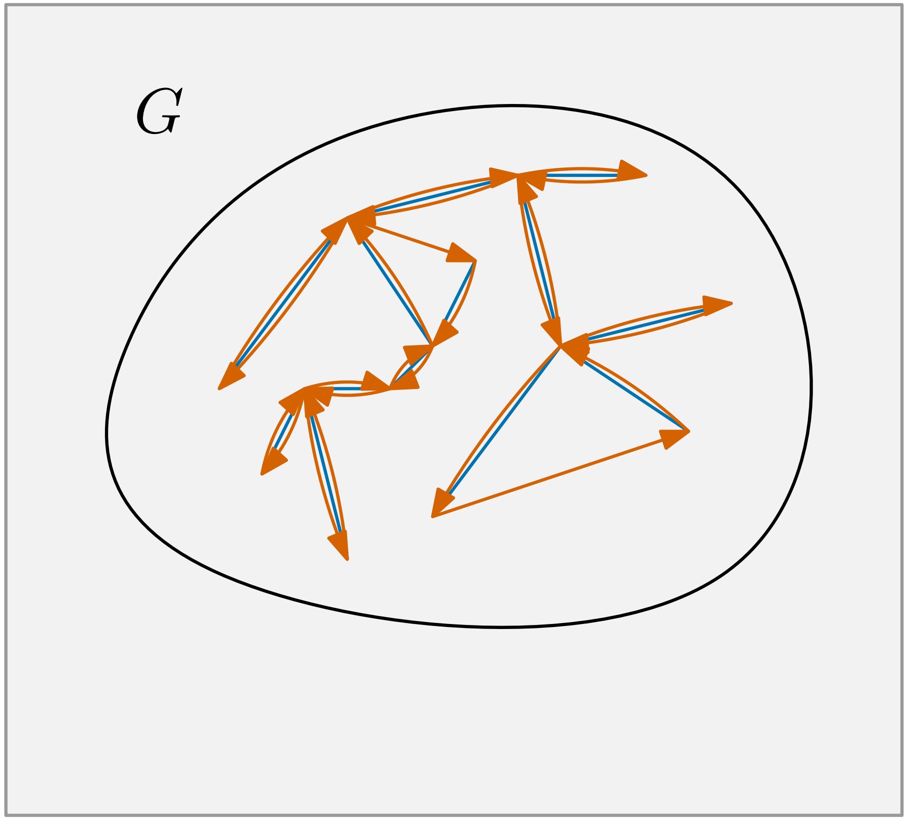

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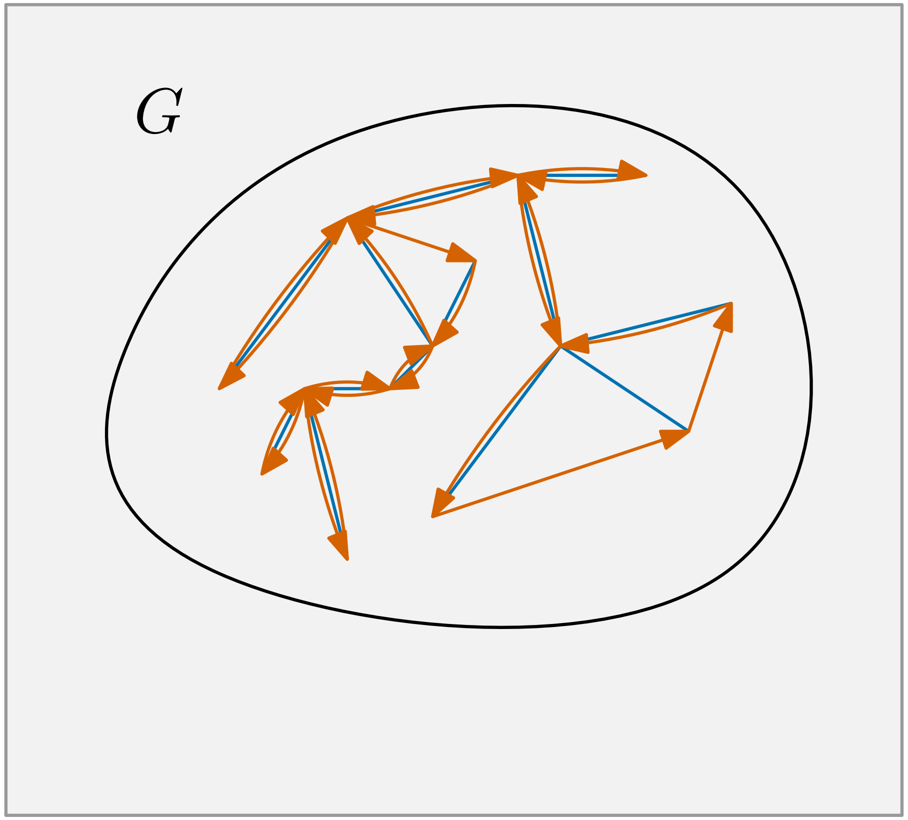

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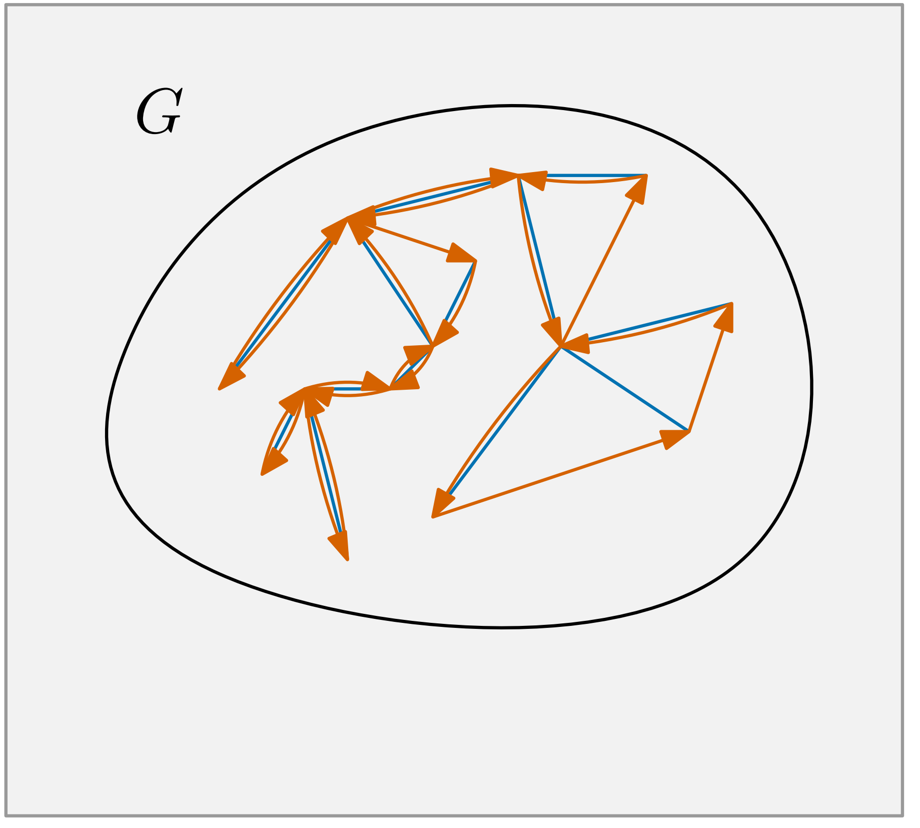

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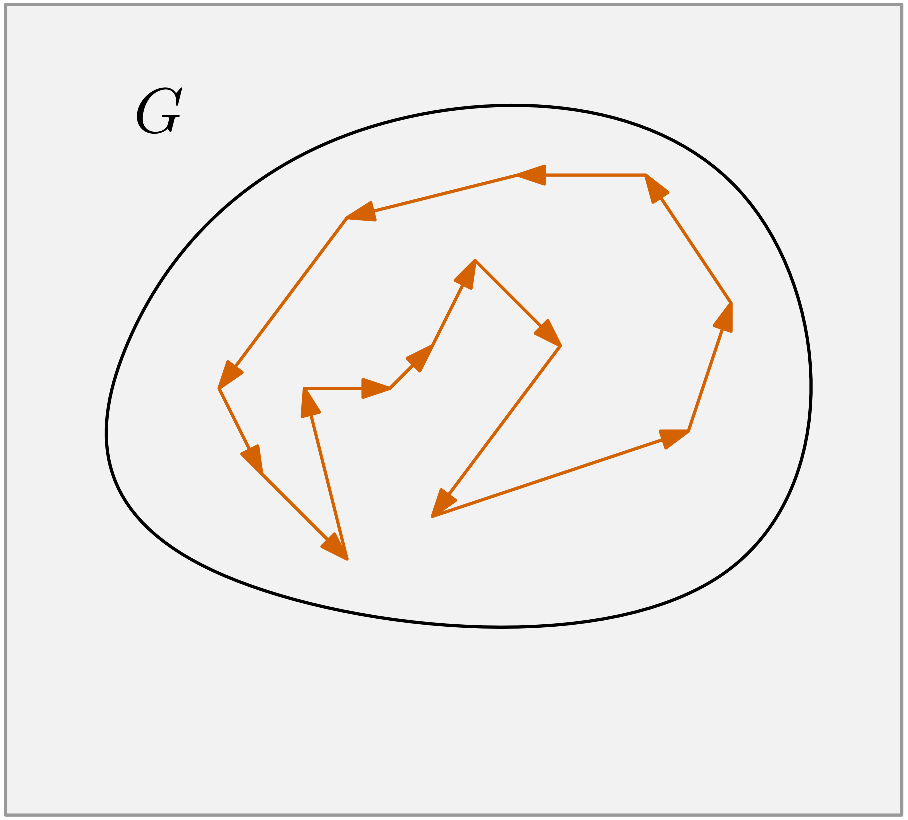

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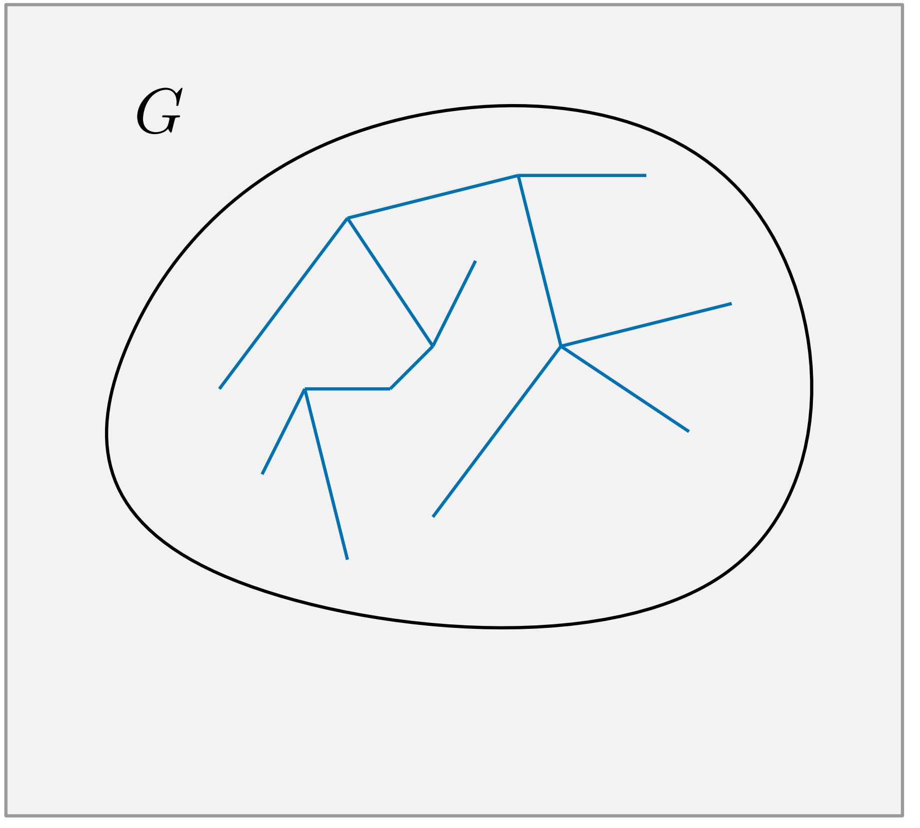

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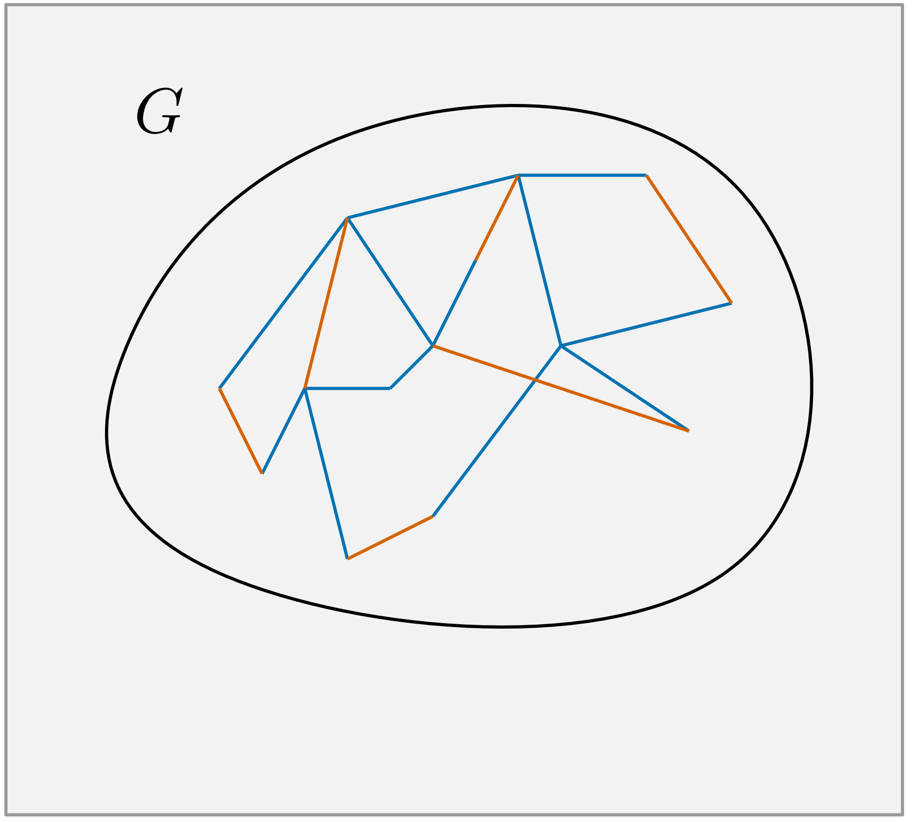

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best lower bounds: $\frac{123}{122}$ vs. $\frac{75}{74}$ (Karpinski et al.)

Our contributions

Goal: FPT runtime

$$f(k) \cdot n^{\mathcal{O}(1)}$$

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- experimental evaluation

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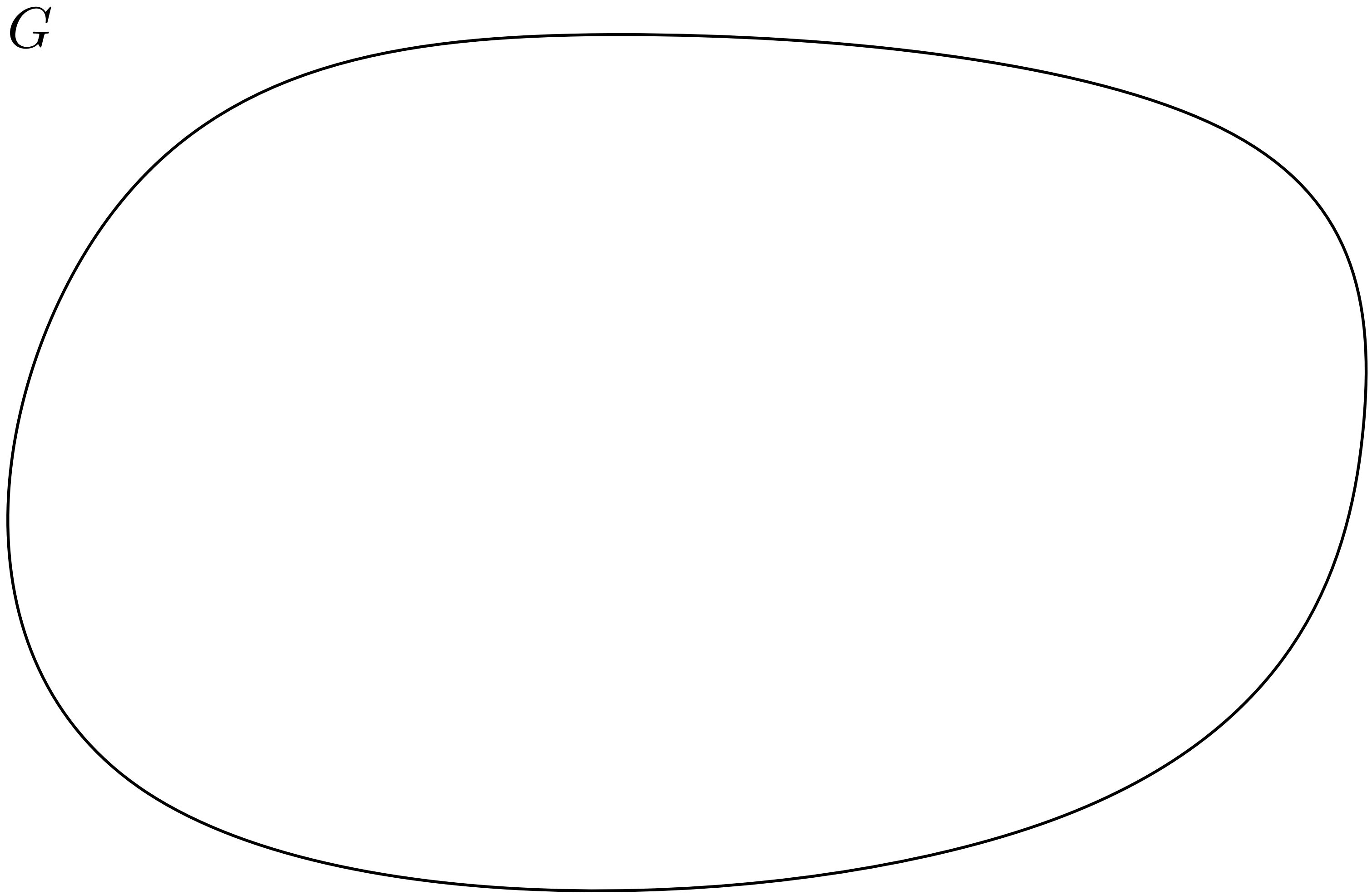

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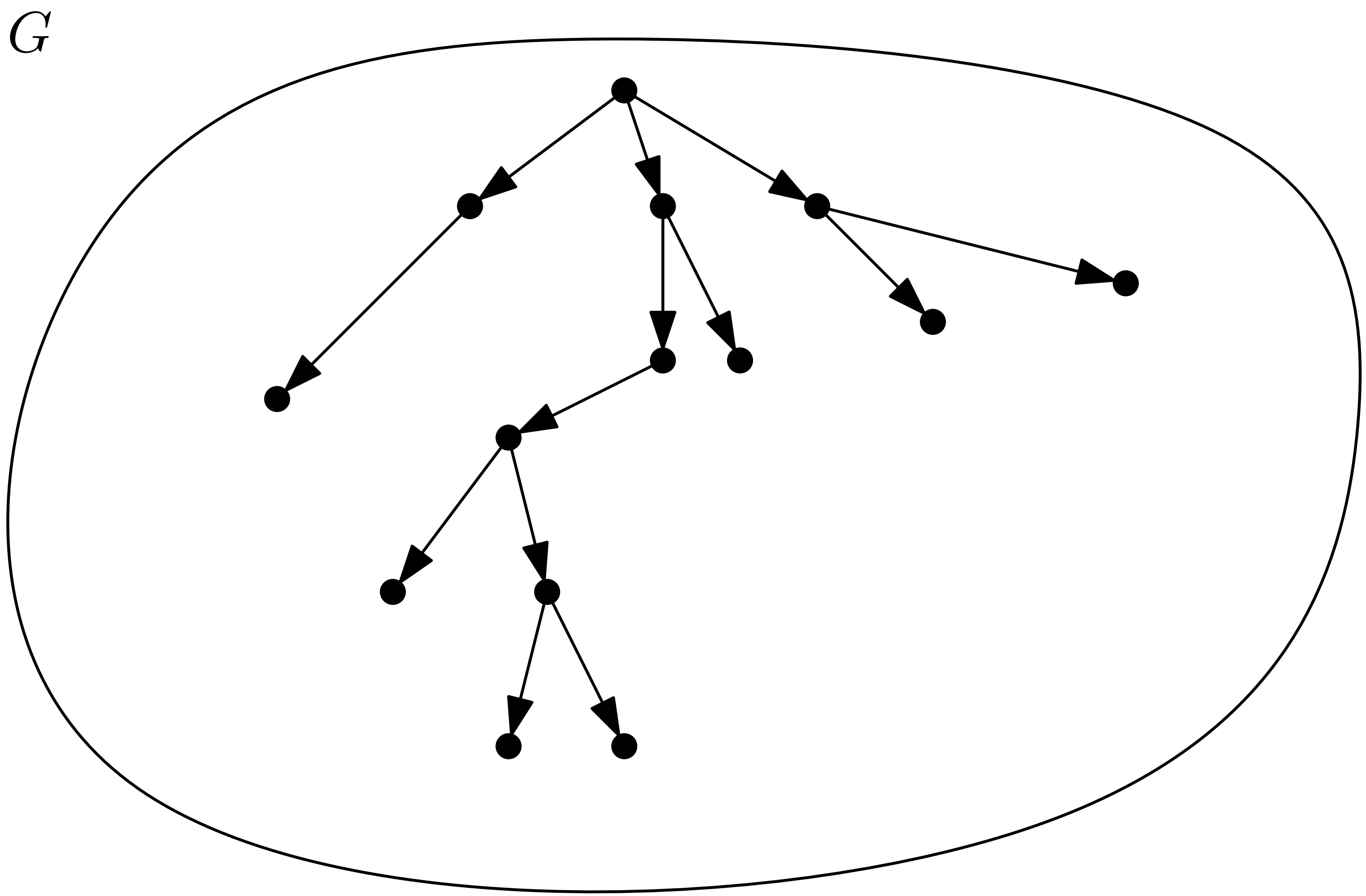

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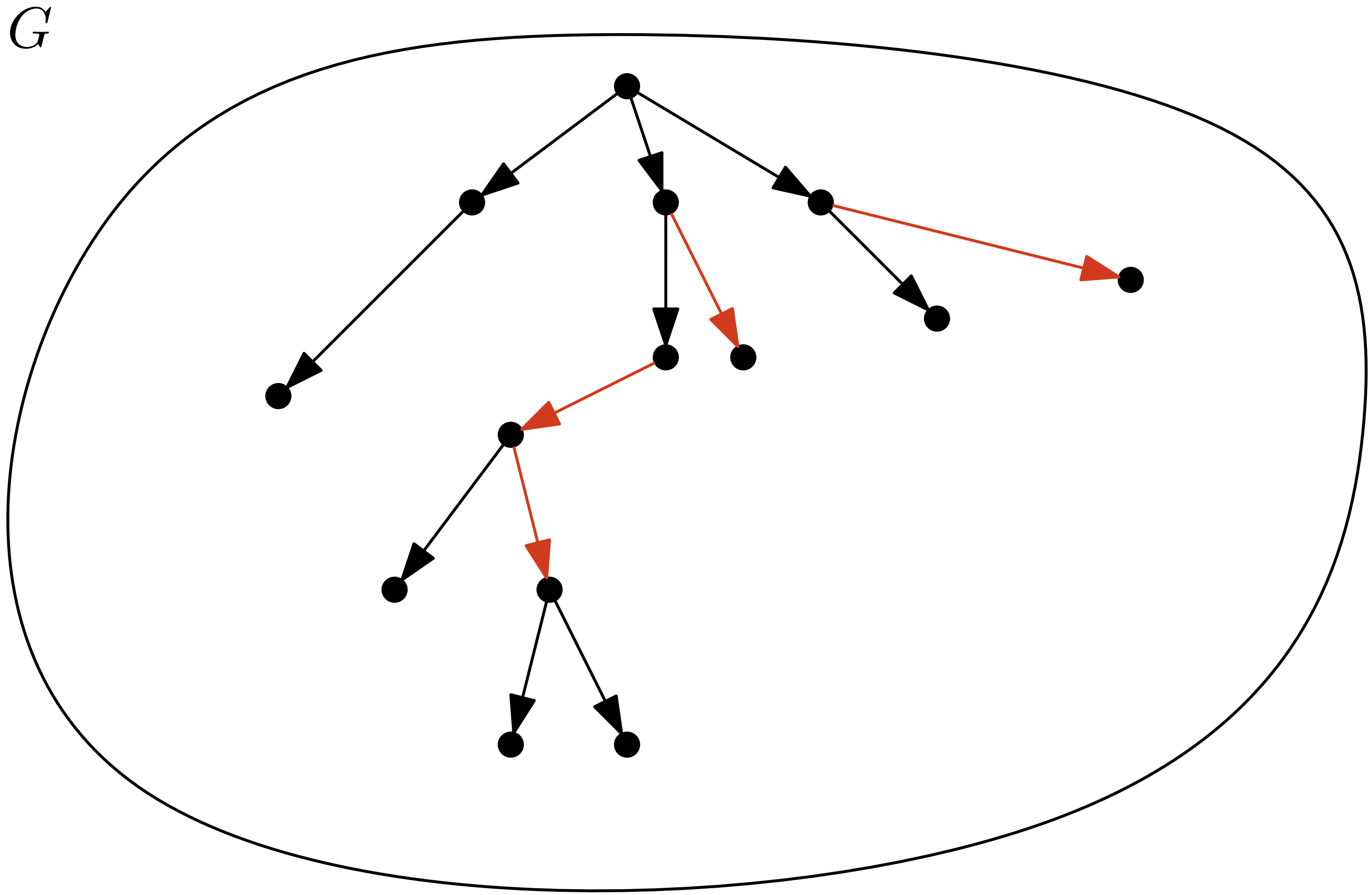

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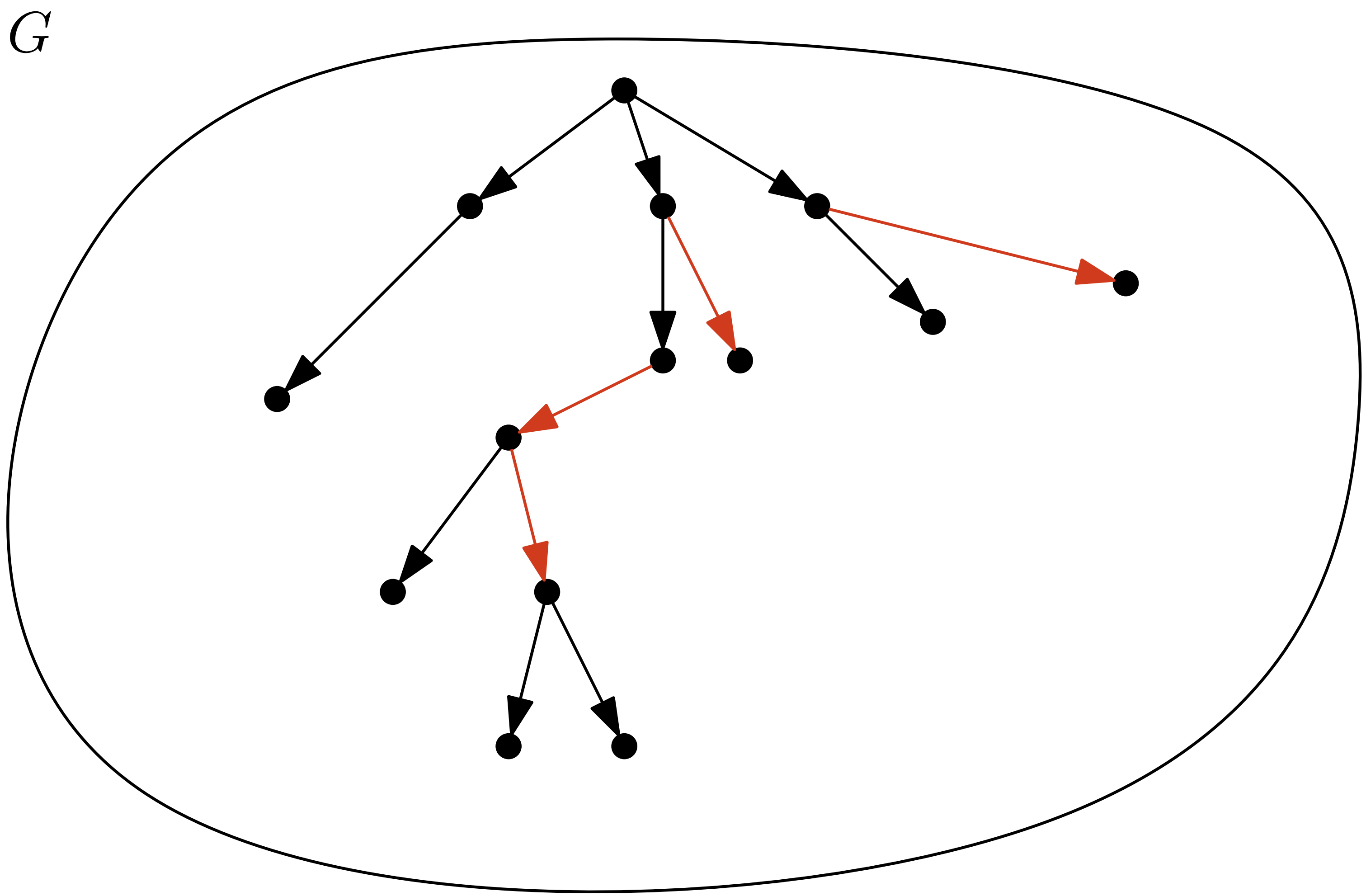
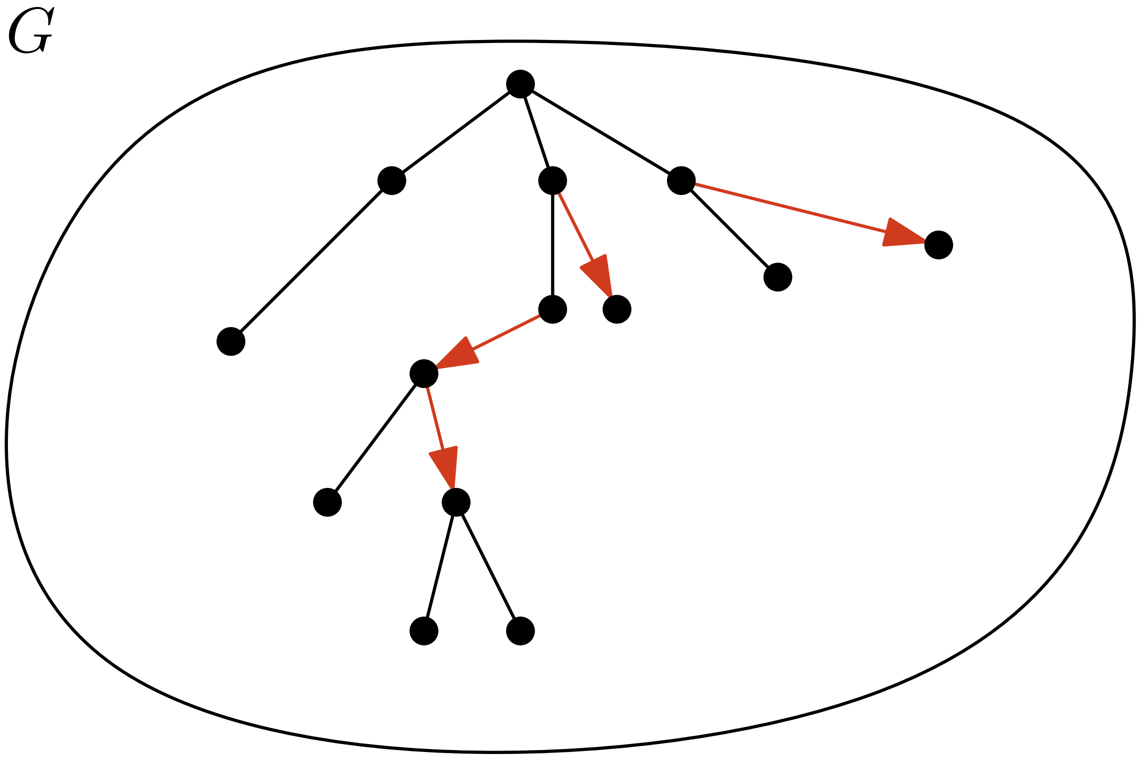


Def. One-way arc: arc (u, v) in tree with $c(u, v) < c(v, u)$

Generalized Tree Doubling Algorithm



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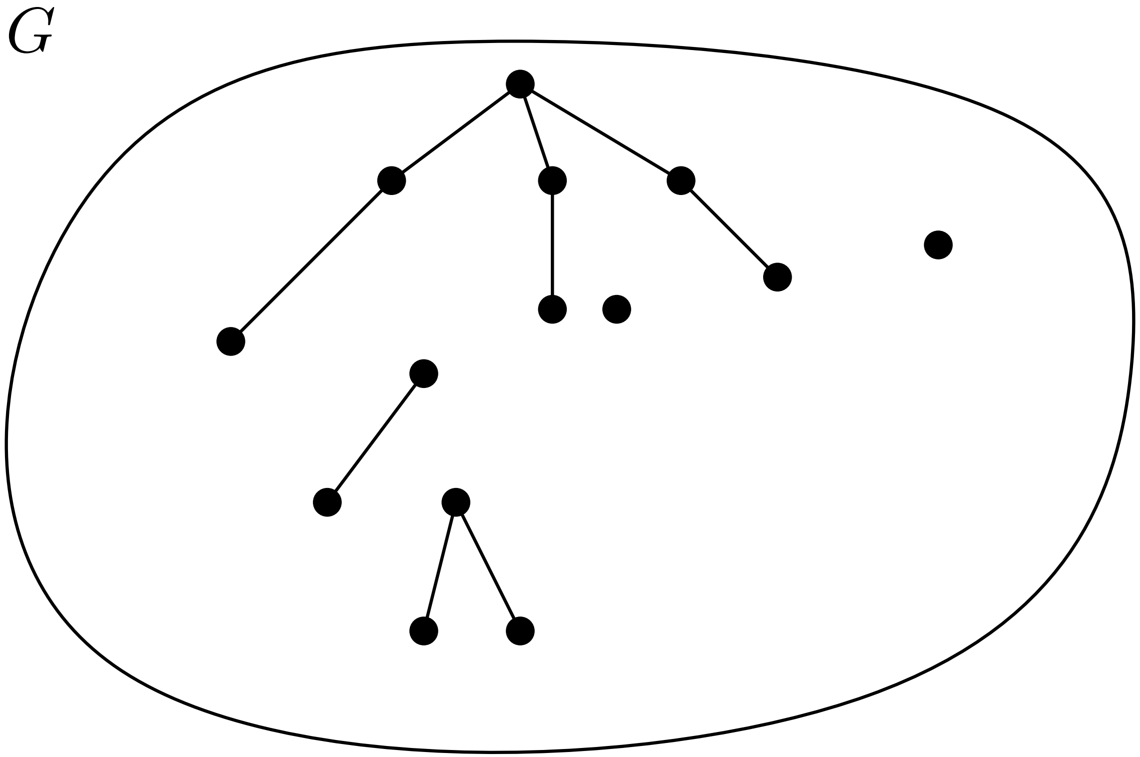

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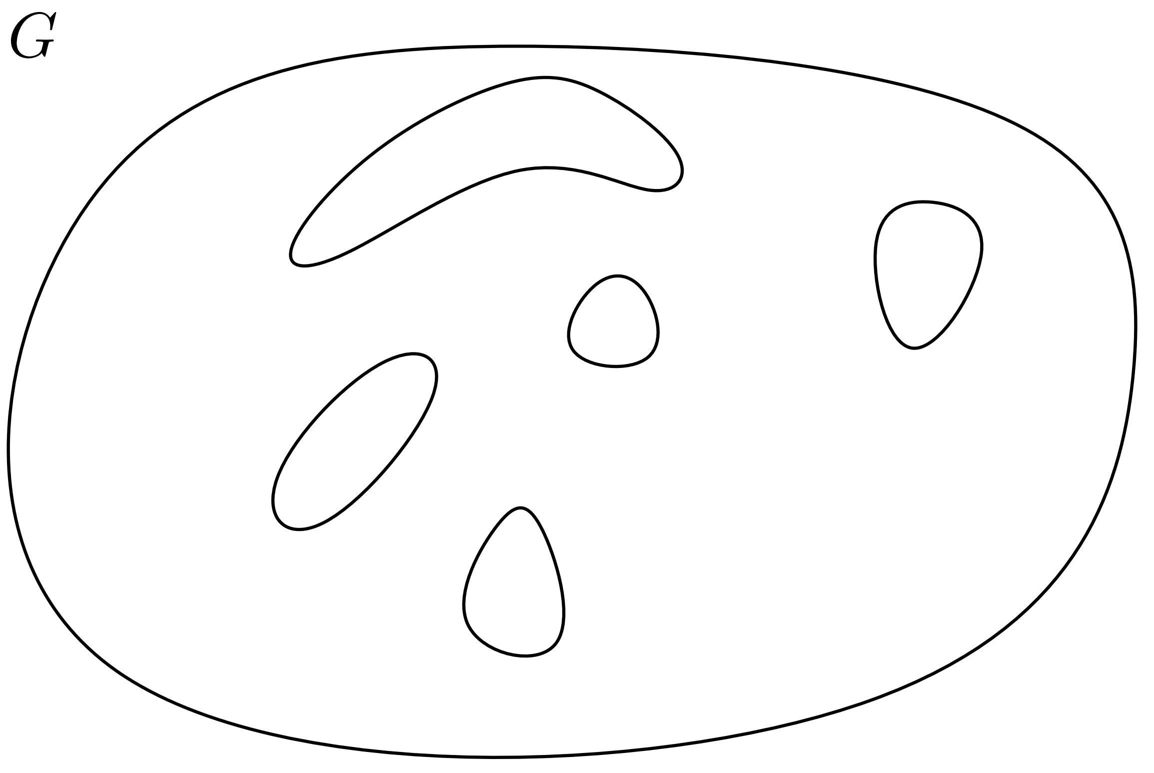

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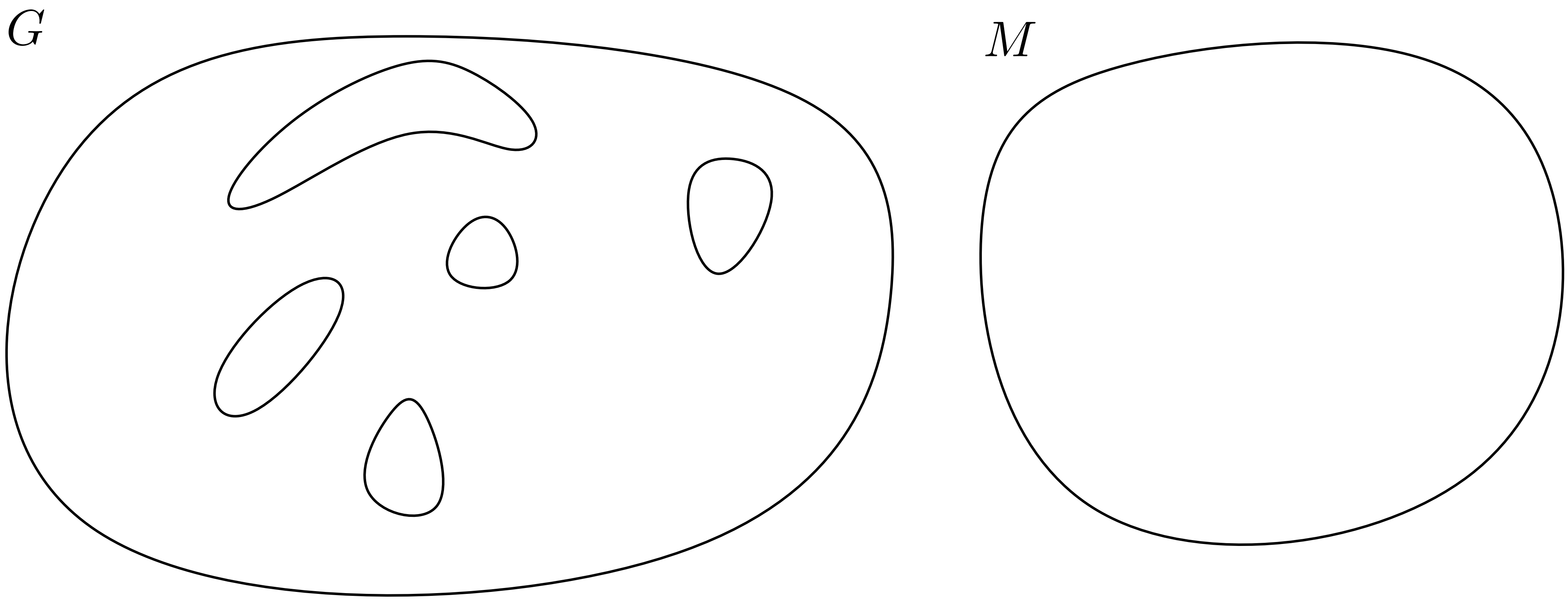

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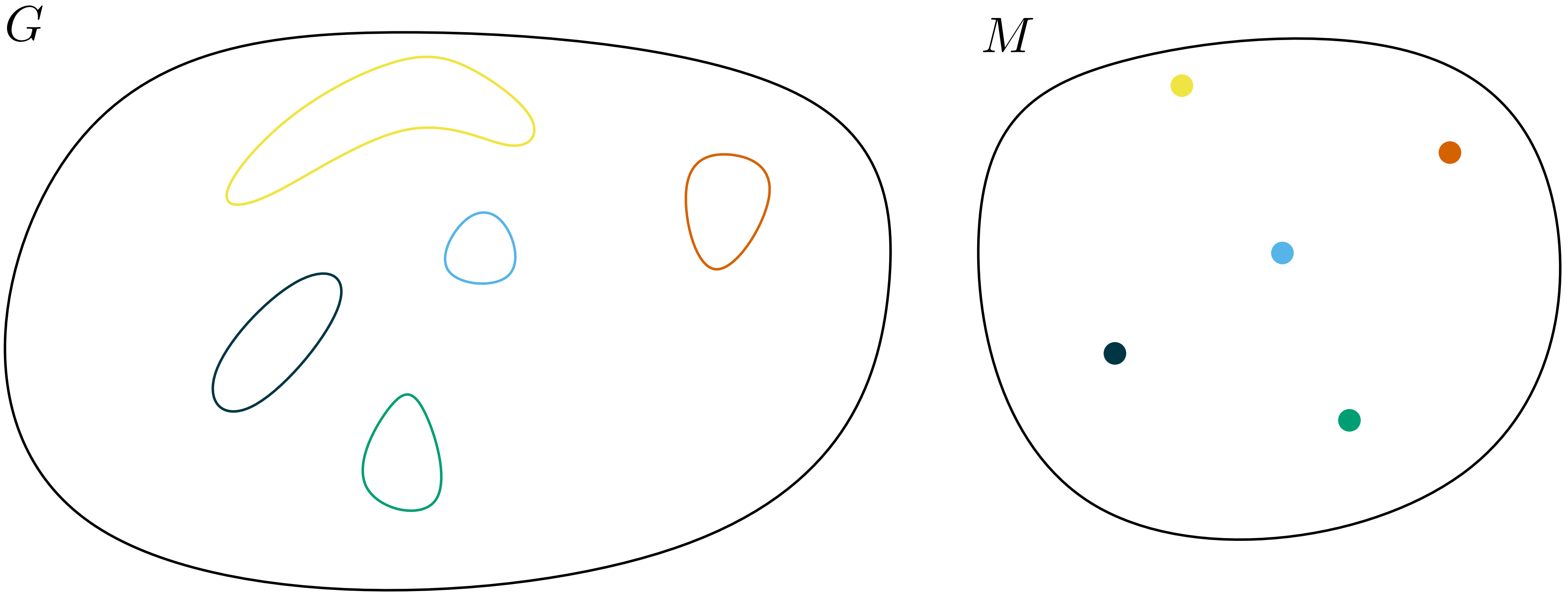

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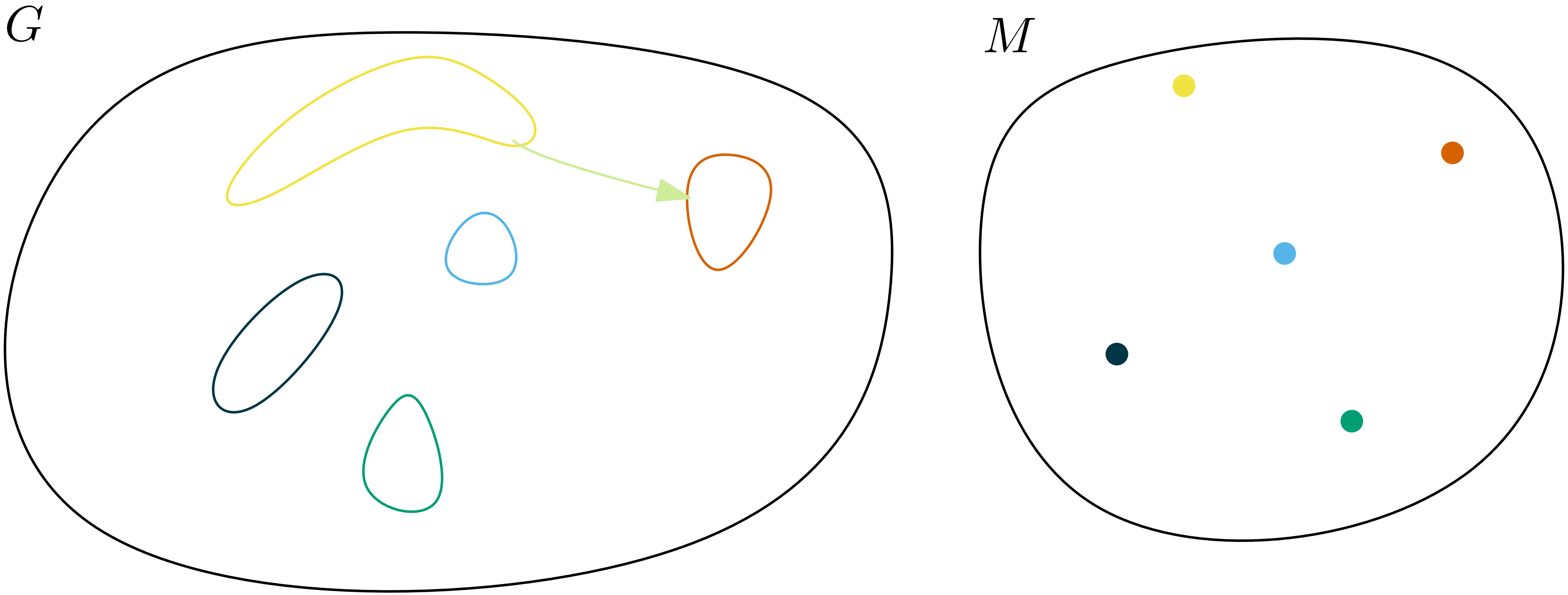

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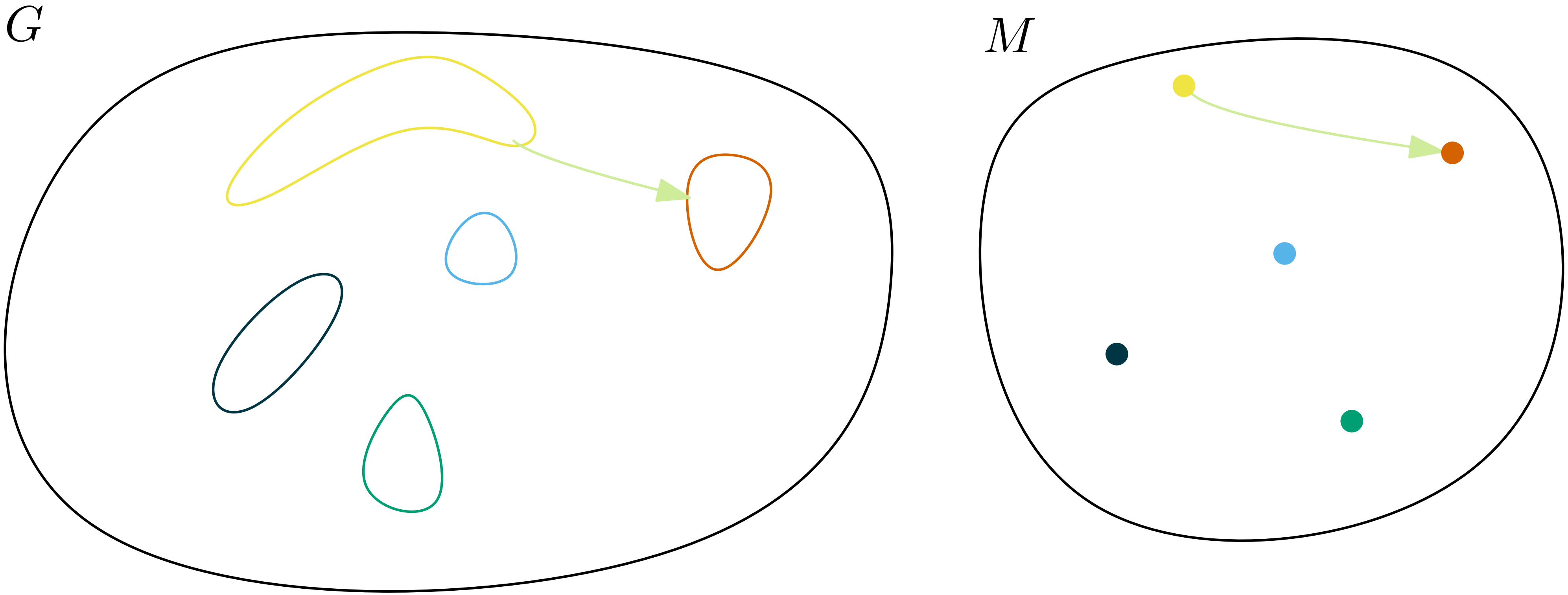

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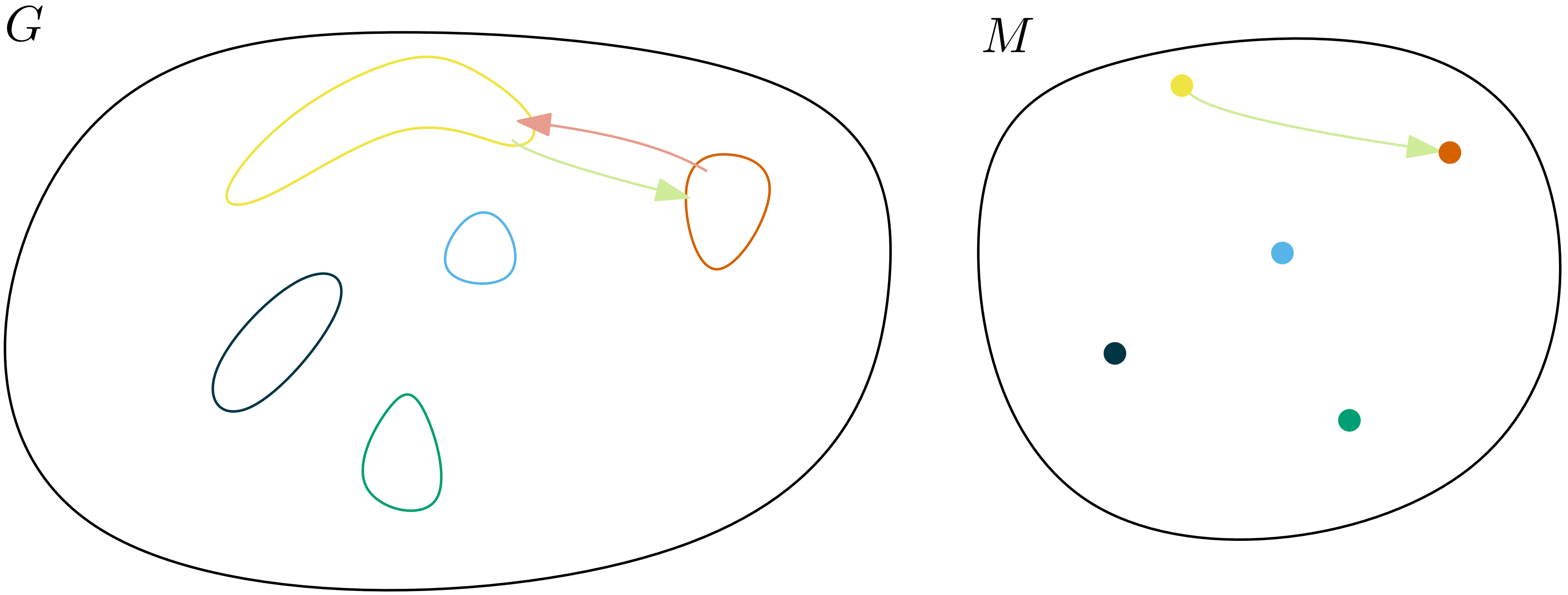

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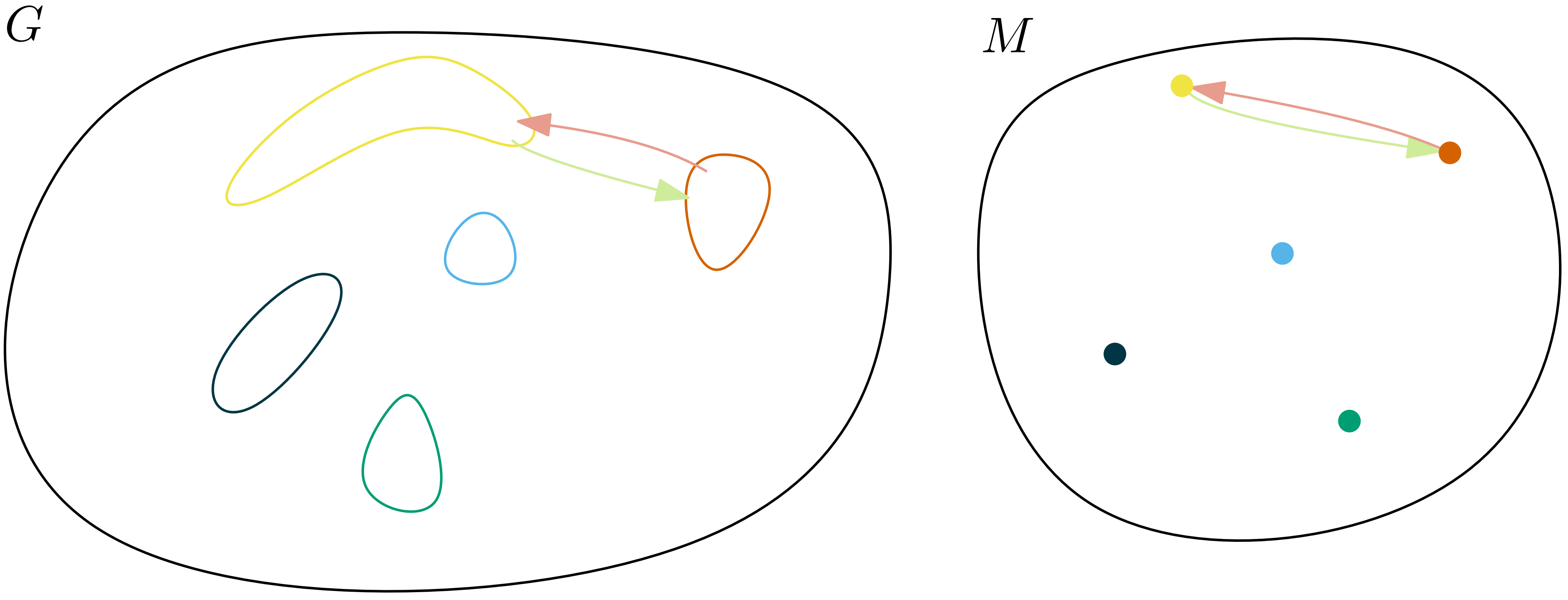

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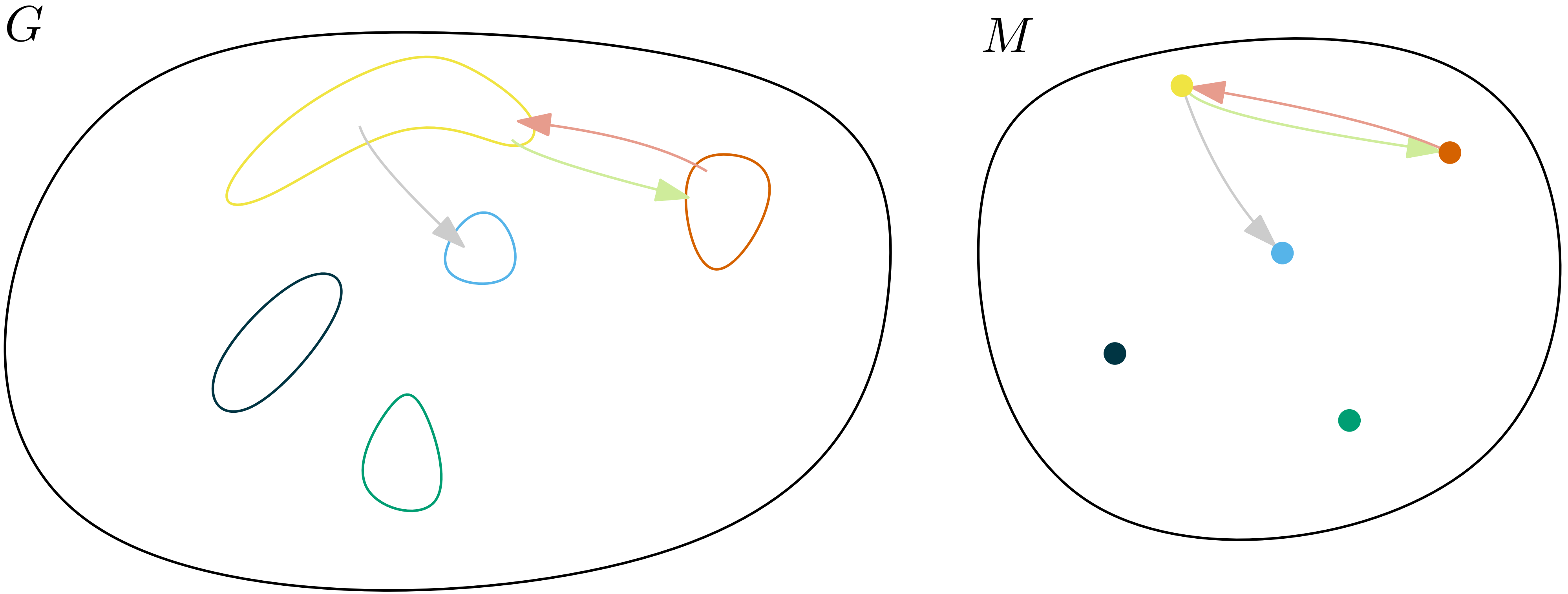

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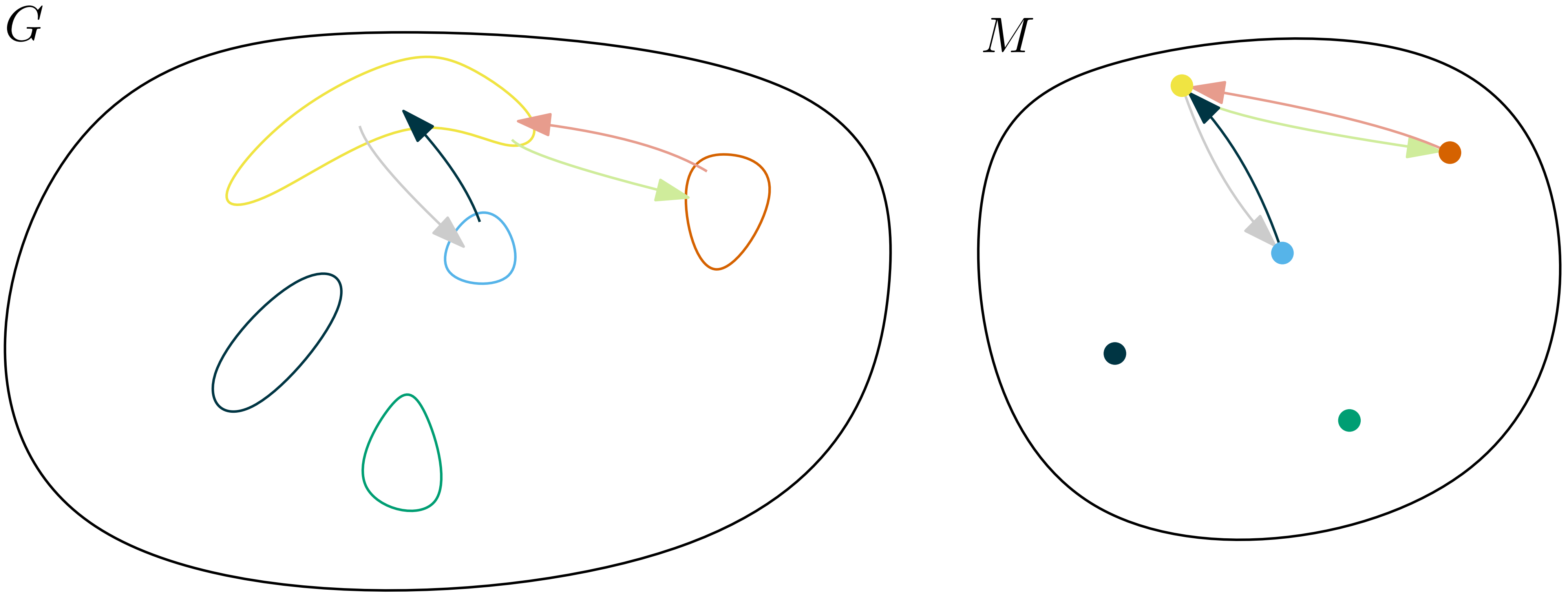

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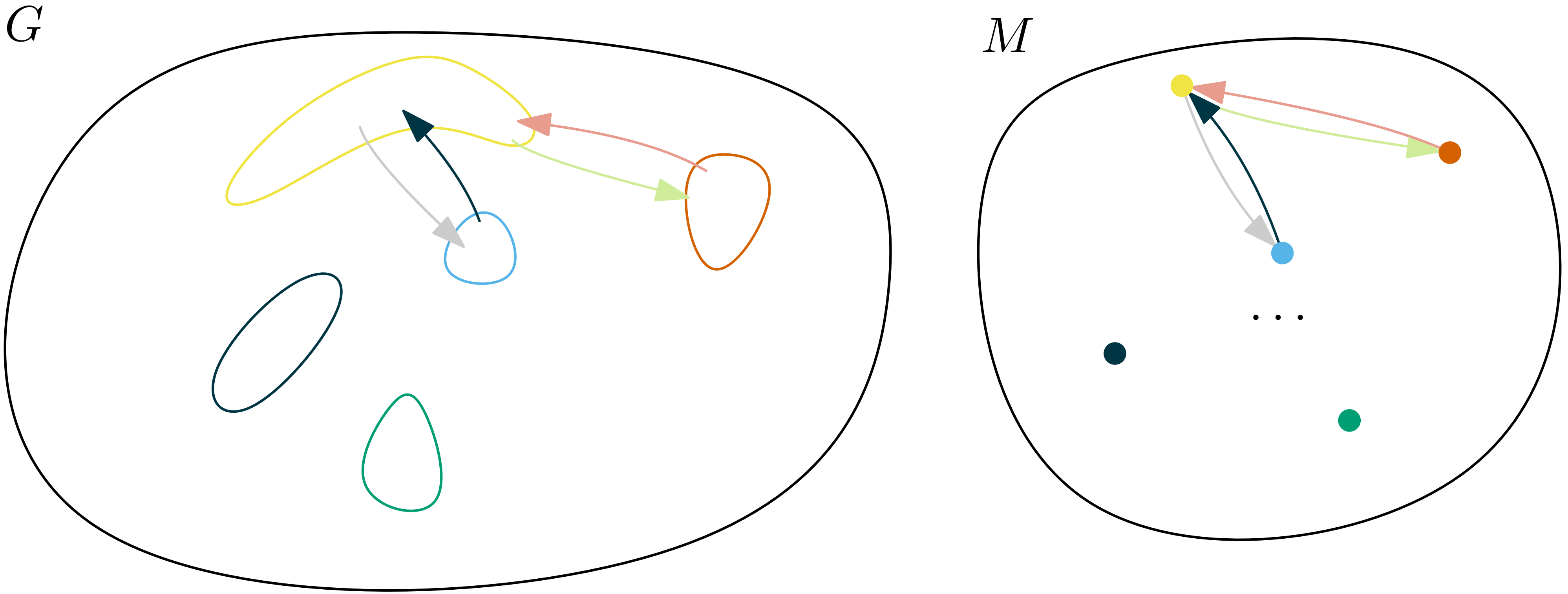

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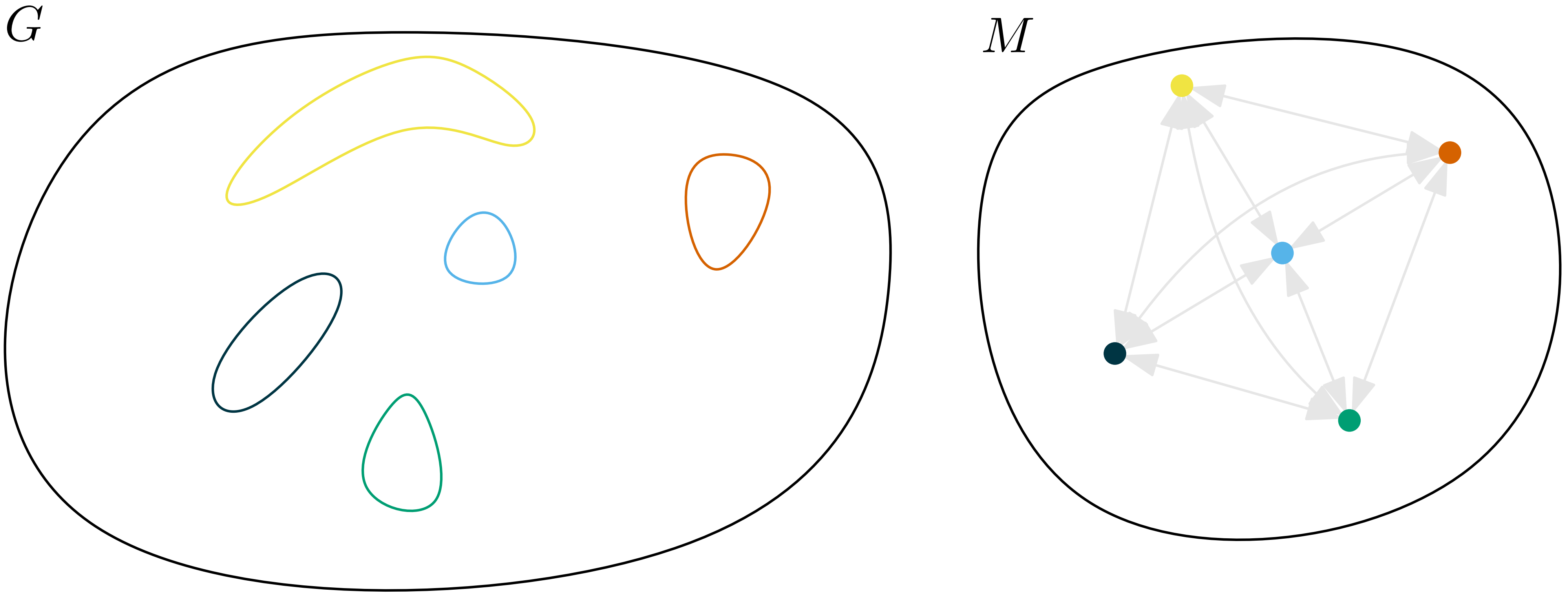

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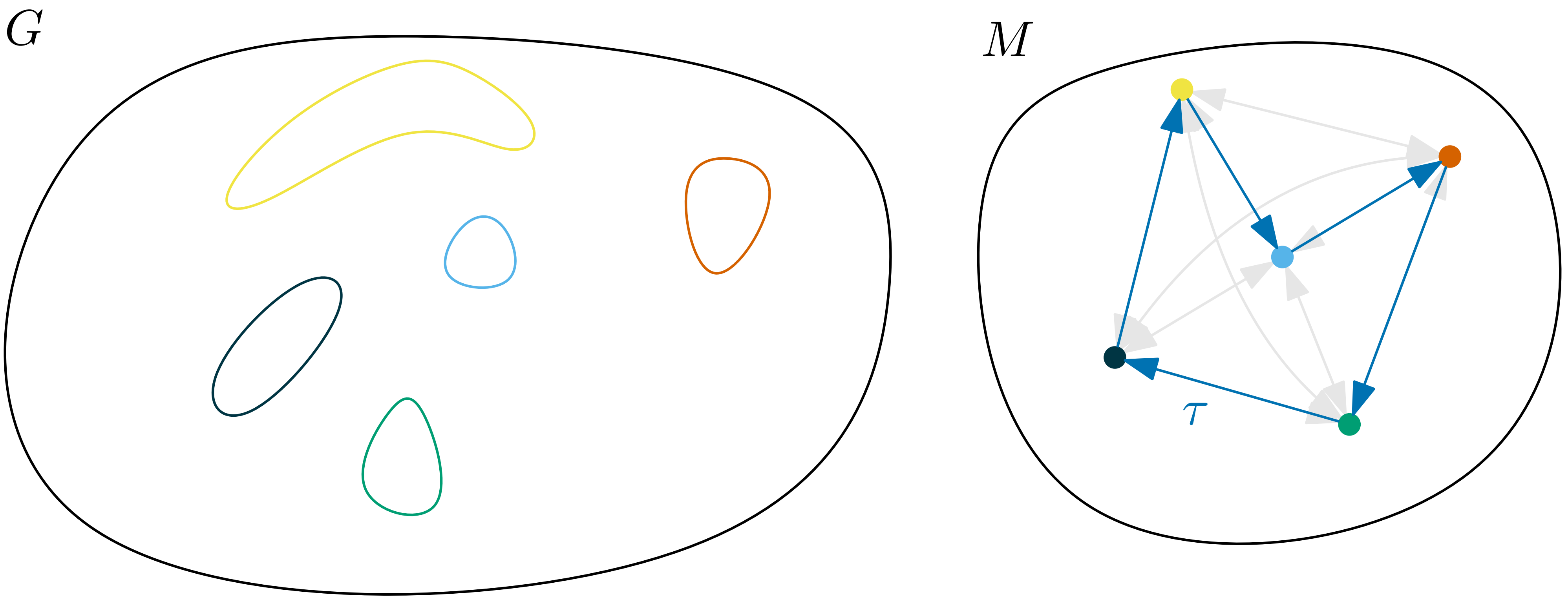

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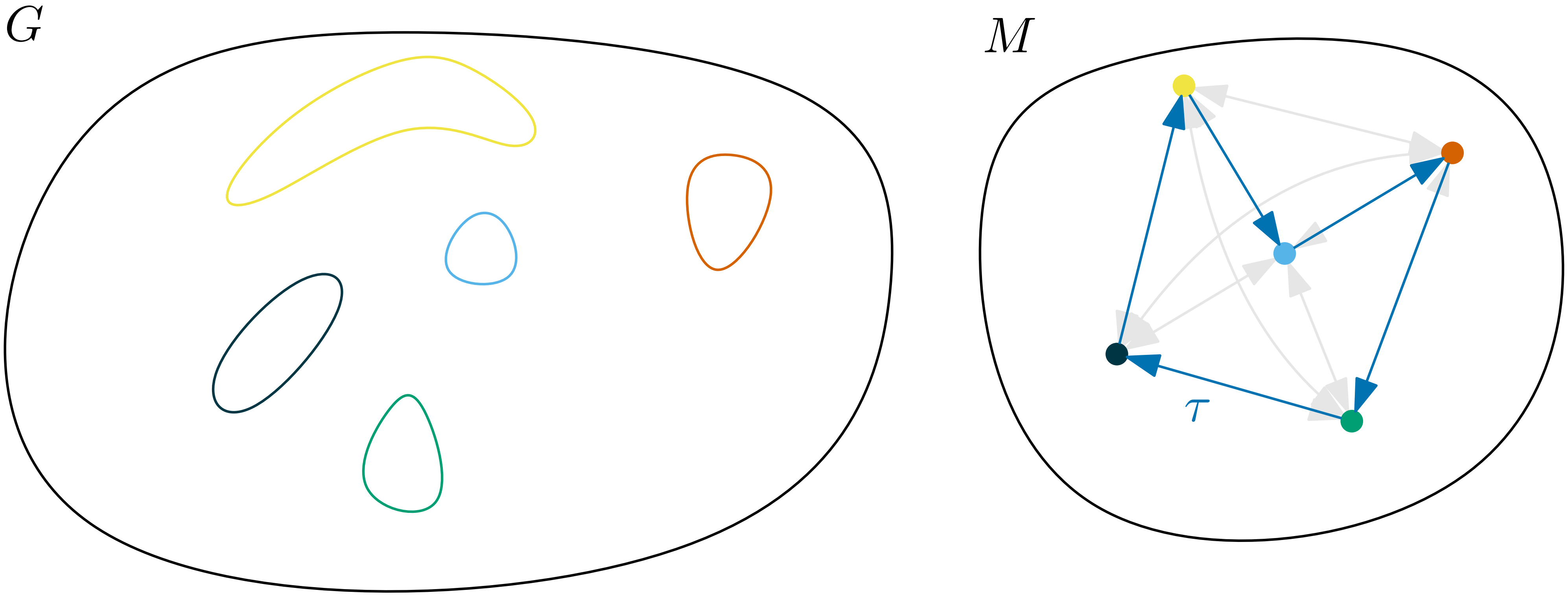
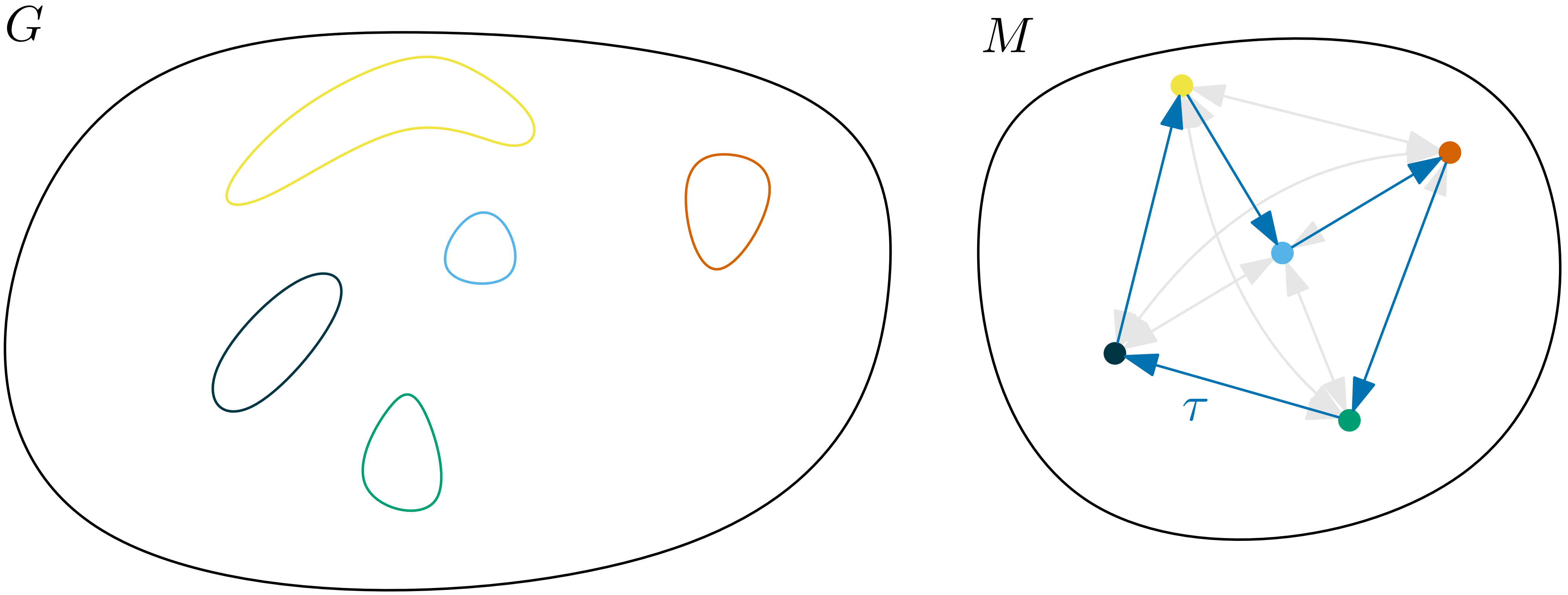


Question: Can we upper bound the cost of τ?

Generalized Tree Doubling Algorithm



Question: Can we upper bound the cost of τ, given that M is minor of G?

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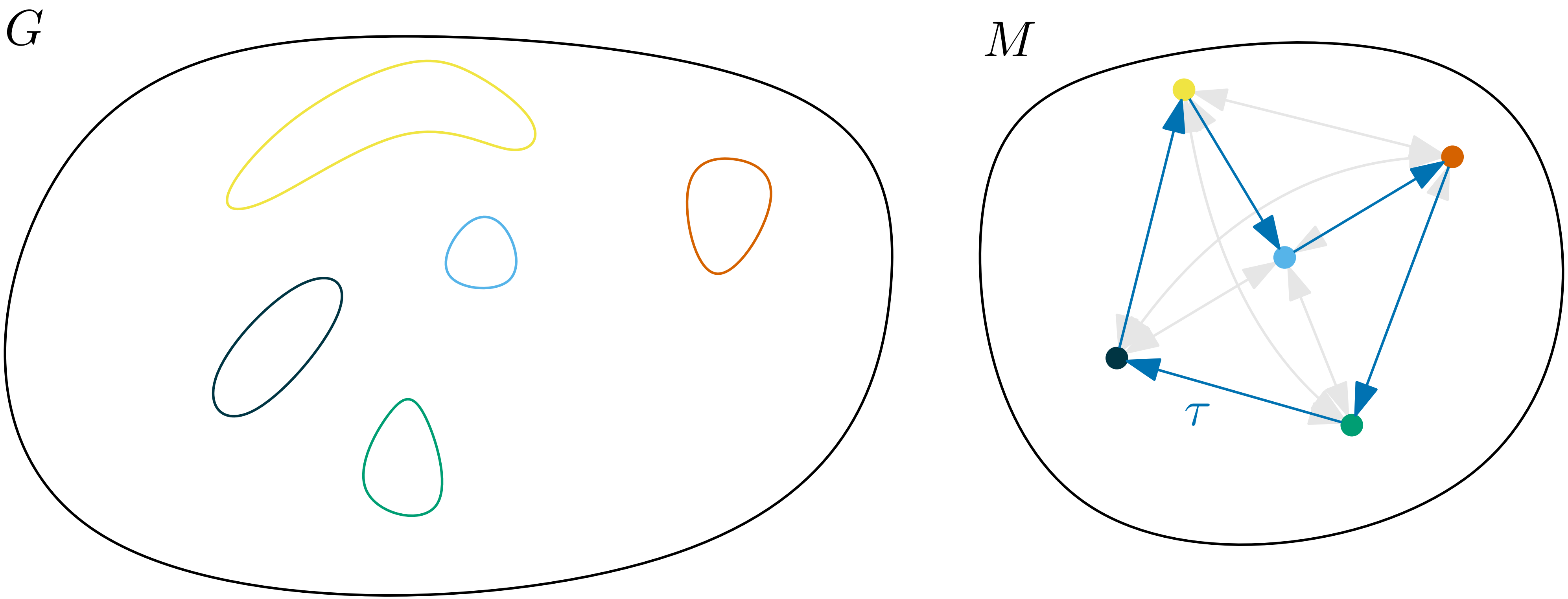

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Lemma: Let H be a minor of metric graph G, then $TSP(H) \leq TSP(G)$.

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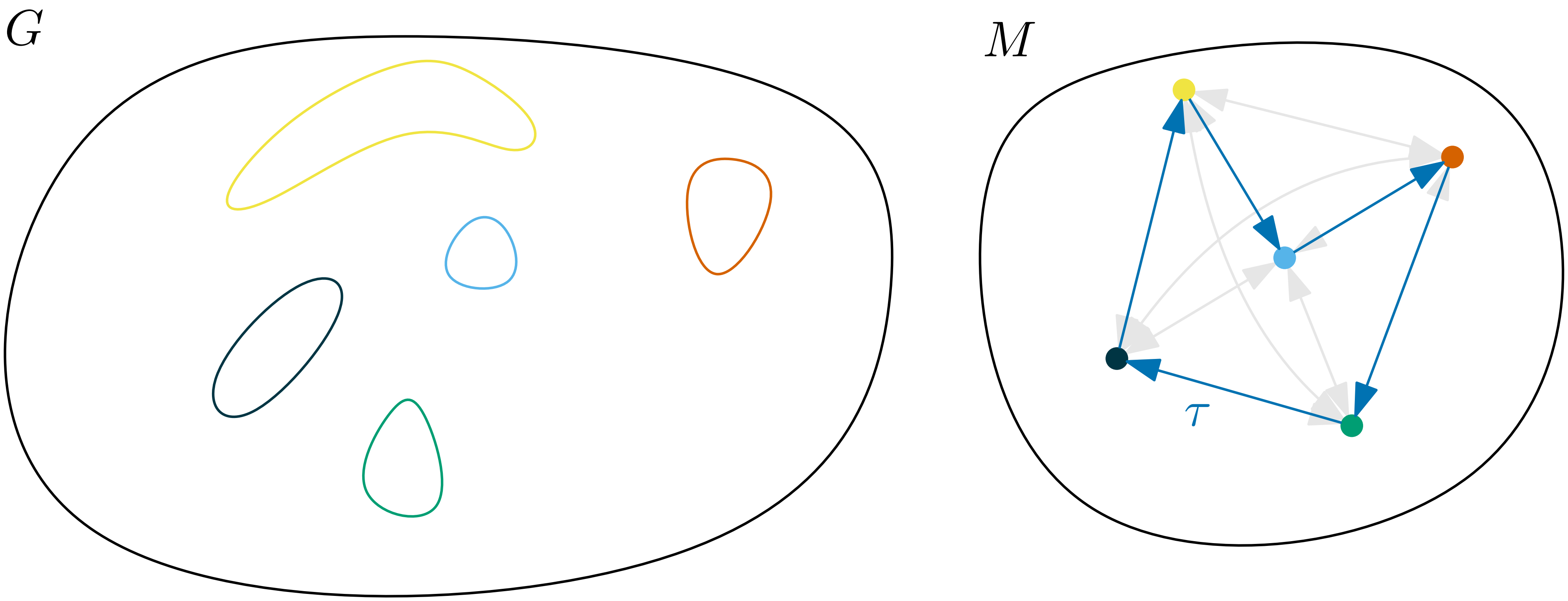

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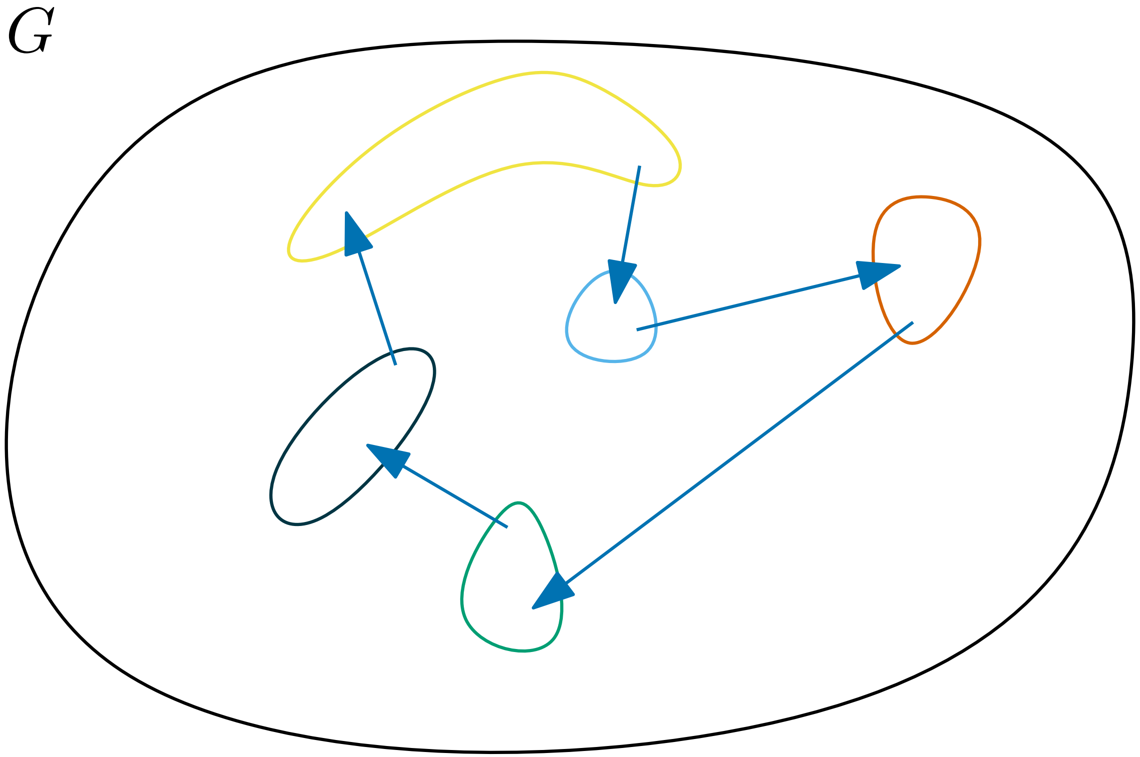
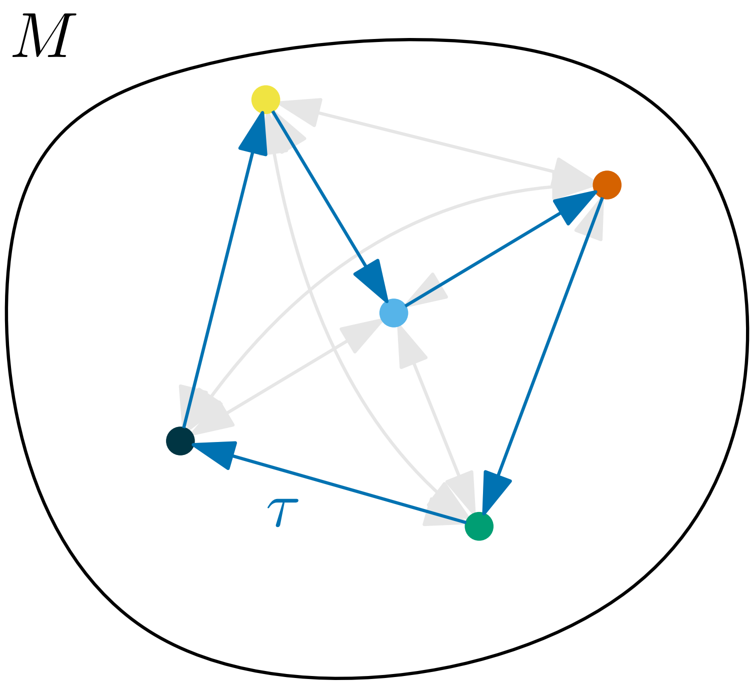
$$cost(\tau) = TSP(M) \leq TSP(G)$$

Generalized Tree Doubling Algorithm





Question: Can we upper bound the cost of τ, given that M is minor of G?

Lemma: Let H be a minor of metric graph G, then $TSP(H) \leq TSP(G)$.

$$cost(\tau) = TSP(M) \leq TSP(G)$$

Generalized Tree Doubling Algorithm

Now: Connect τ into tour of G

Generalized Tree Doubling Algorithm

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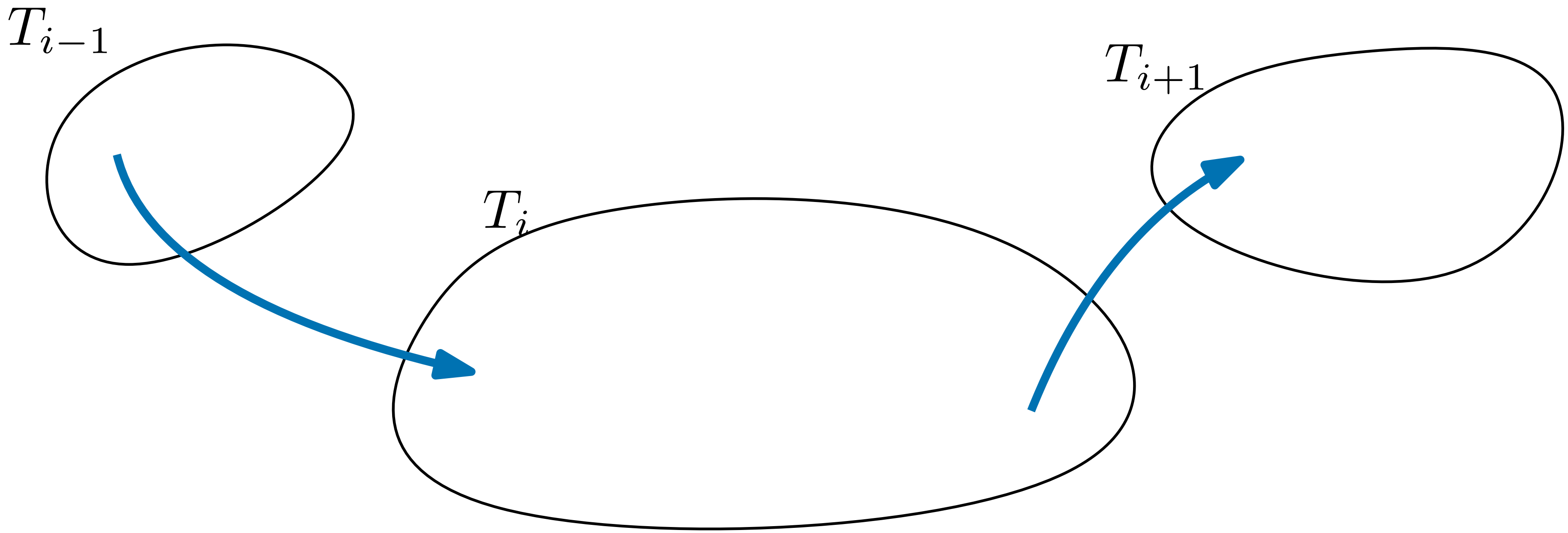
Consider components $T_{i-1}, T_i, T_{i+1} \in \tau$:



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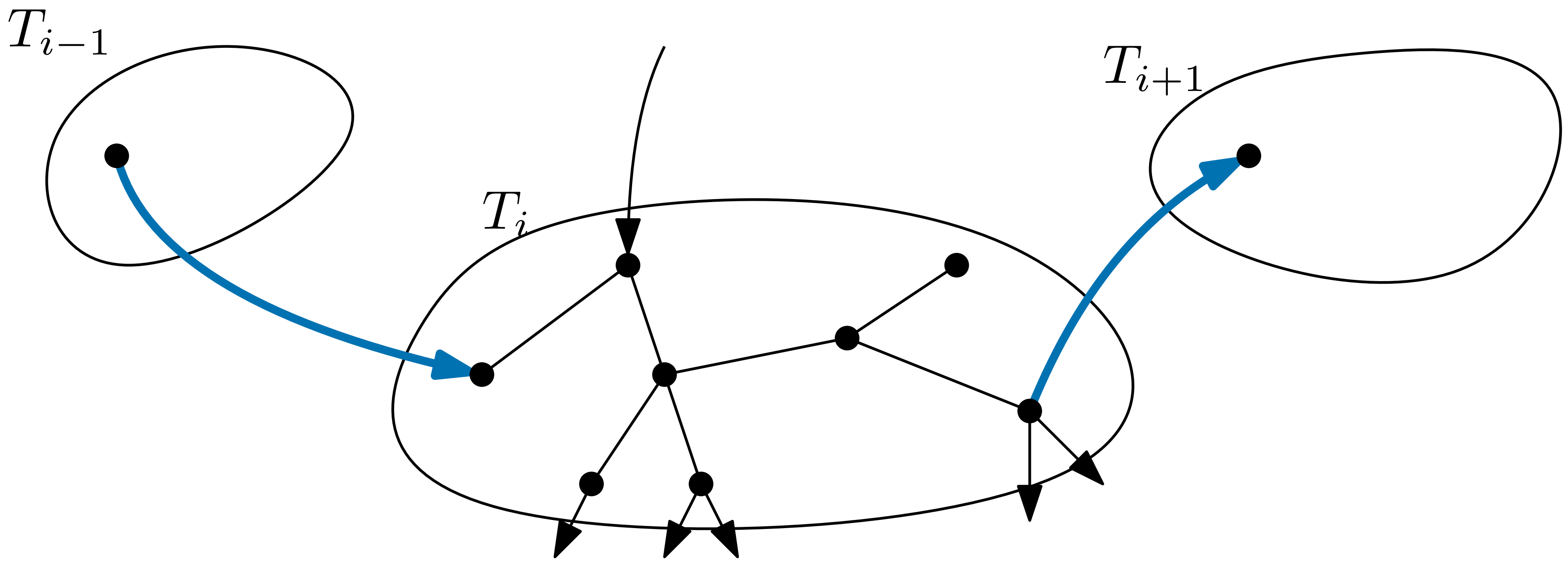

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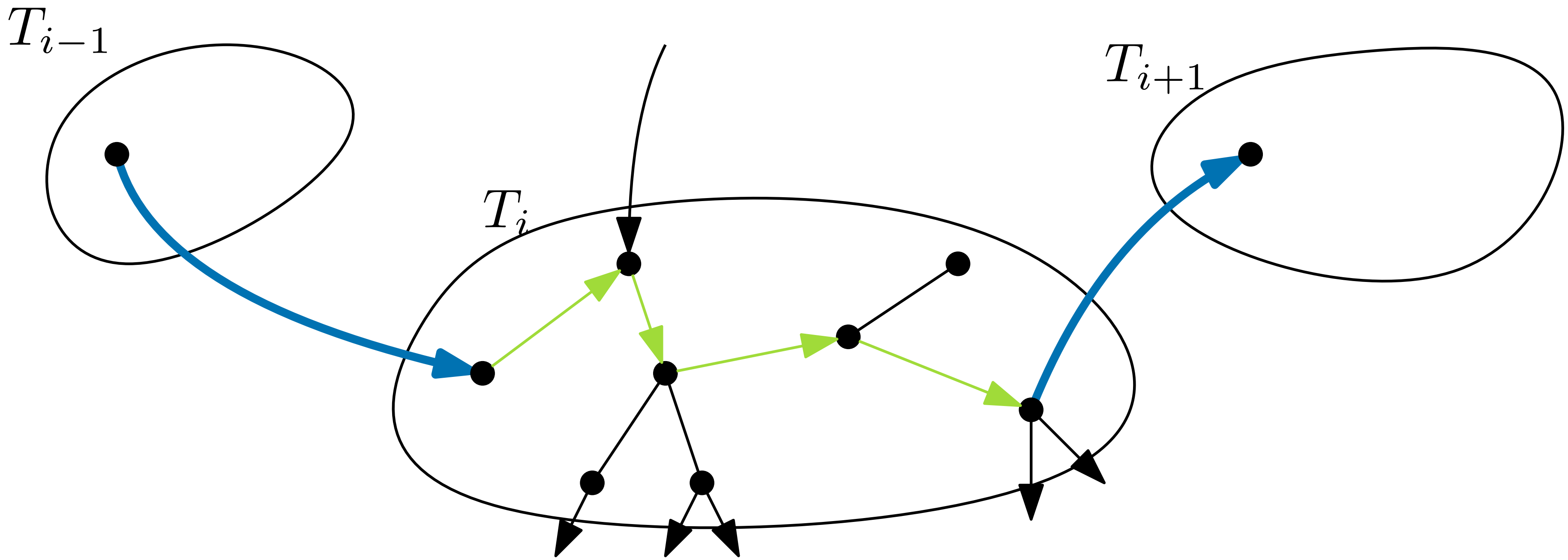

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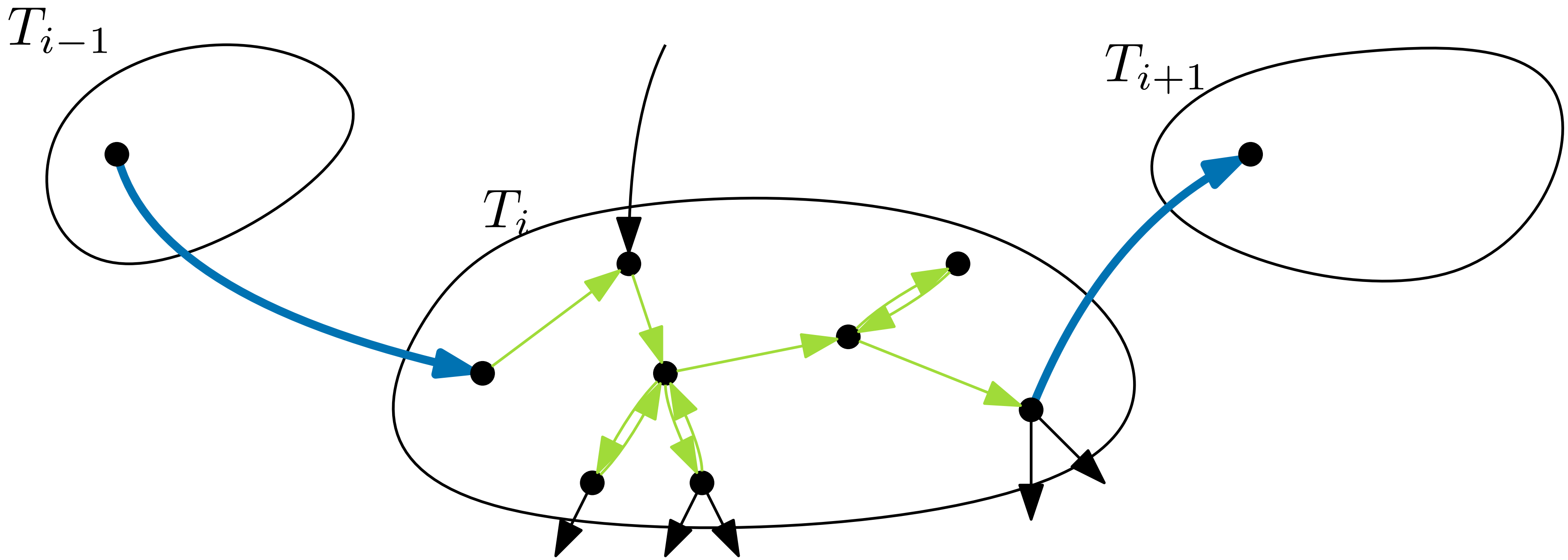

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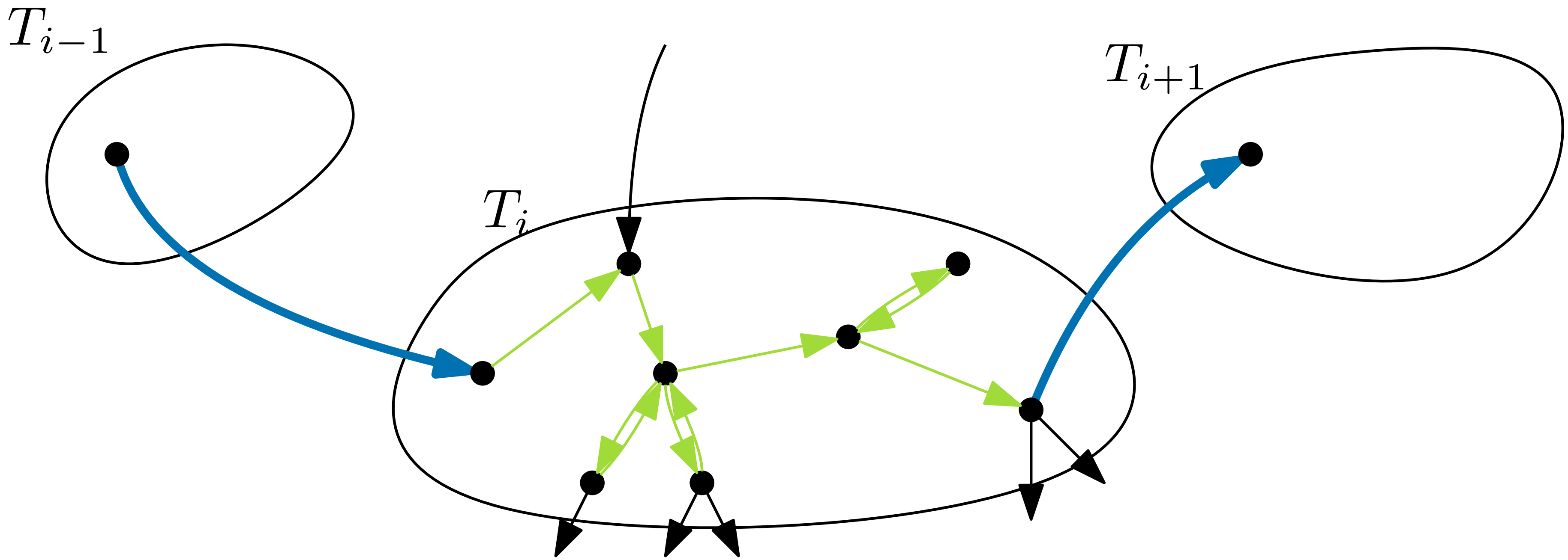

Cost of path p_i through T_i?

Generalized Tree Doubling Algorithm

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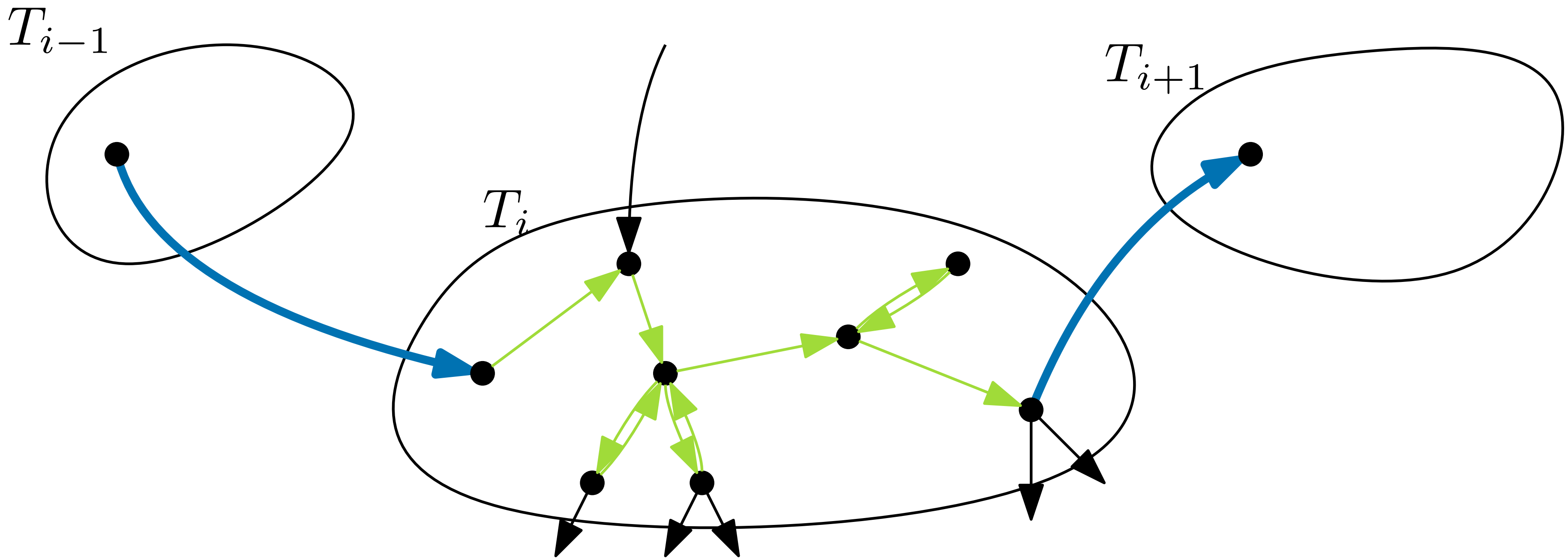

Cost of path p_i through T_i? $c(p_i) \leq 2c(T_i)$

Generalized Tree Doubling Algorithm

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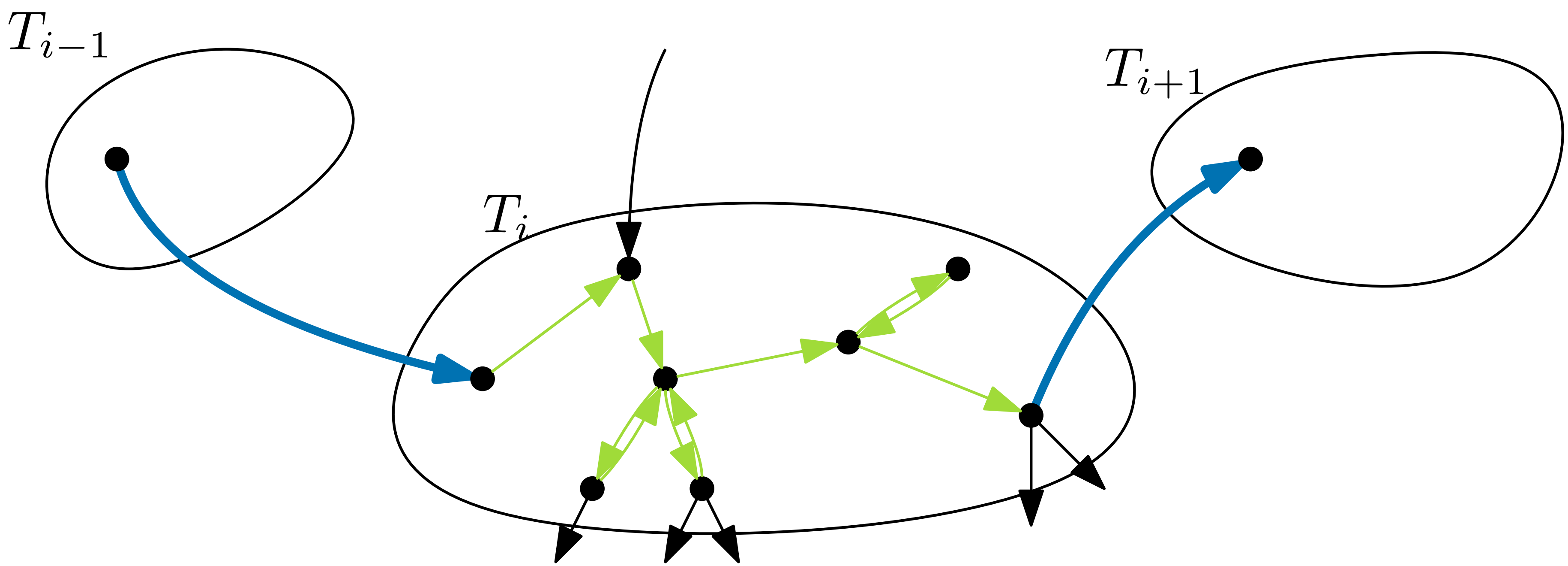

Cost of path p_i through T_i? $c(p_i) \leq 2c(T_i)$

thus $\sum c(p_i) \leq \sum 2c(T_i) \leq 2\,TSP(G)$

Generalized Tree Doubling Algorithm

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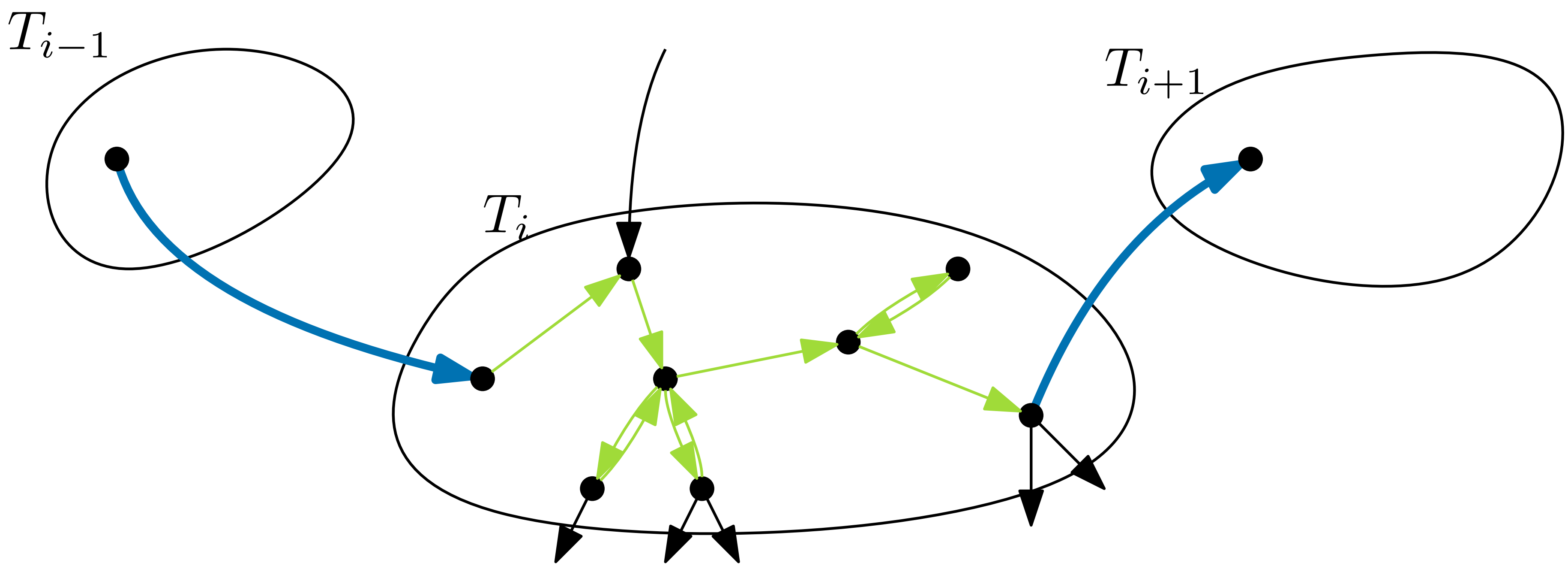

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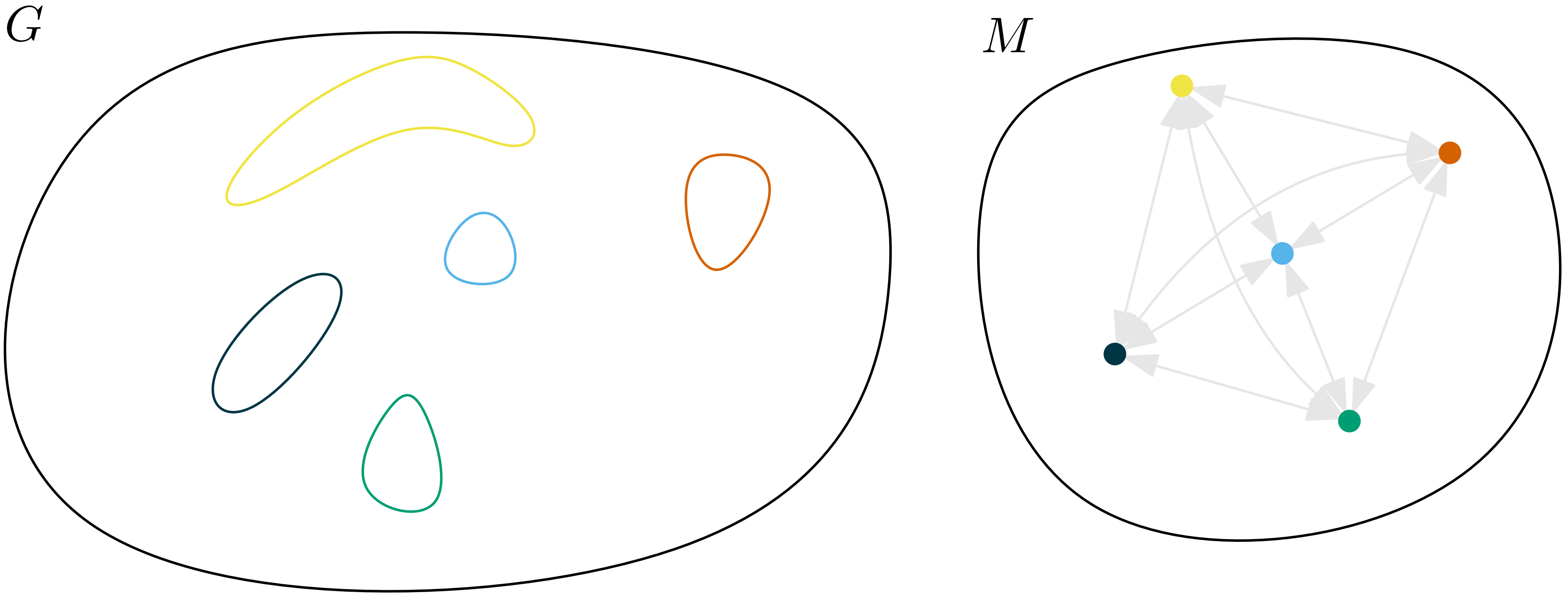
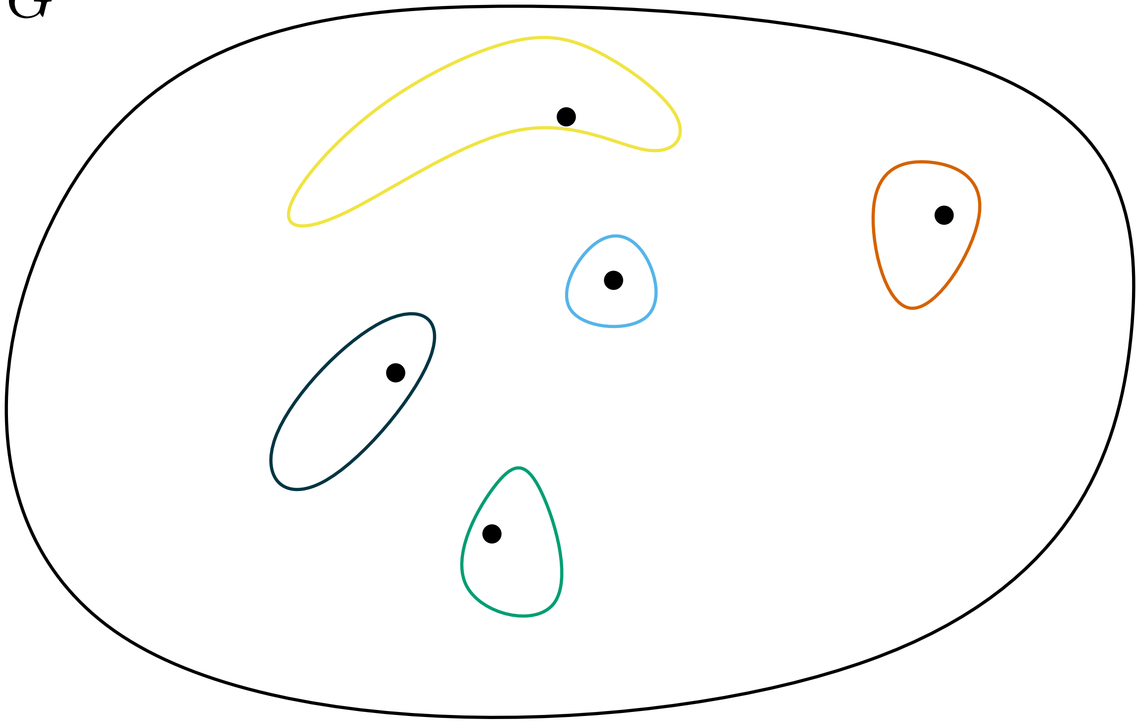
Theorem: The algorithm computes a 3-approx. for ATSP in $\mathcal{O}^*(2^k)$ where k is the number of one-way edges in a given min. spanning arborescence.

Algorithm revisited



Algorithm revisited

G



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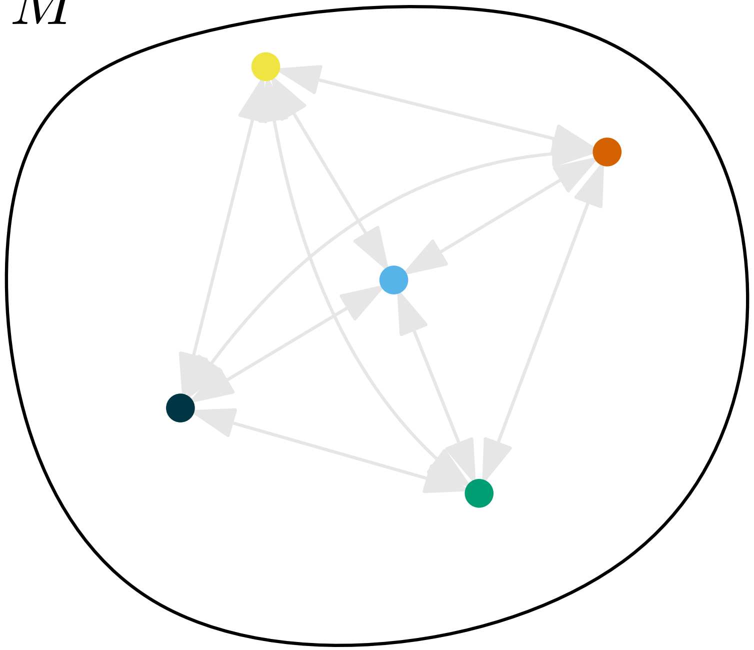

Algorithm revisited

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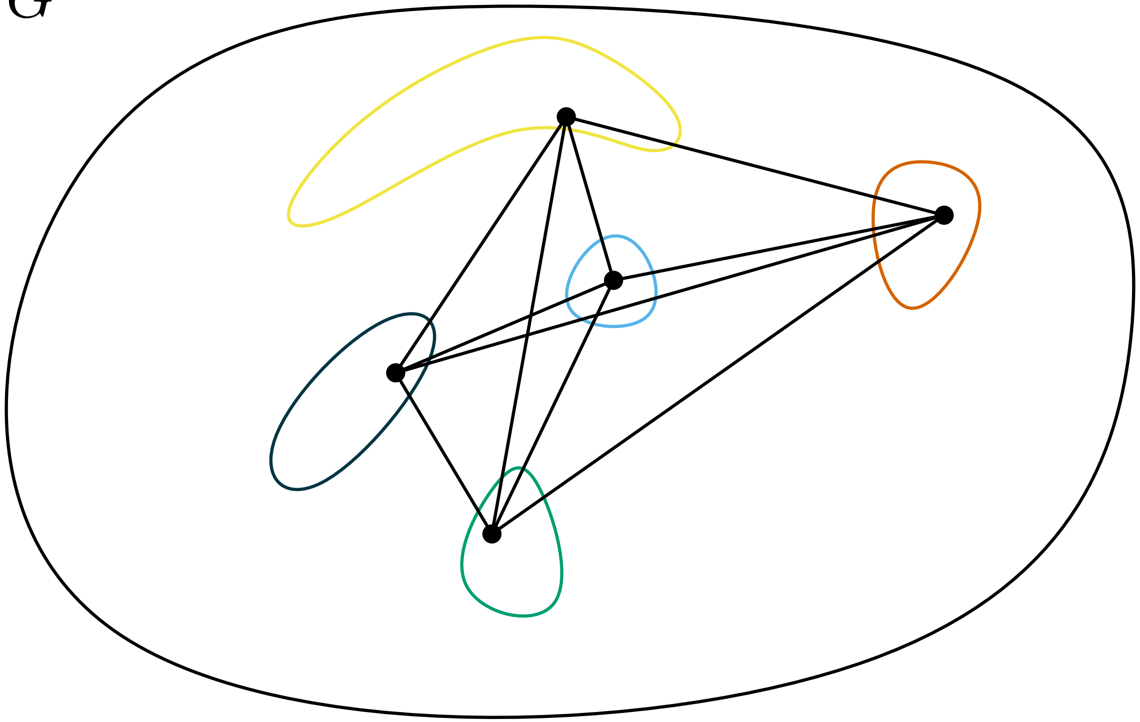

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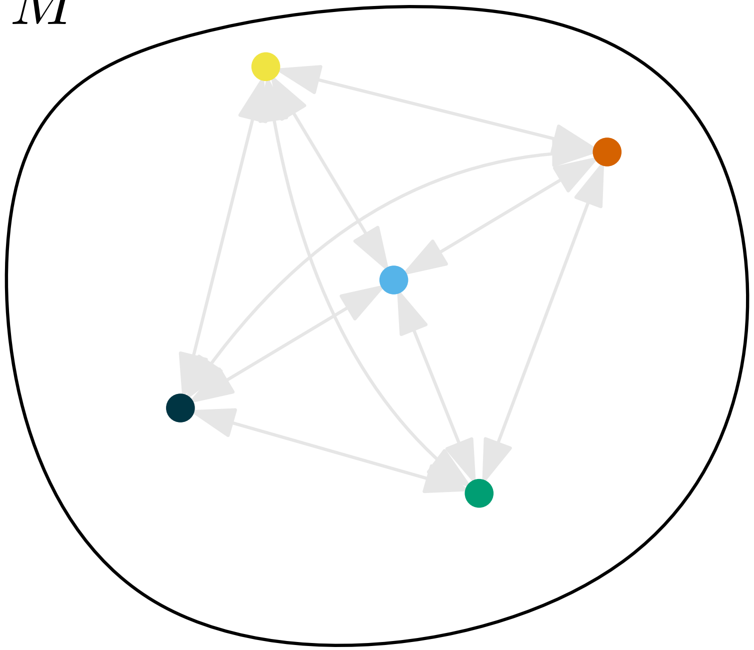

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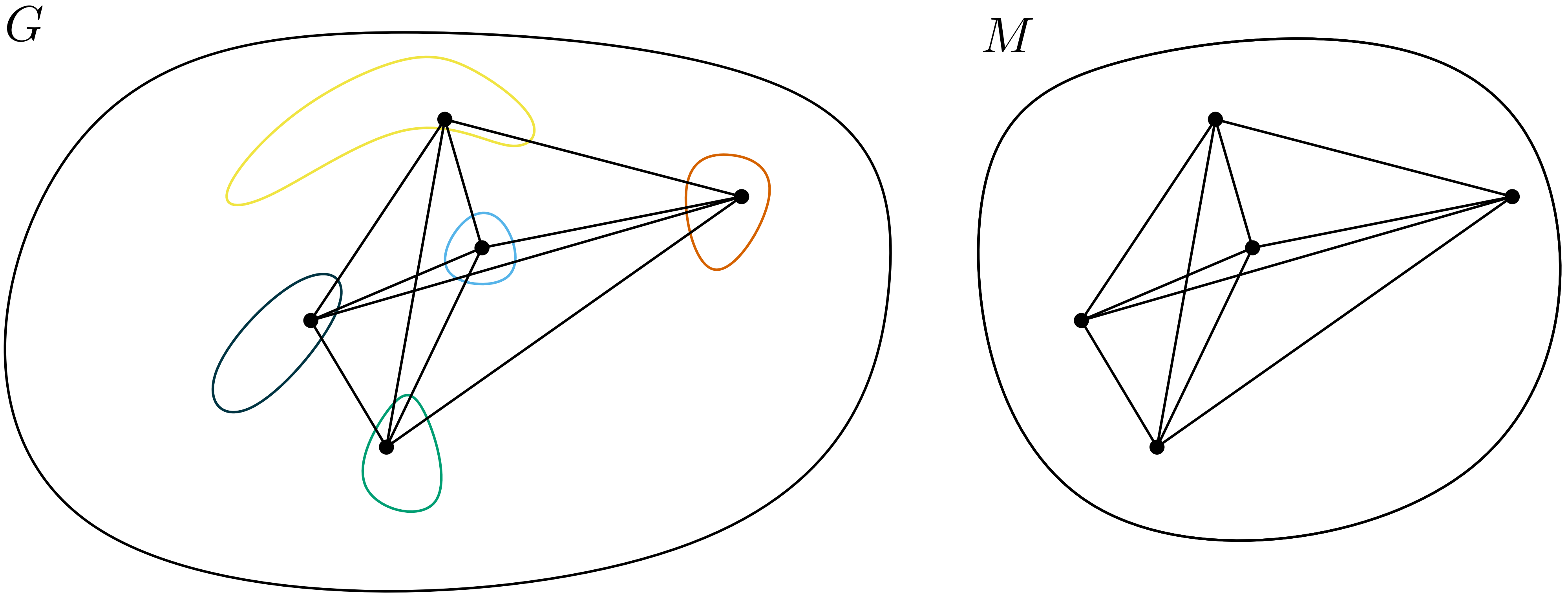

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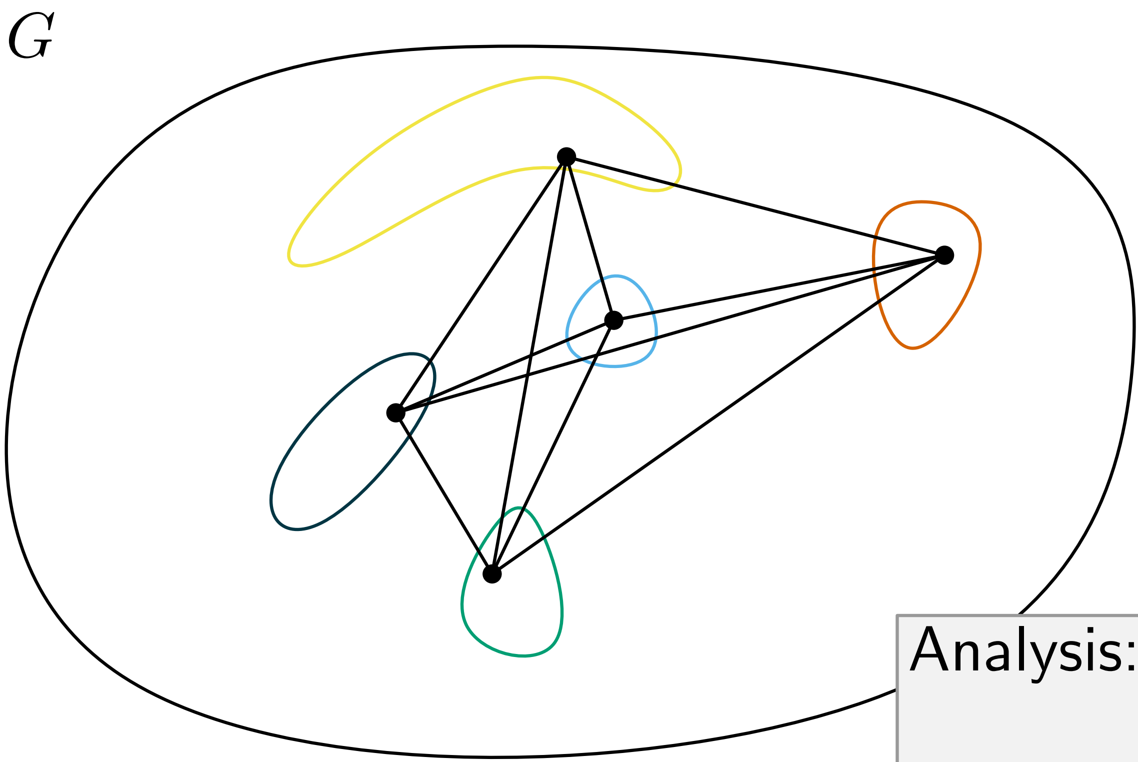
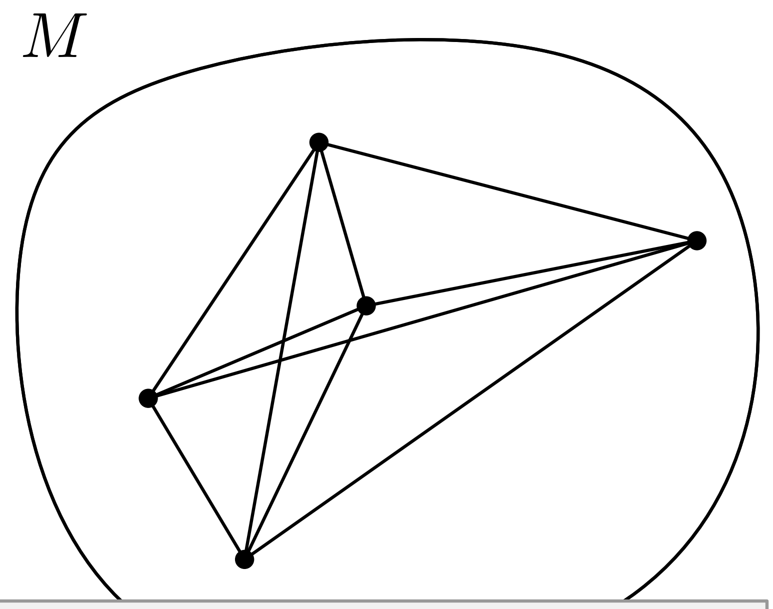
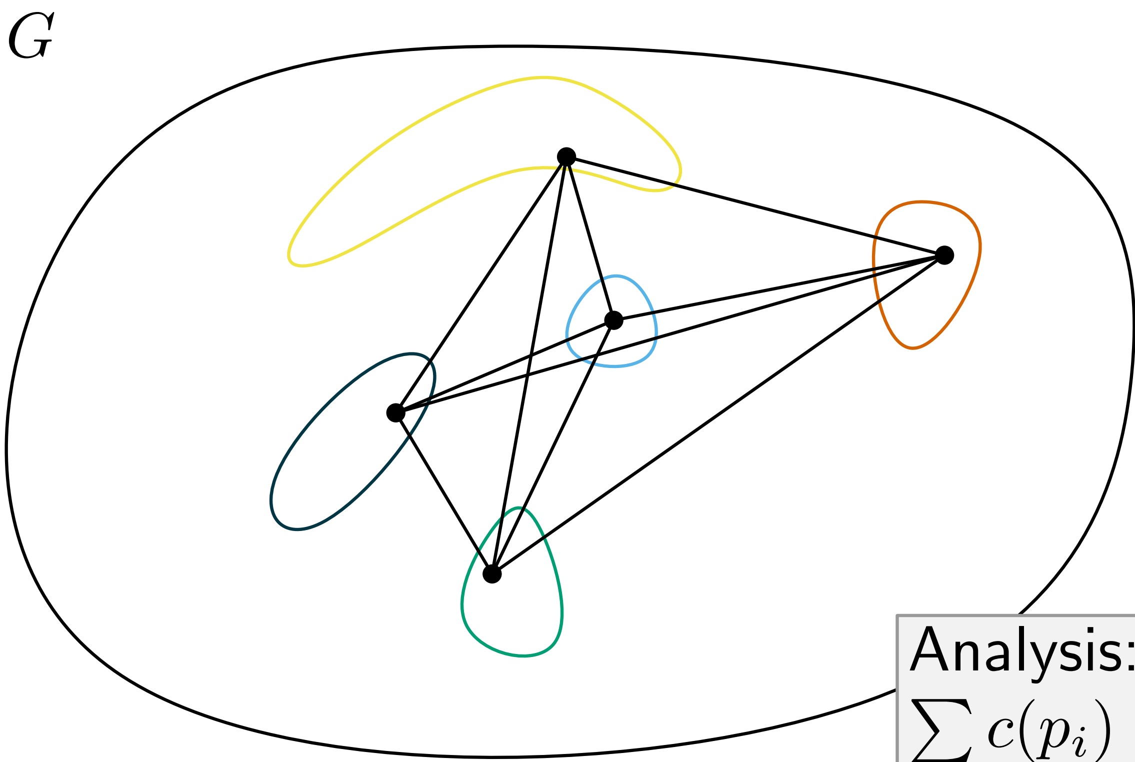
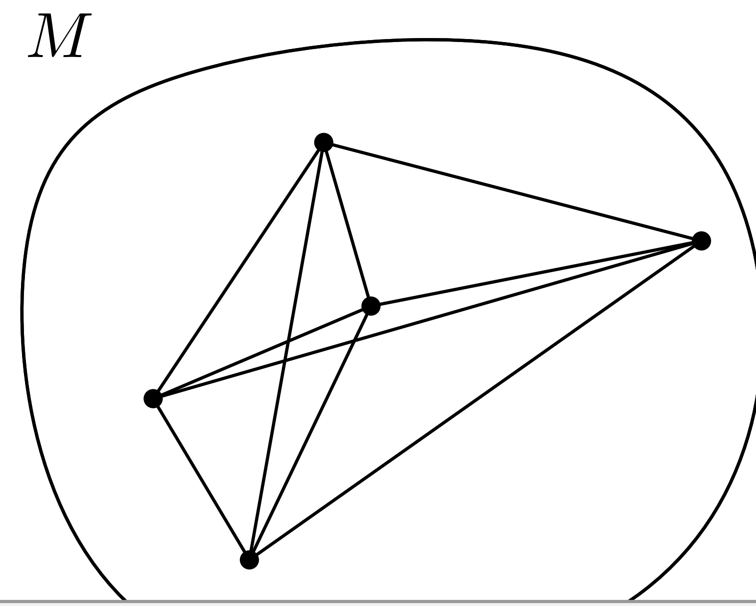




Analysis:

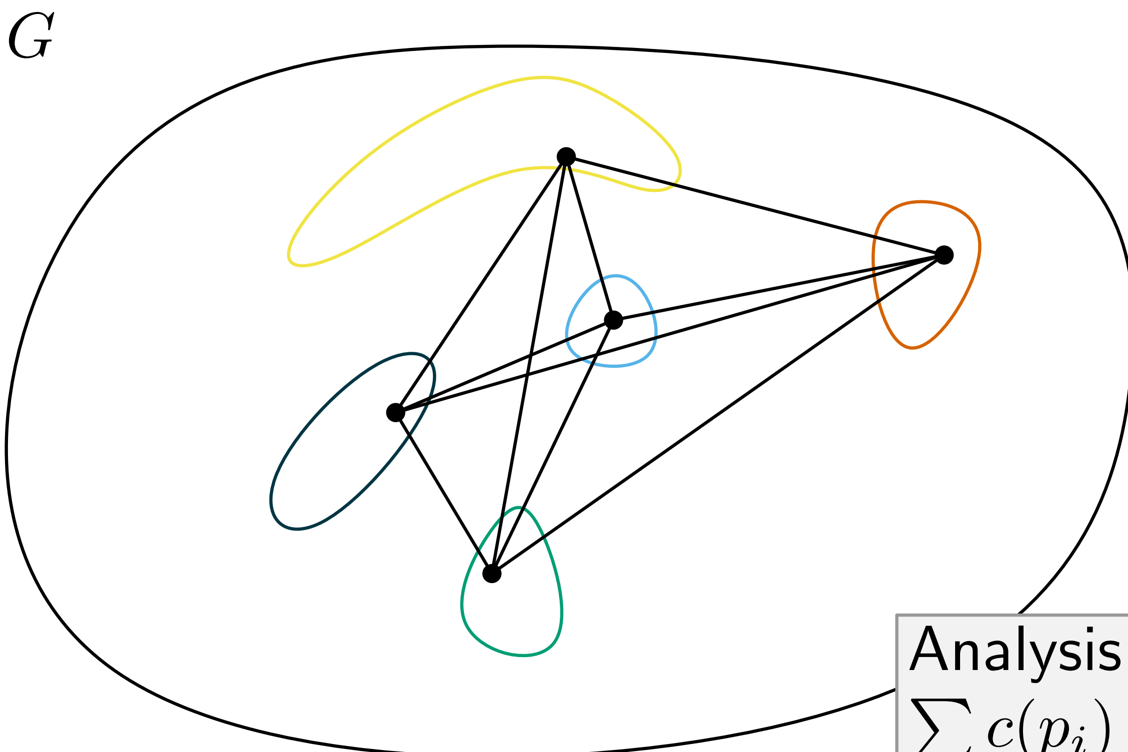
Algorithm revisited

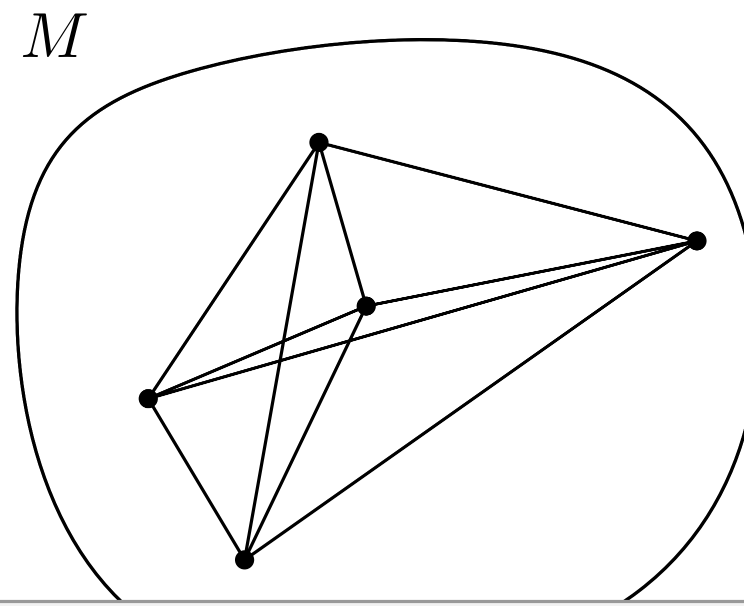




Analysis:
$\sum c(p_i) \leq \sum 2c(T_i) \leq 2TSP(G)$ still holds

Algorithm revisited





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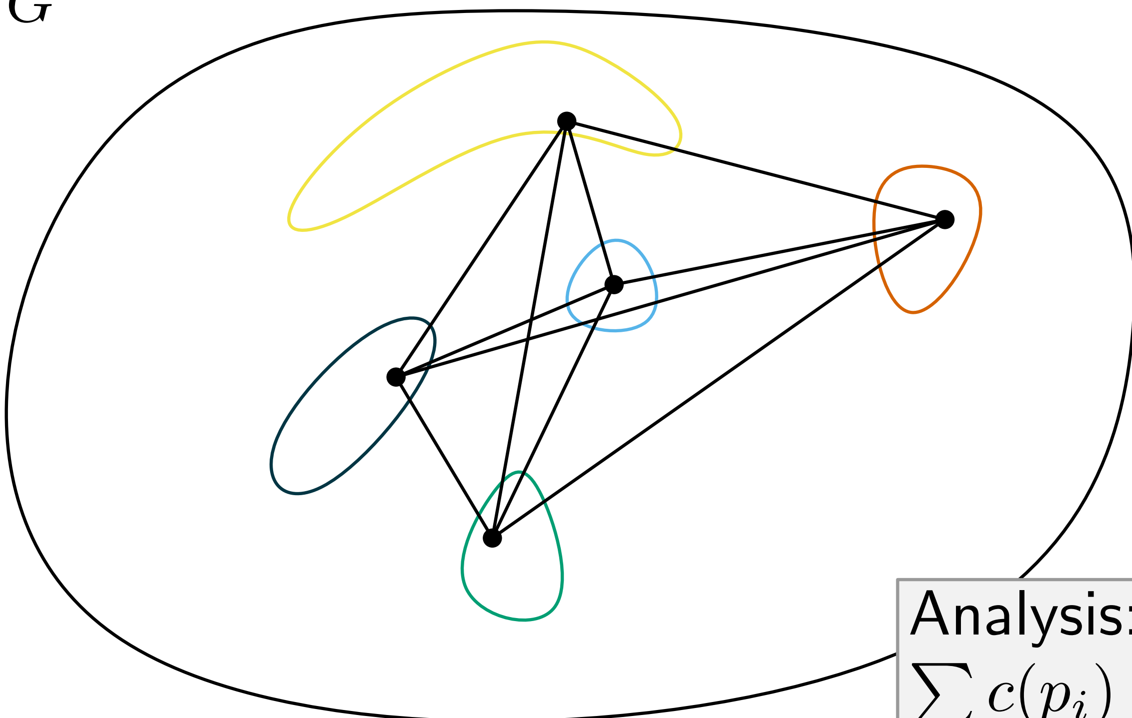
Algorithm revisited

G



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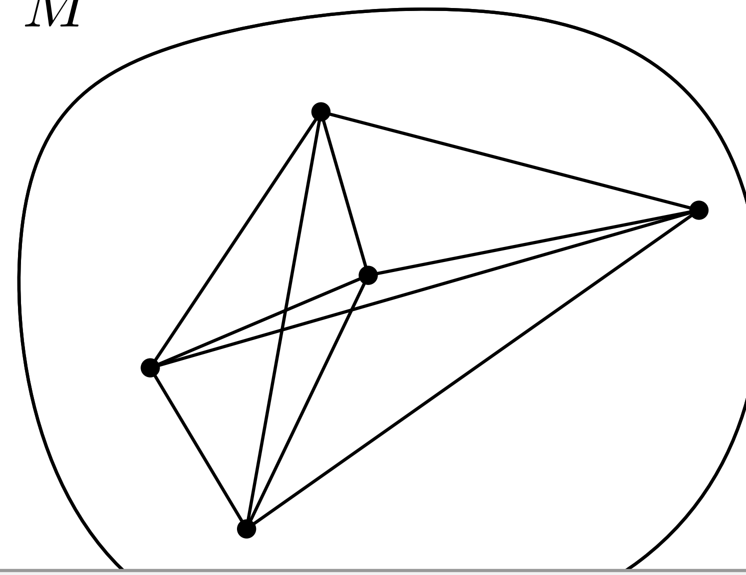

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$\rightarrow$ 3-approximation

Approximation guarantee vs runtime

Problem: the instances are too asymmetric

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Idea: ignore asymmetries up to a factor of β

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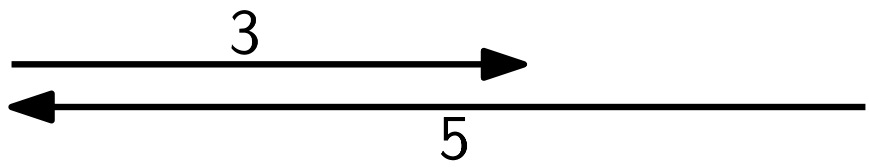

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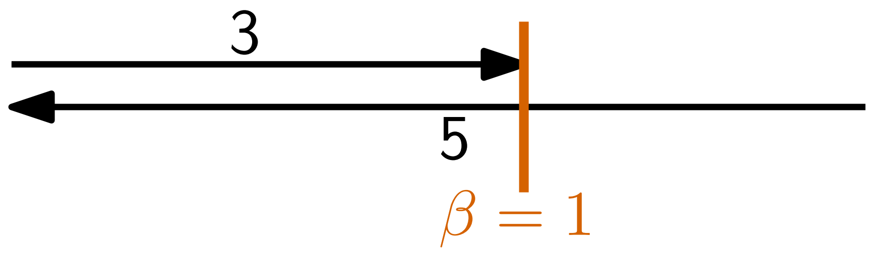

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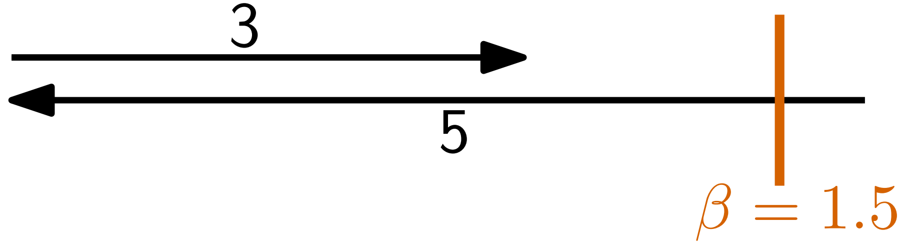

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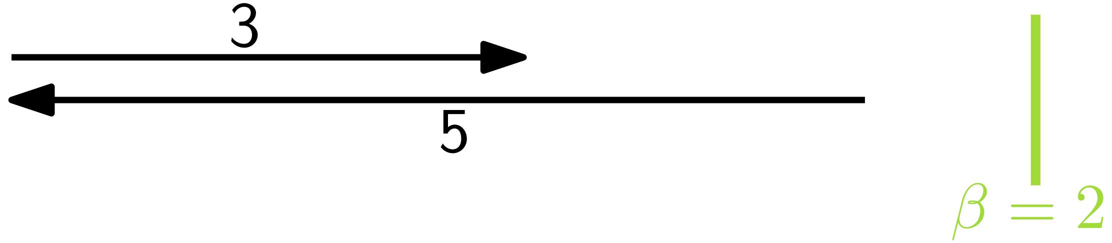

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Generalized Christofides algorithm

Theorem: Metric ATSP can be $(\frac{7}{4} + \frac{3}{4}\beta)$-approximated in $\mathcal{O}^*(2^{k_\beta})$, where k_β is the size of a vertex cover for the graph induced by all β-asymmetric links, for any $\beta \geq 1$.

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Generalized tree doubling algorithm

Theorem: Metric ATSP can be $(2 + \beta)$-approximated in $\mathcal{O}^*(2^{k_\beta})$, where k_β is the number of β-one-way arcs in a given minimum spanning arborescence, for any $\beta \geq 1$.

Evaluation

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- TSPLIB: most common TSP benchmark; 19 asymmetric instances

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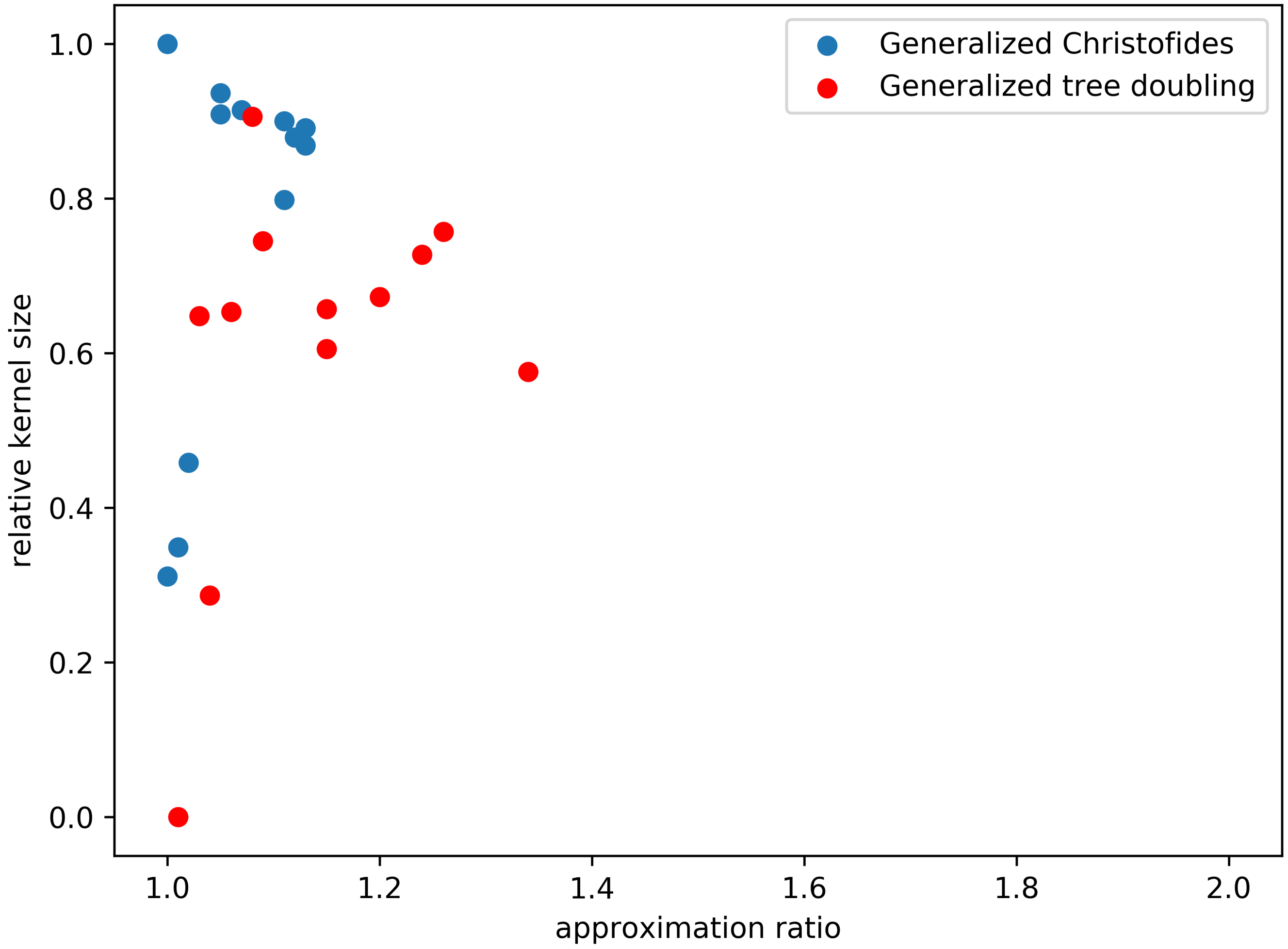

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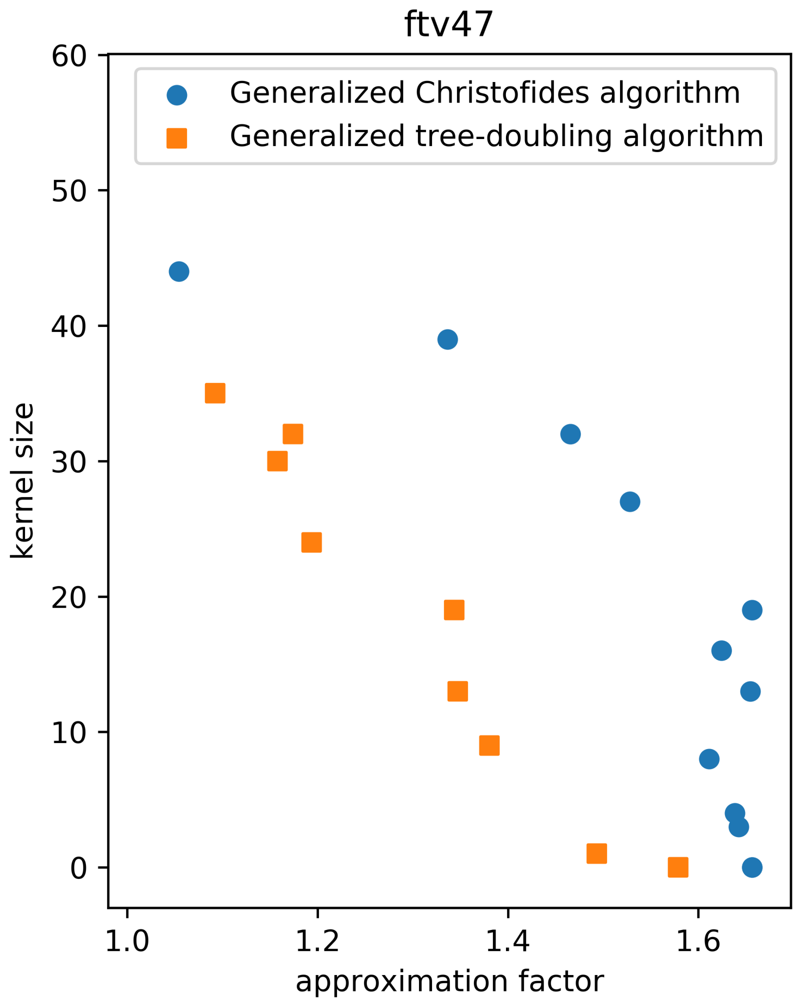

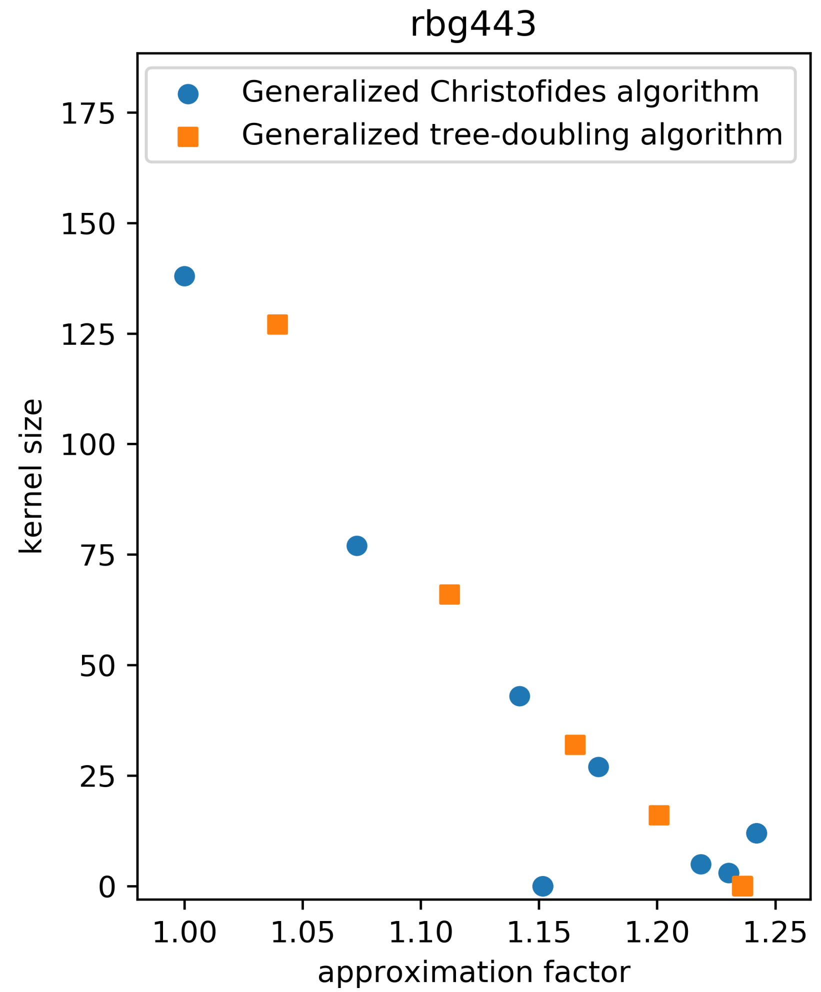

Evaluation

- road networks by Rodríguez and Ruiz

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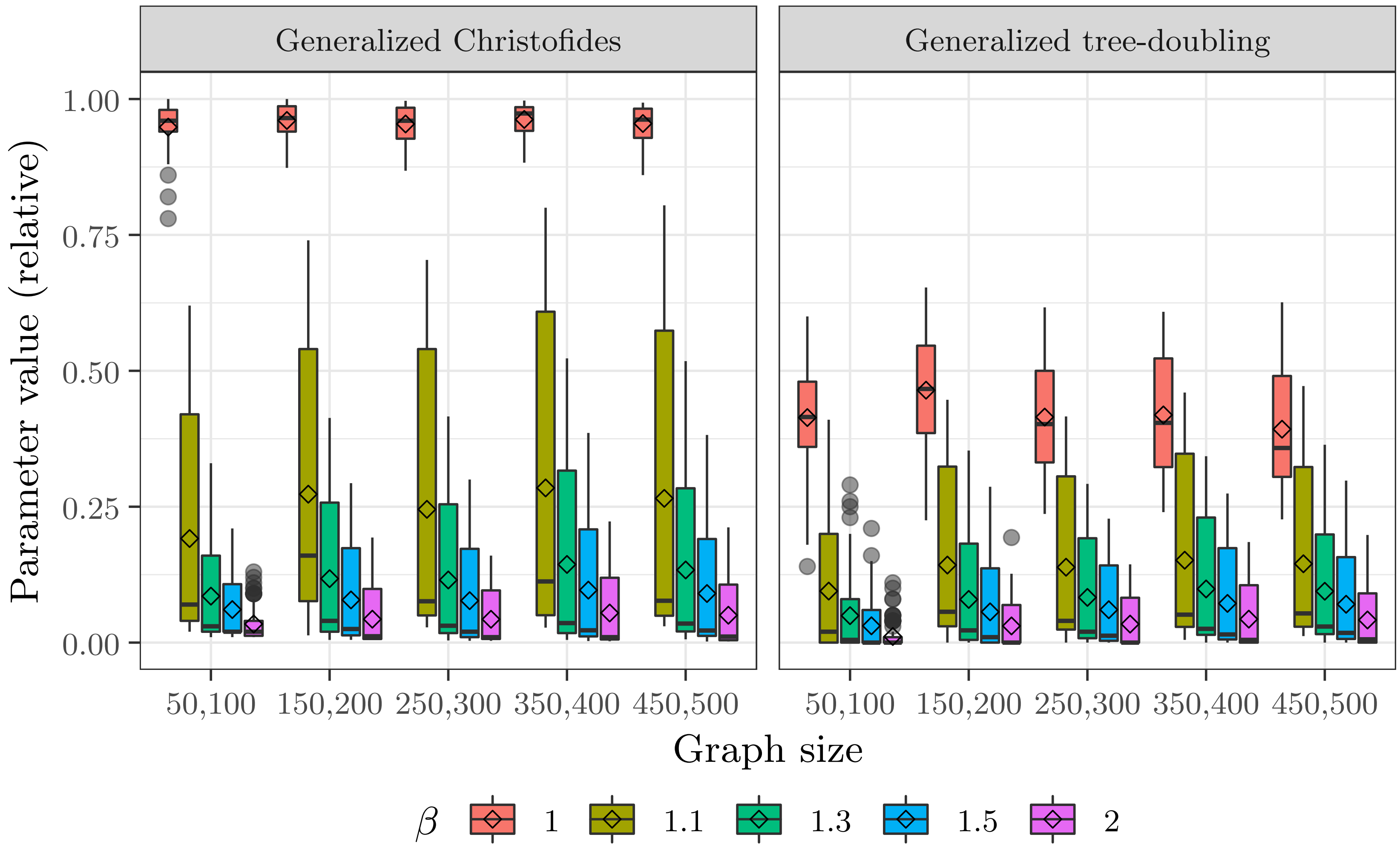
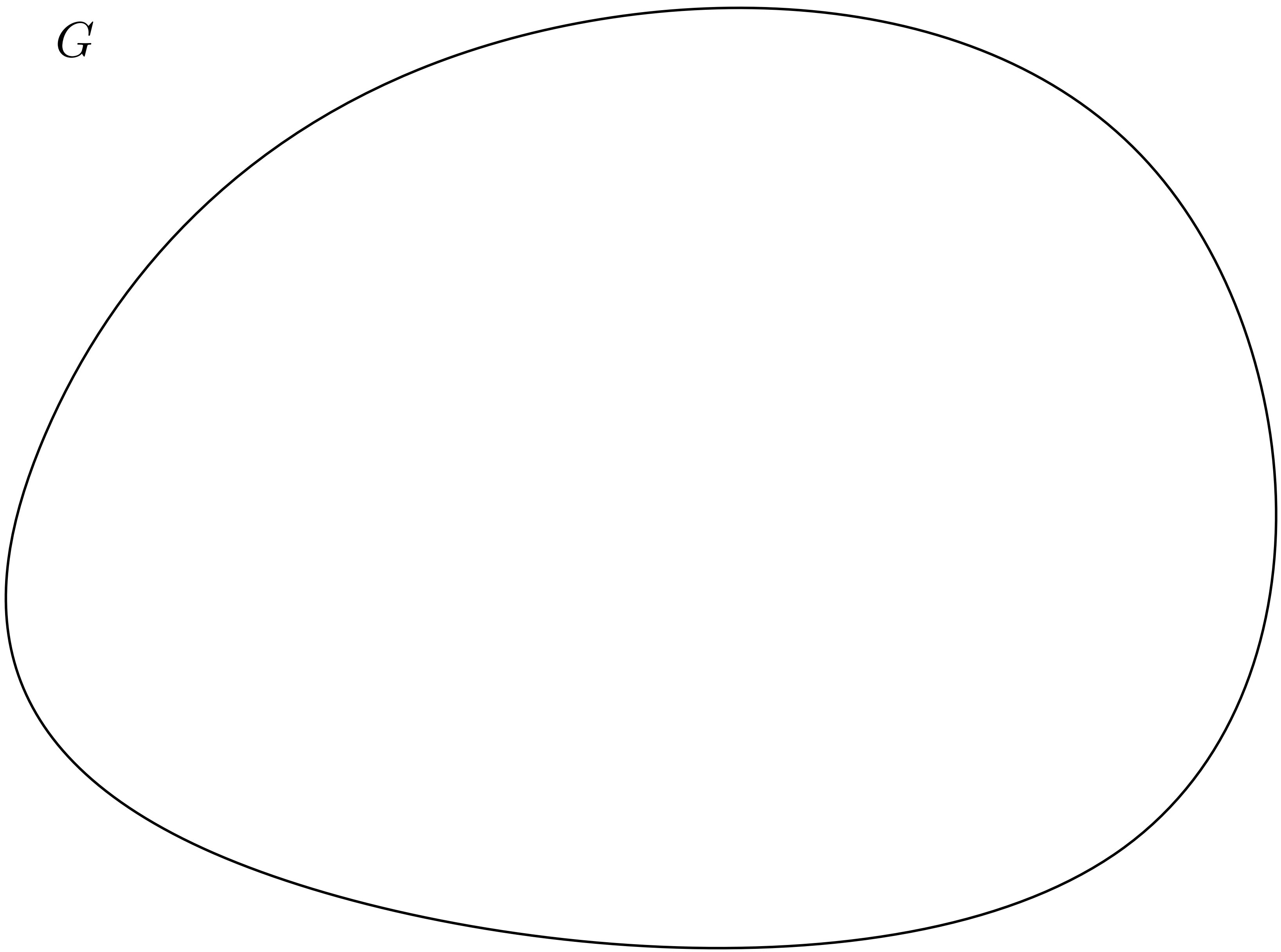


The generalized Christofides algorithm



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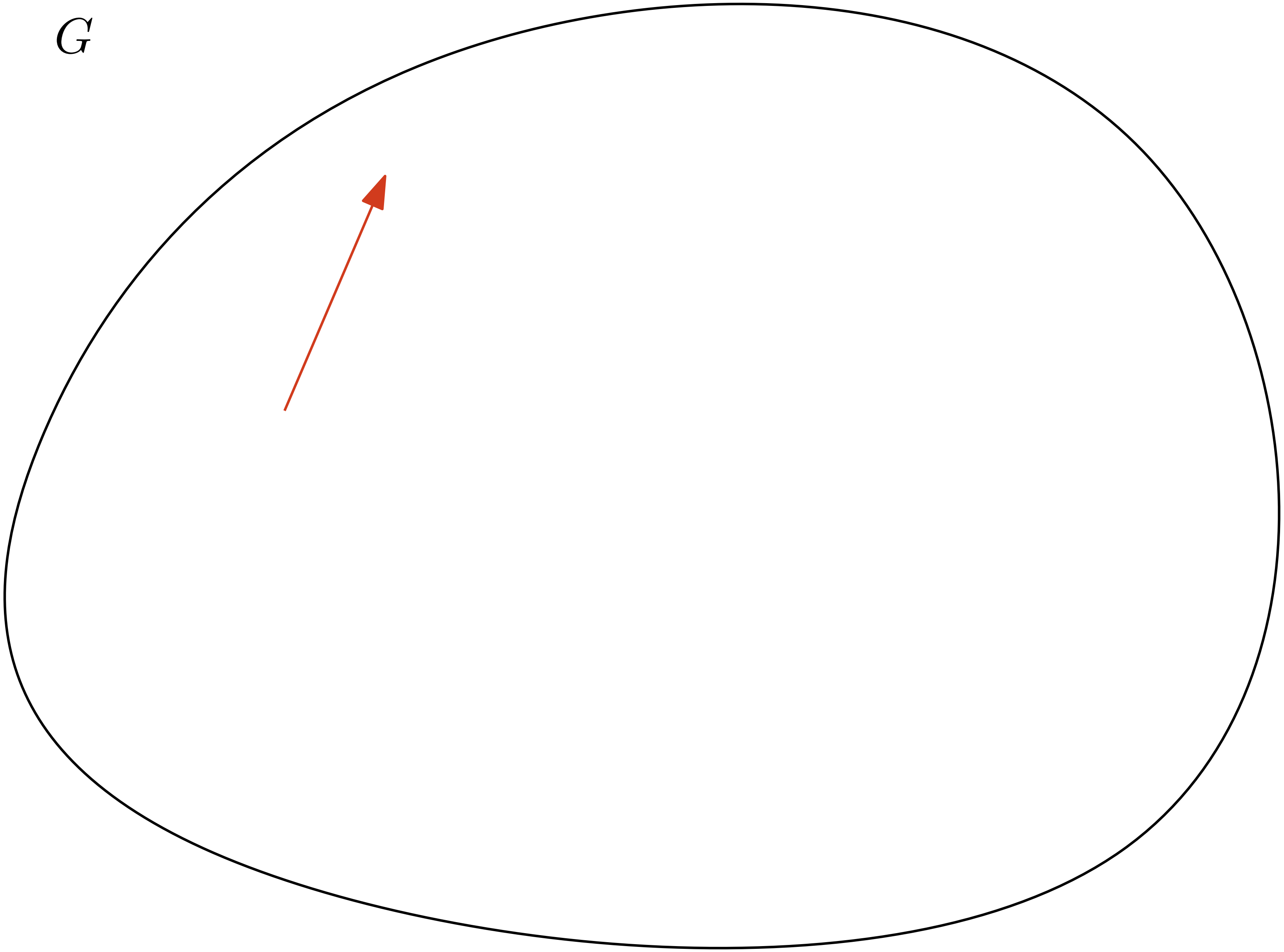

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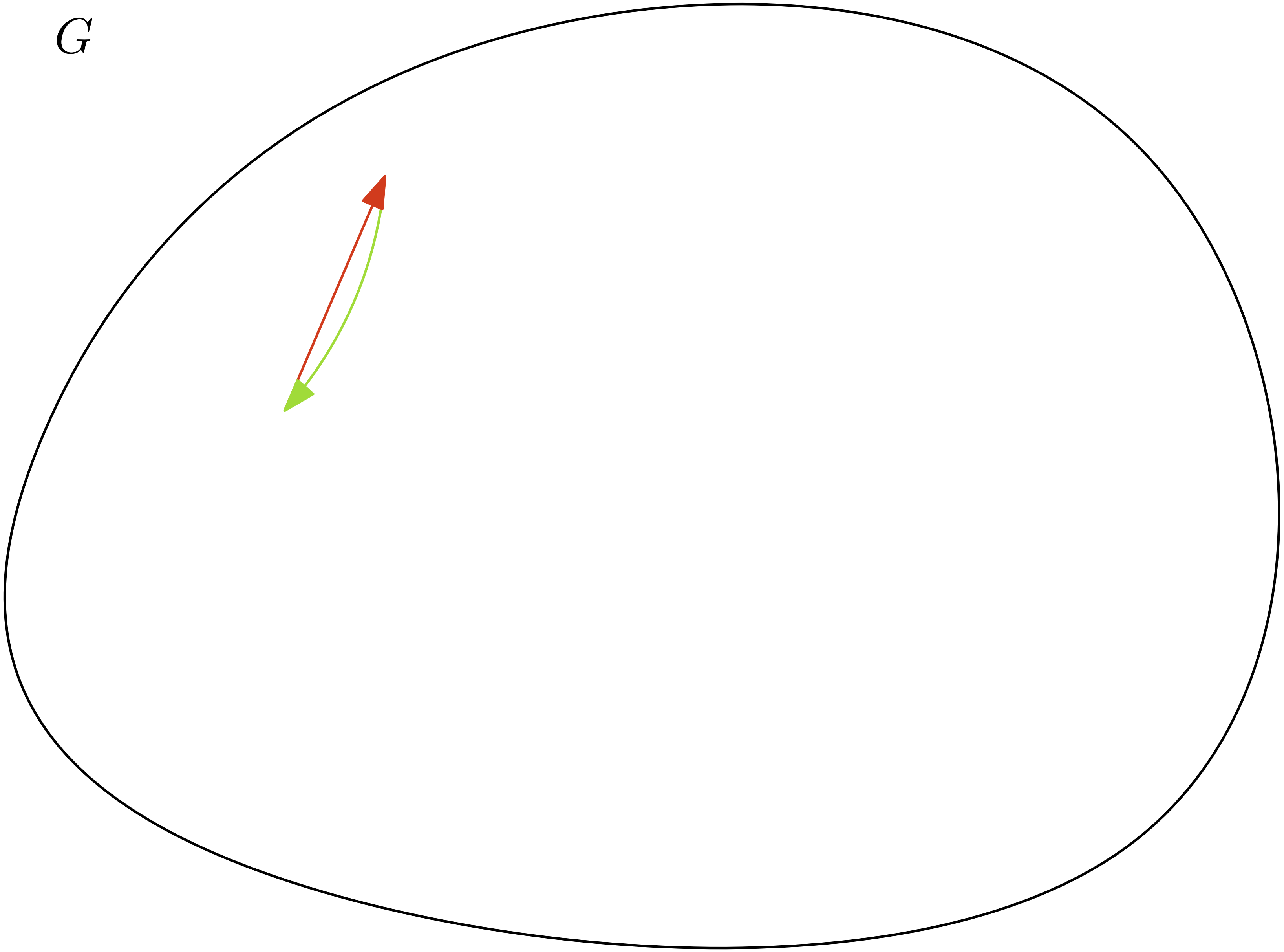

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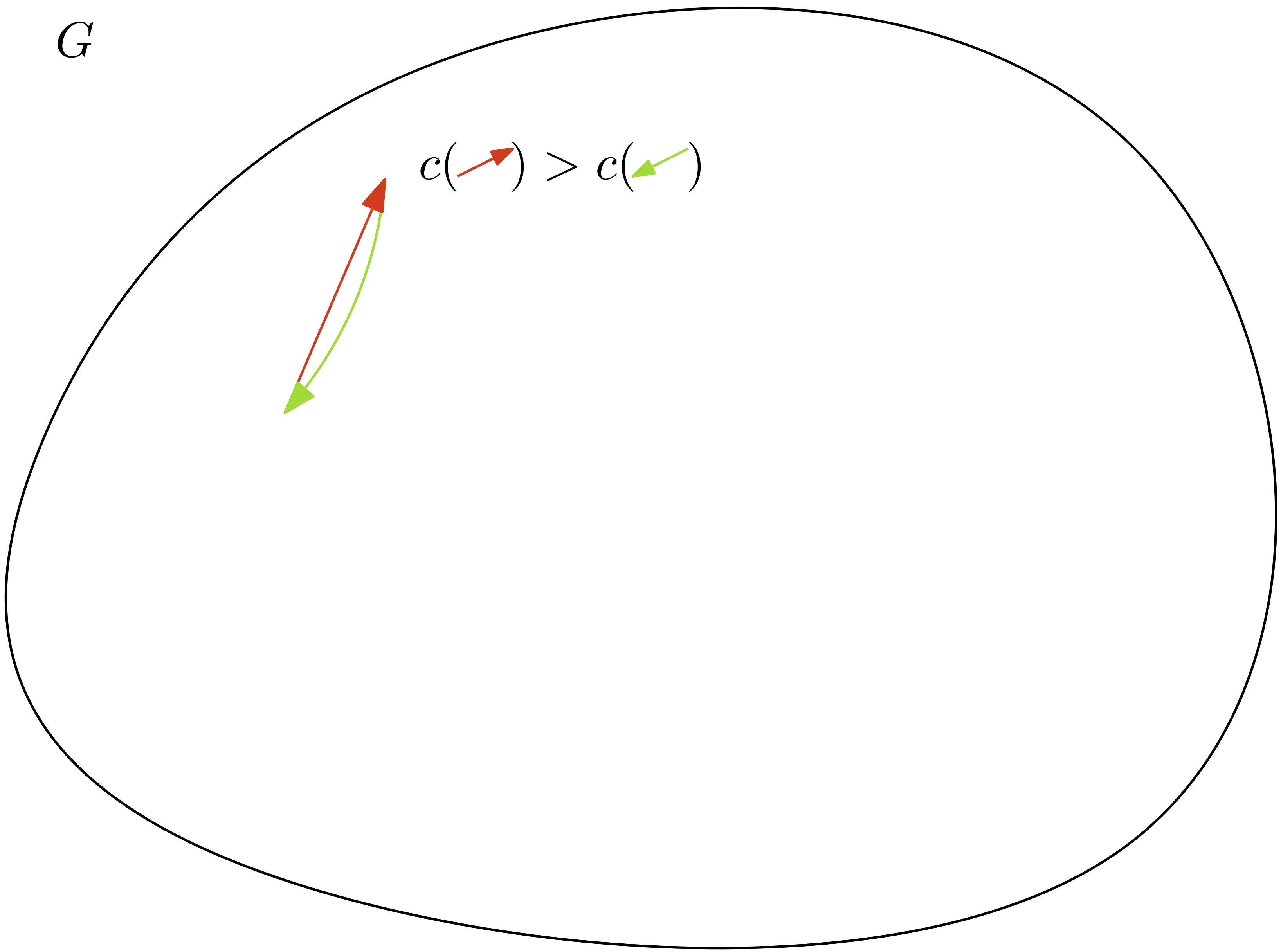

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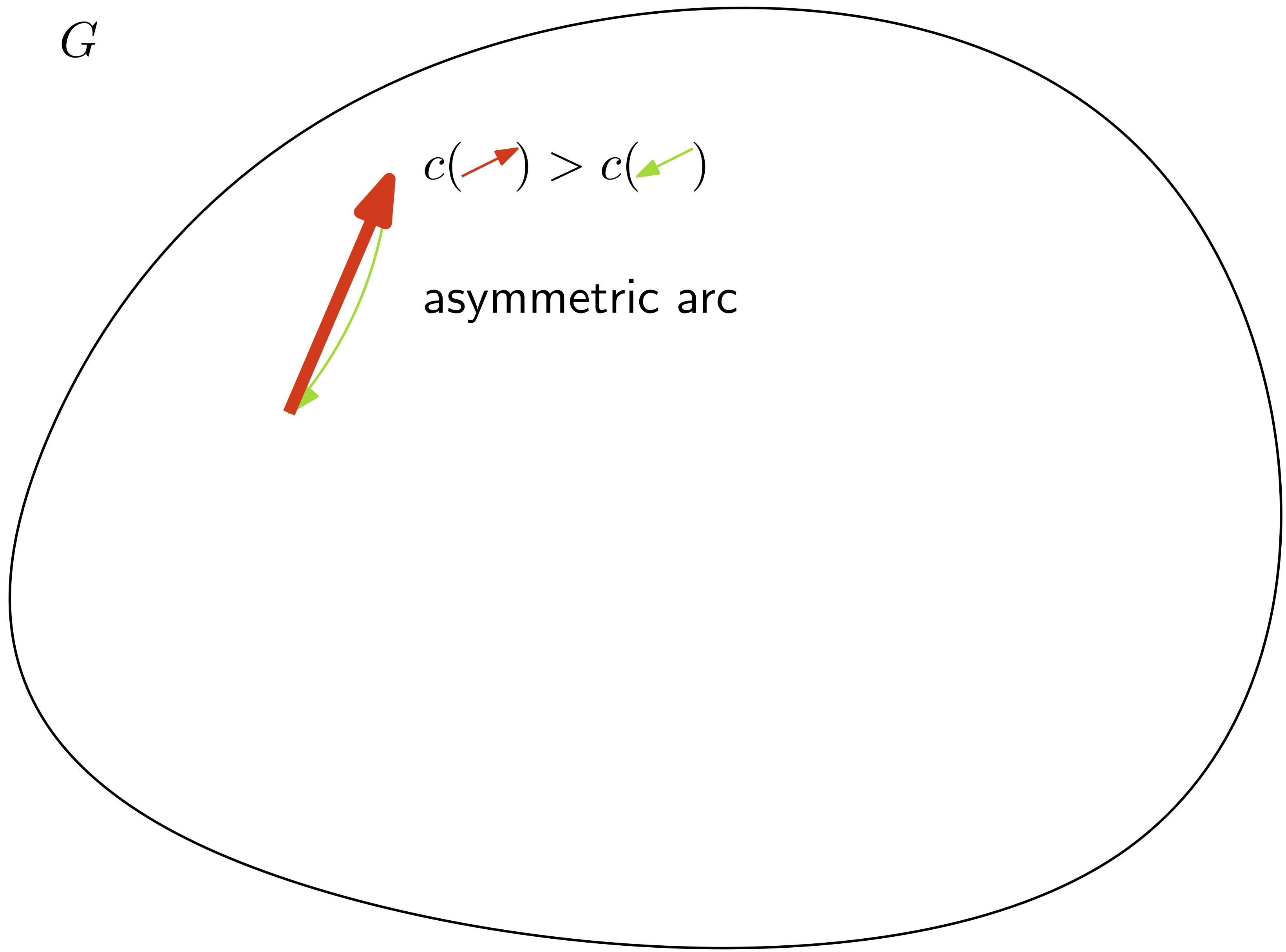

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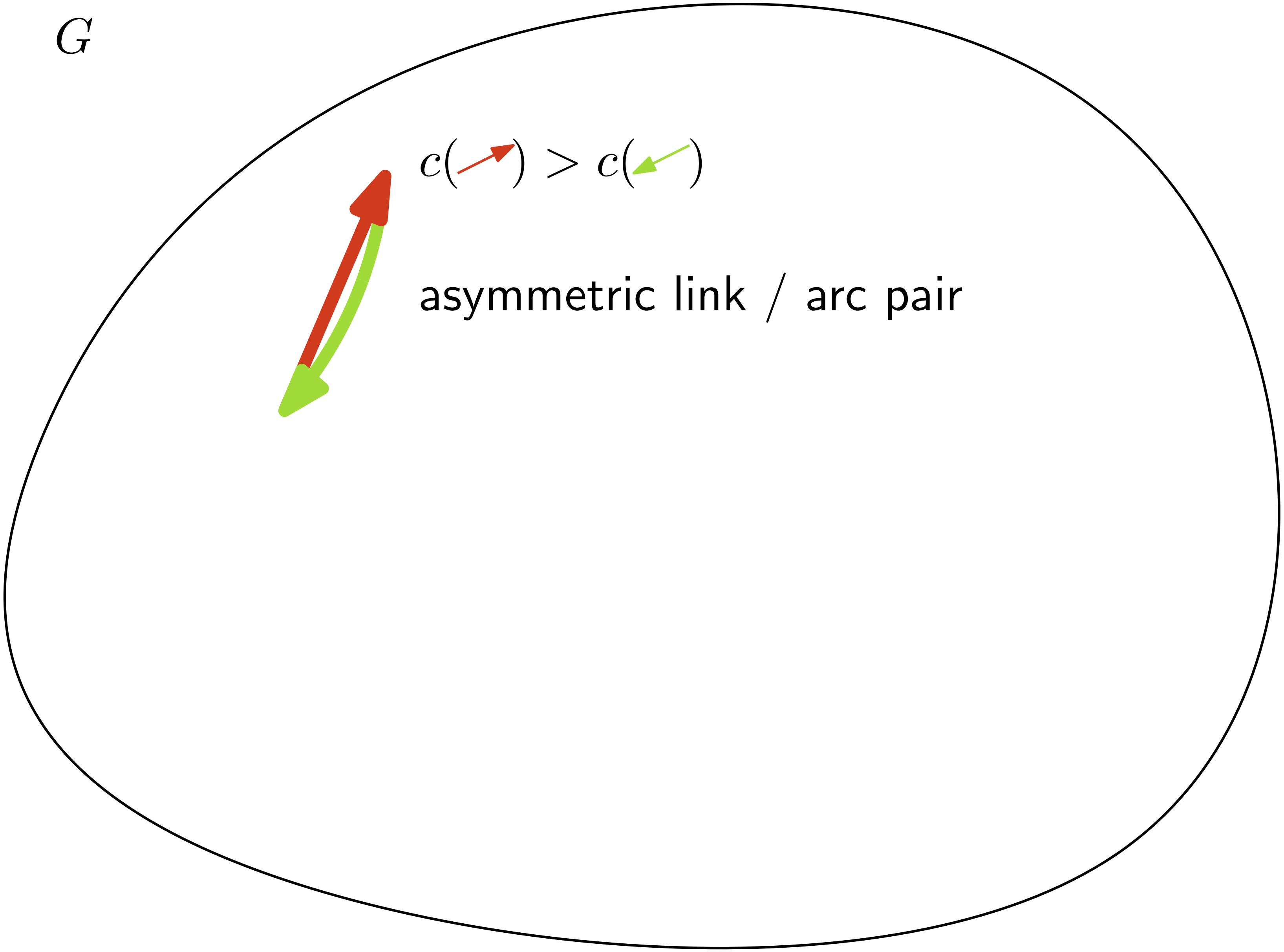

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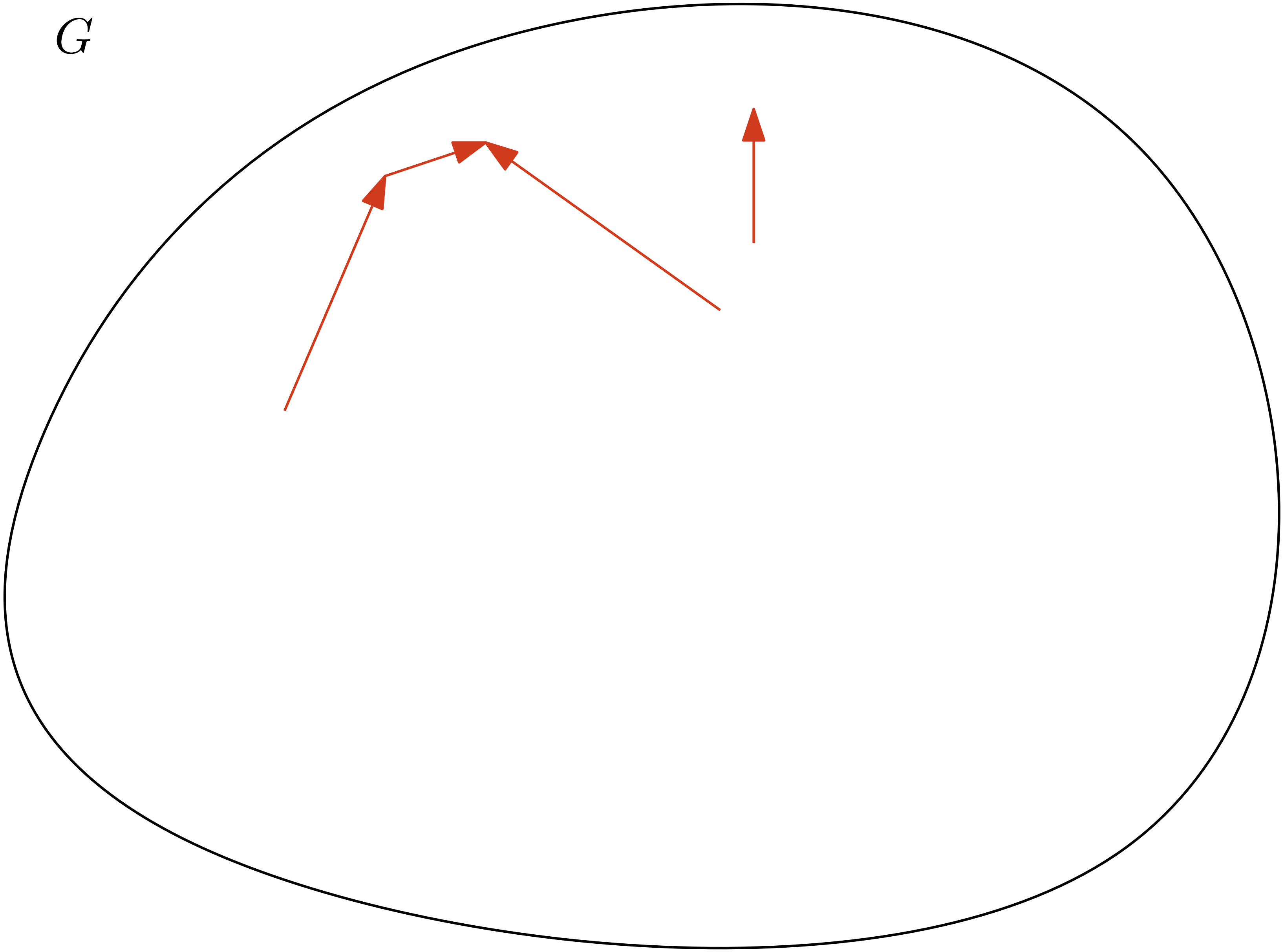

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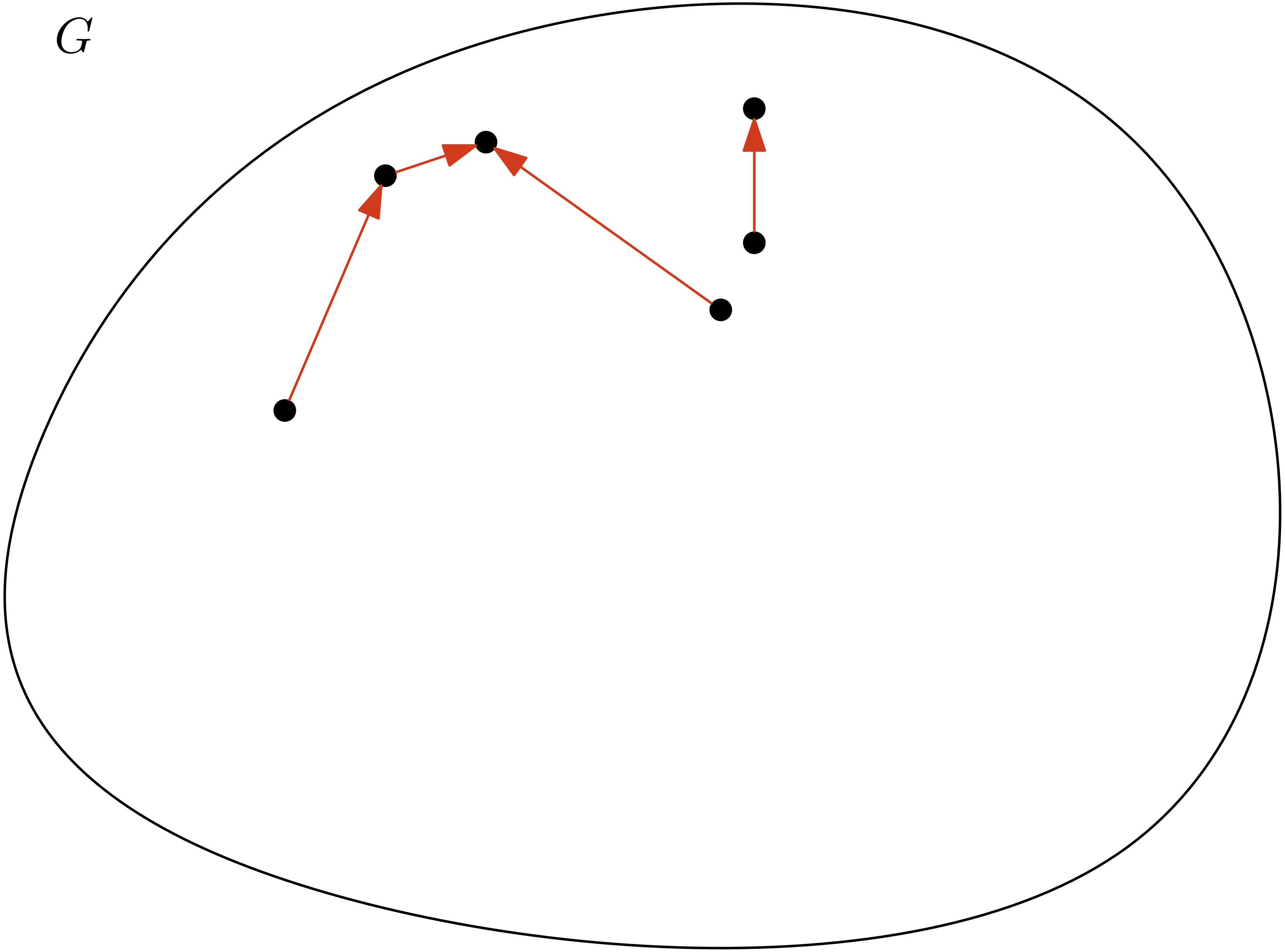

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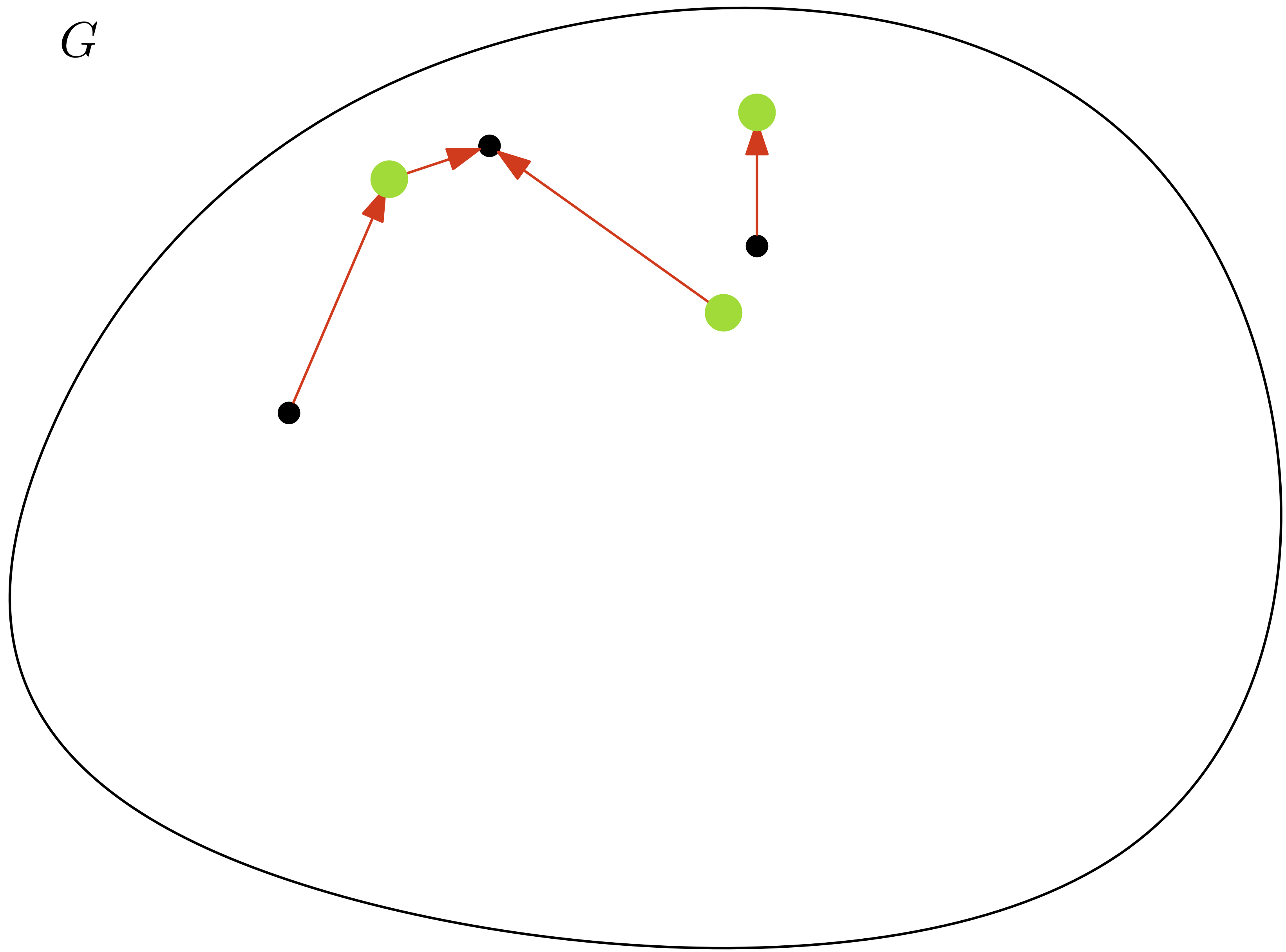

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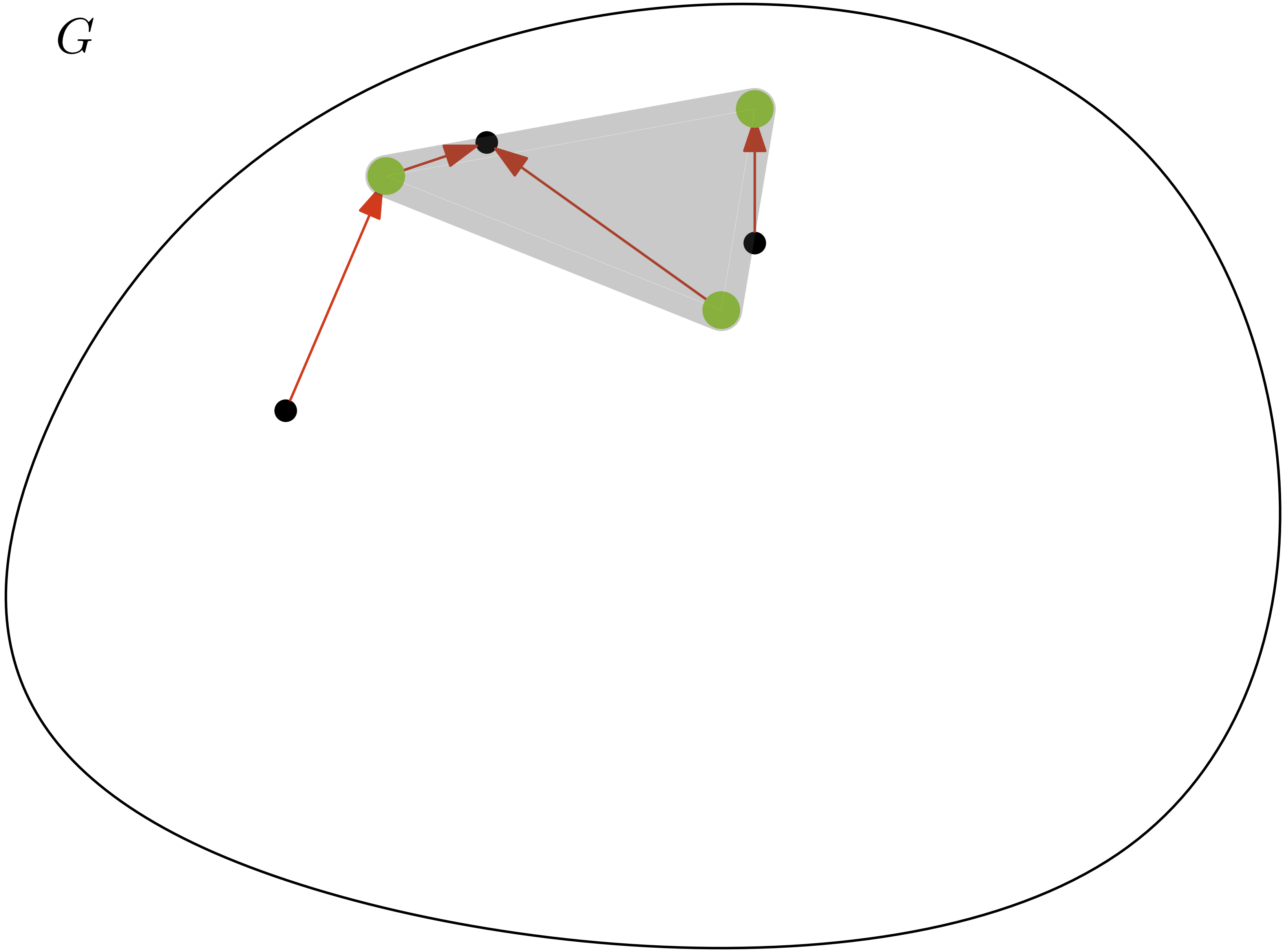

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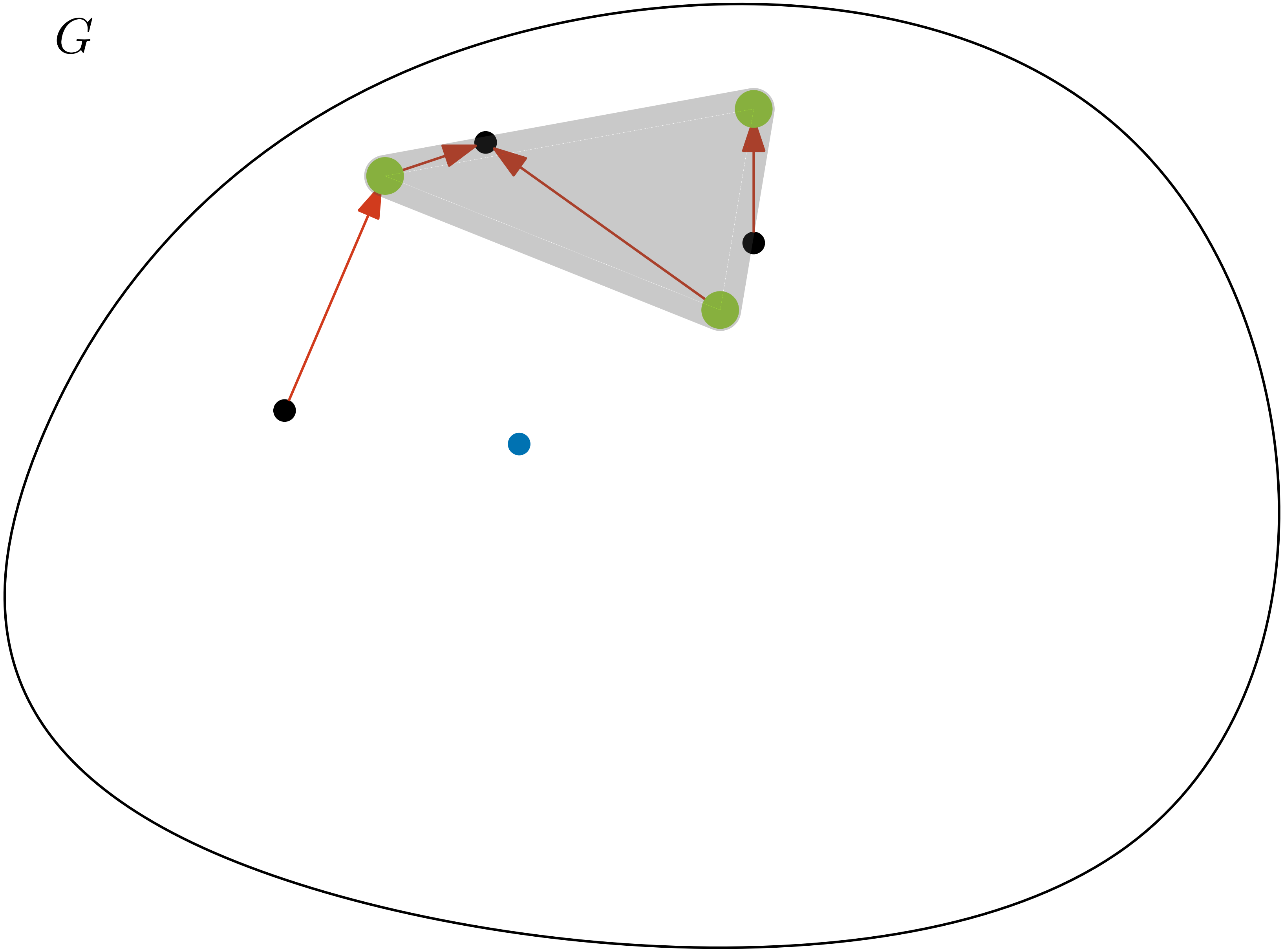

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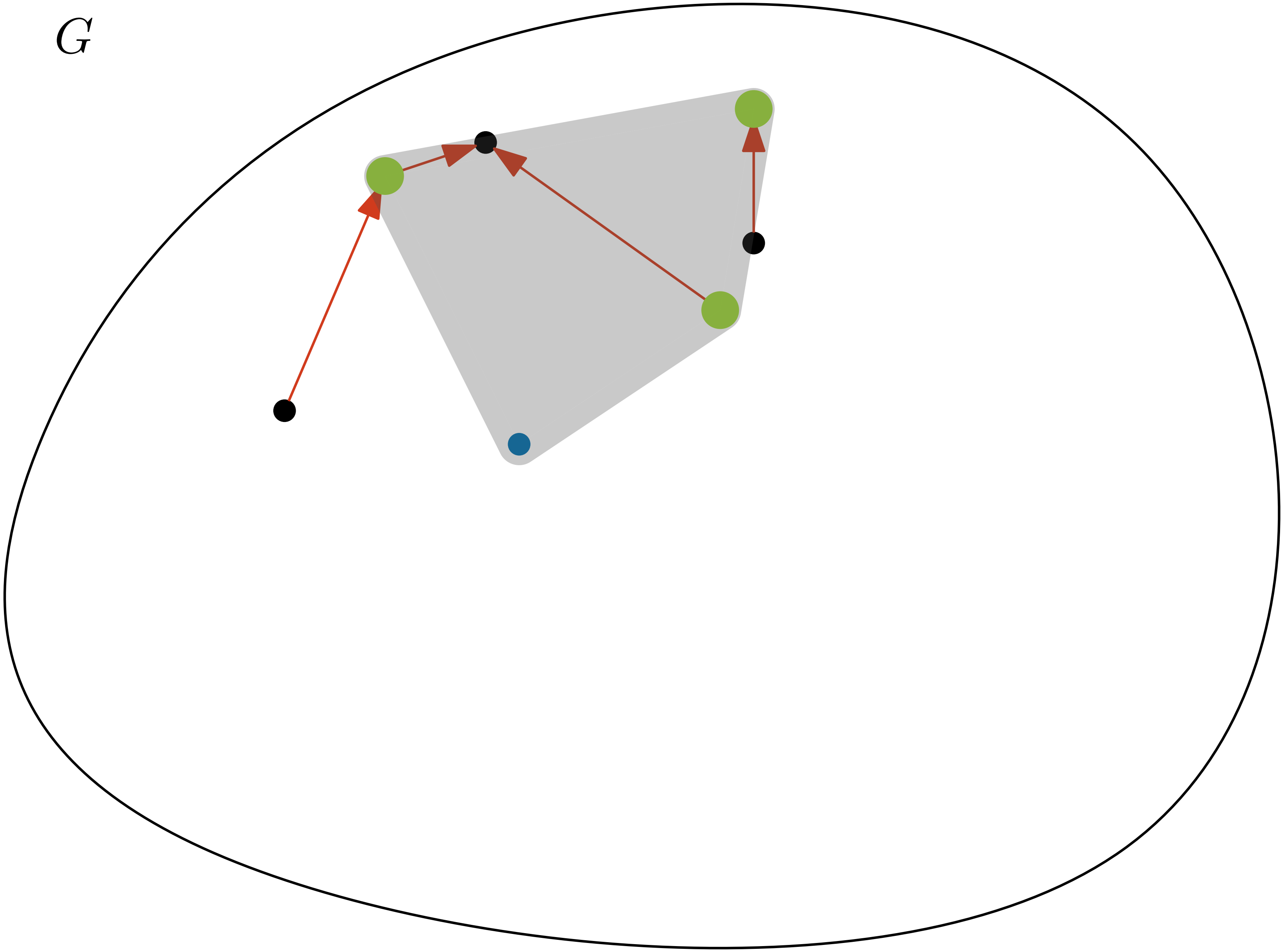

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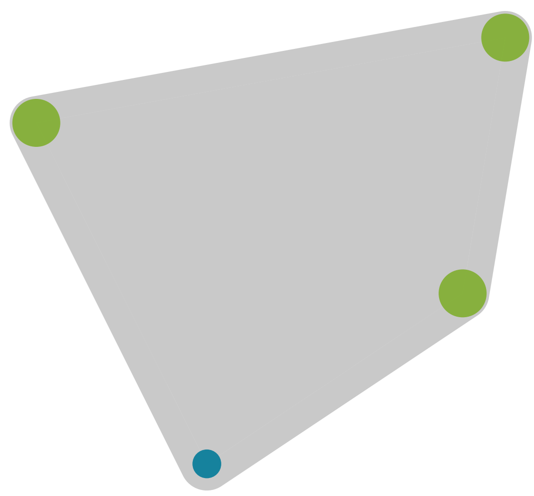
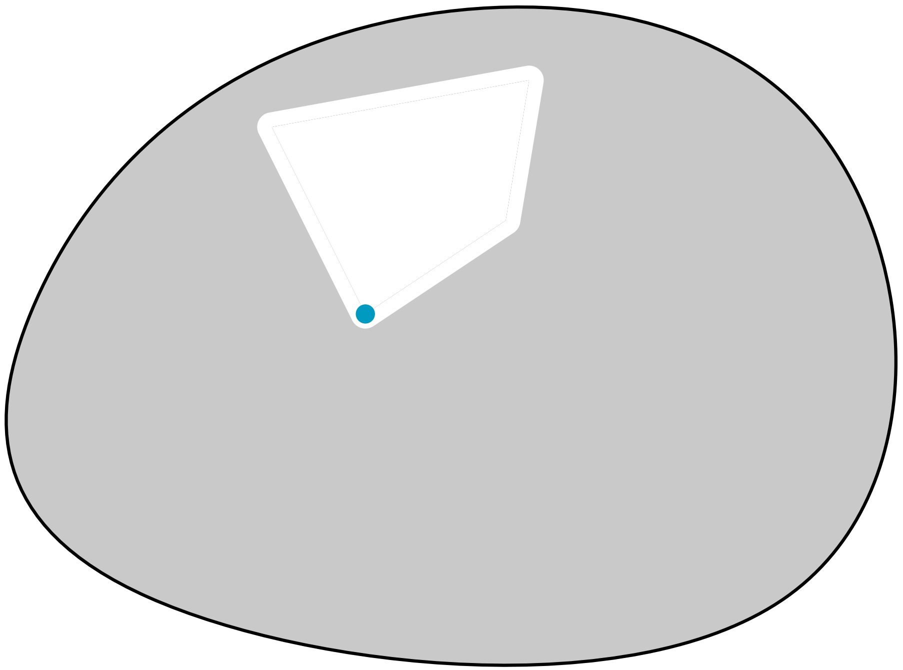


asymmetric subgraph



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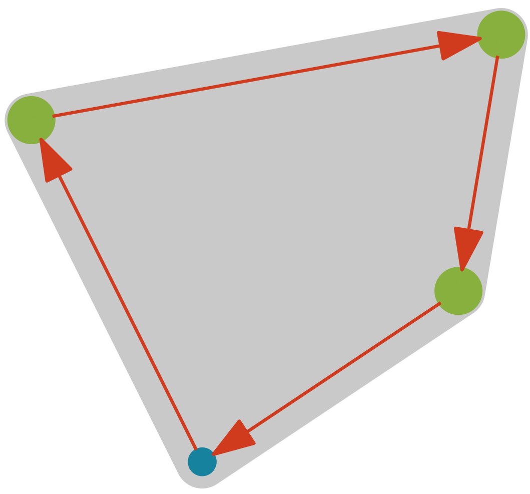

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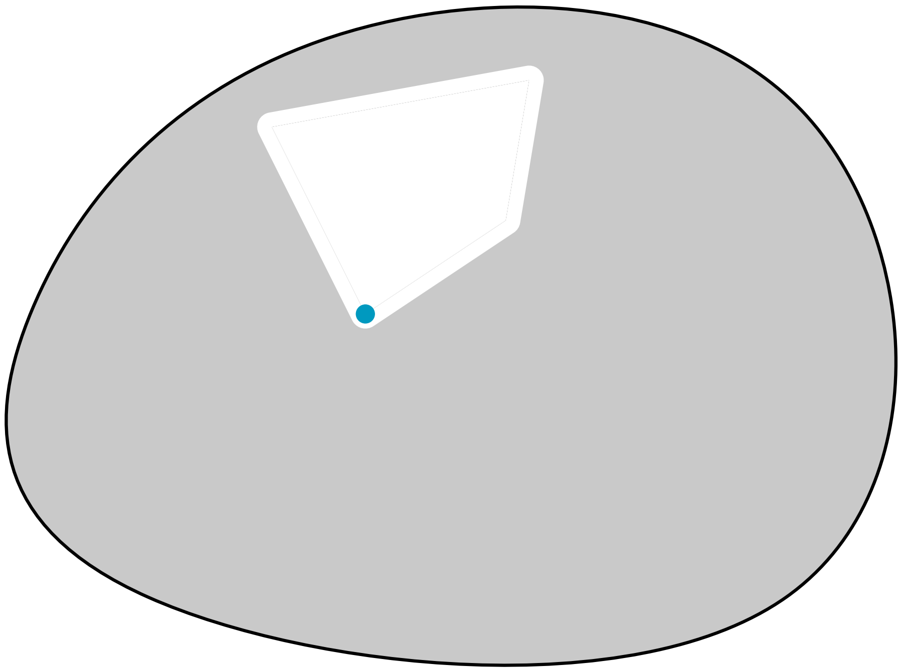

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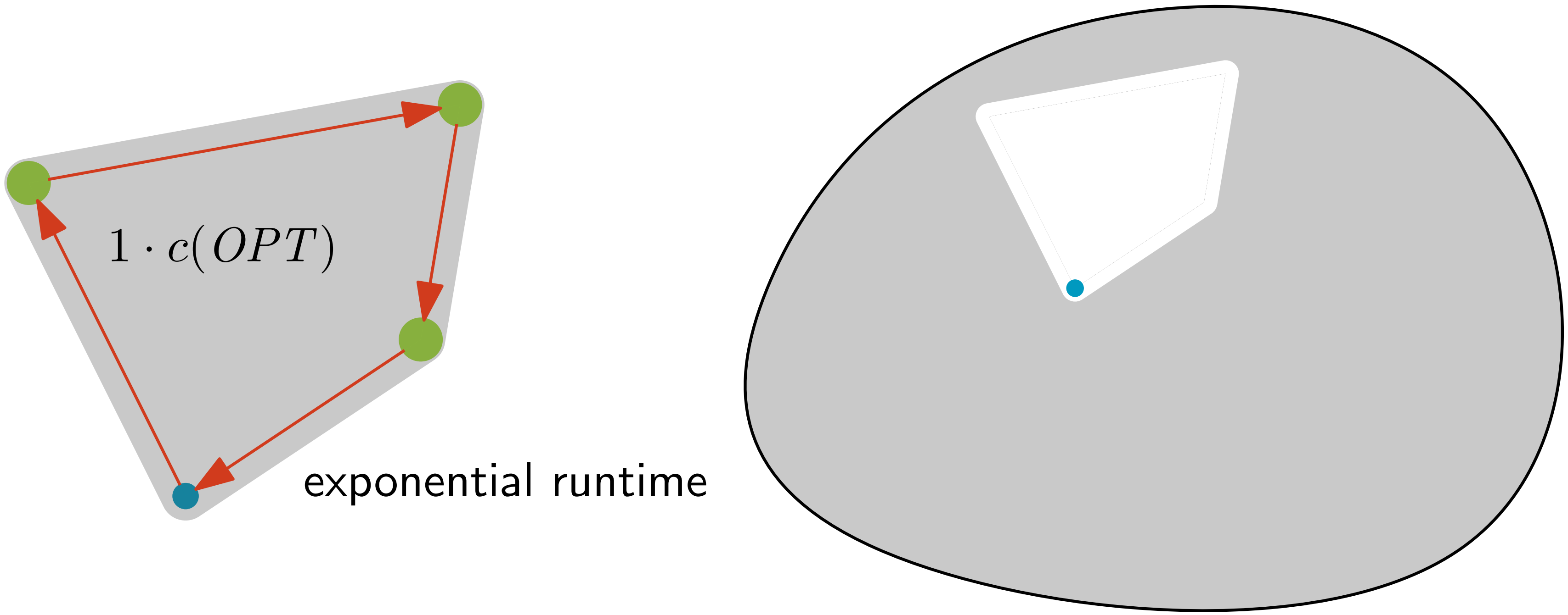

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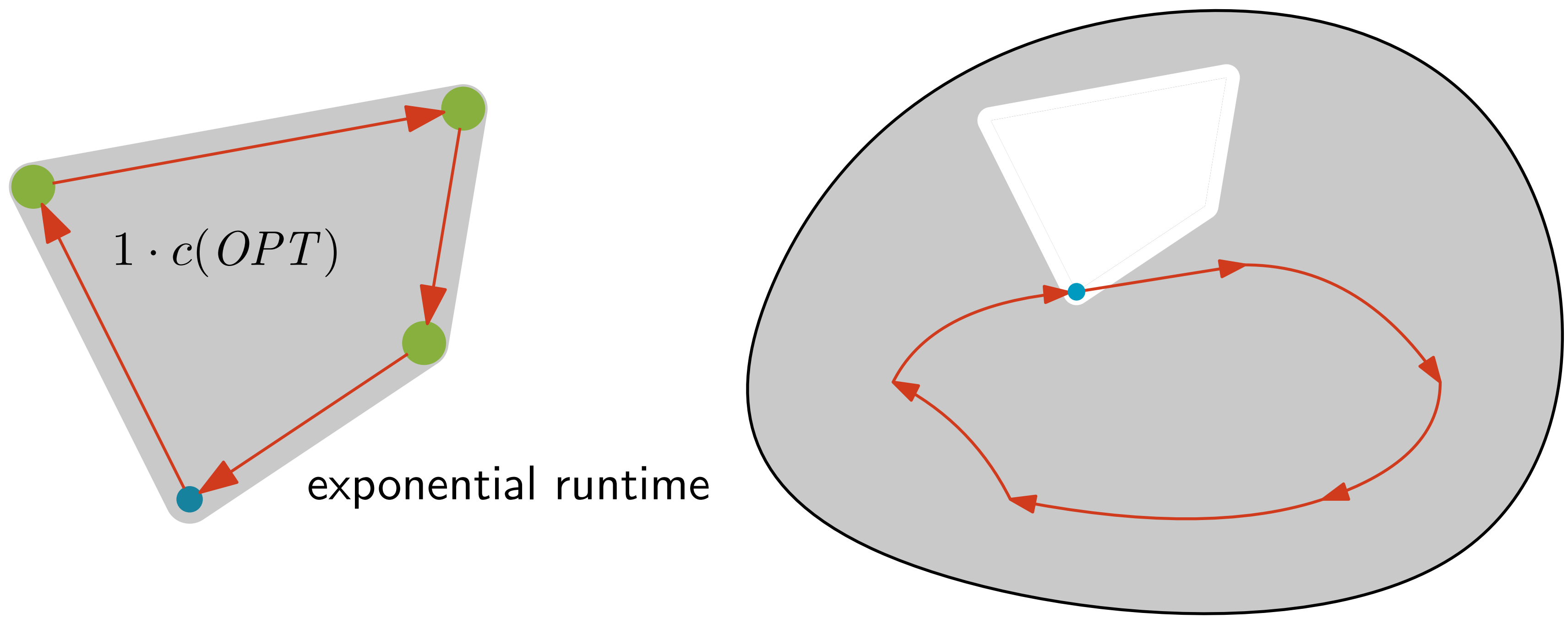

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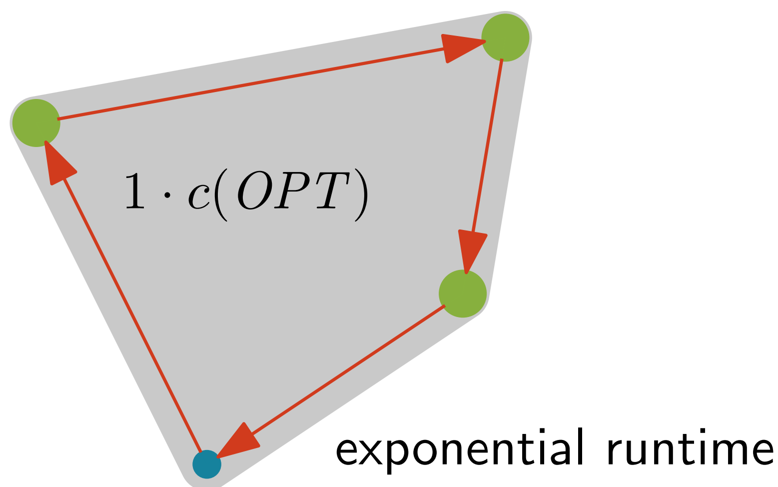

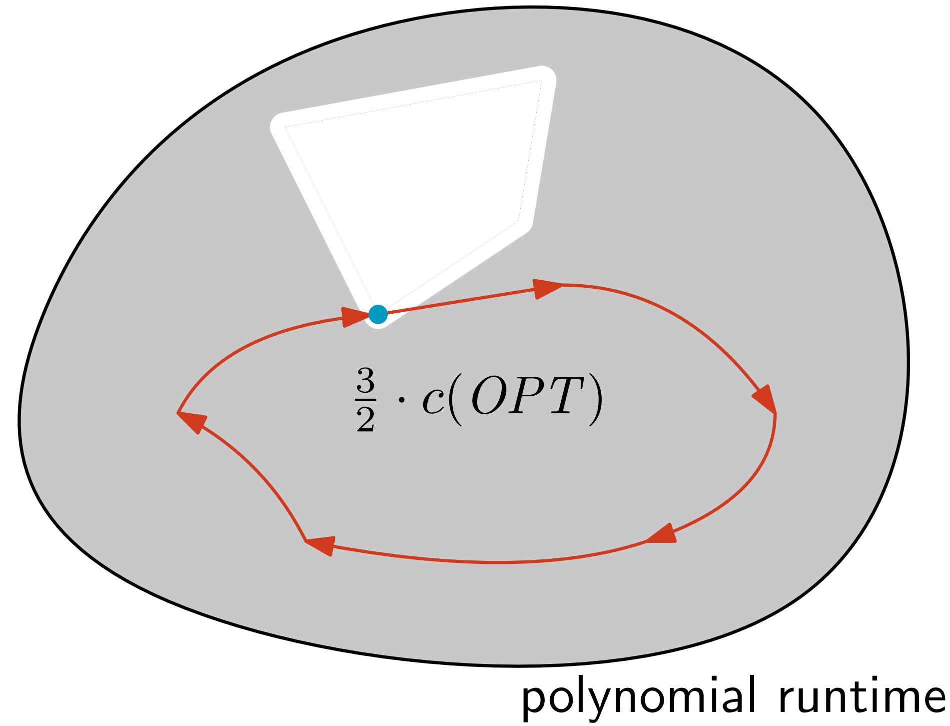

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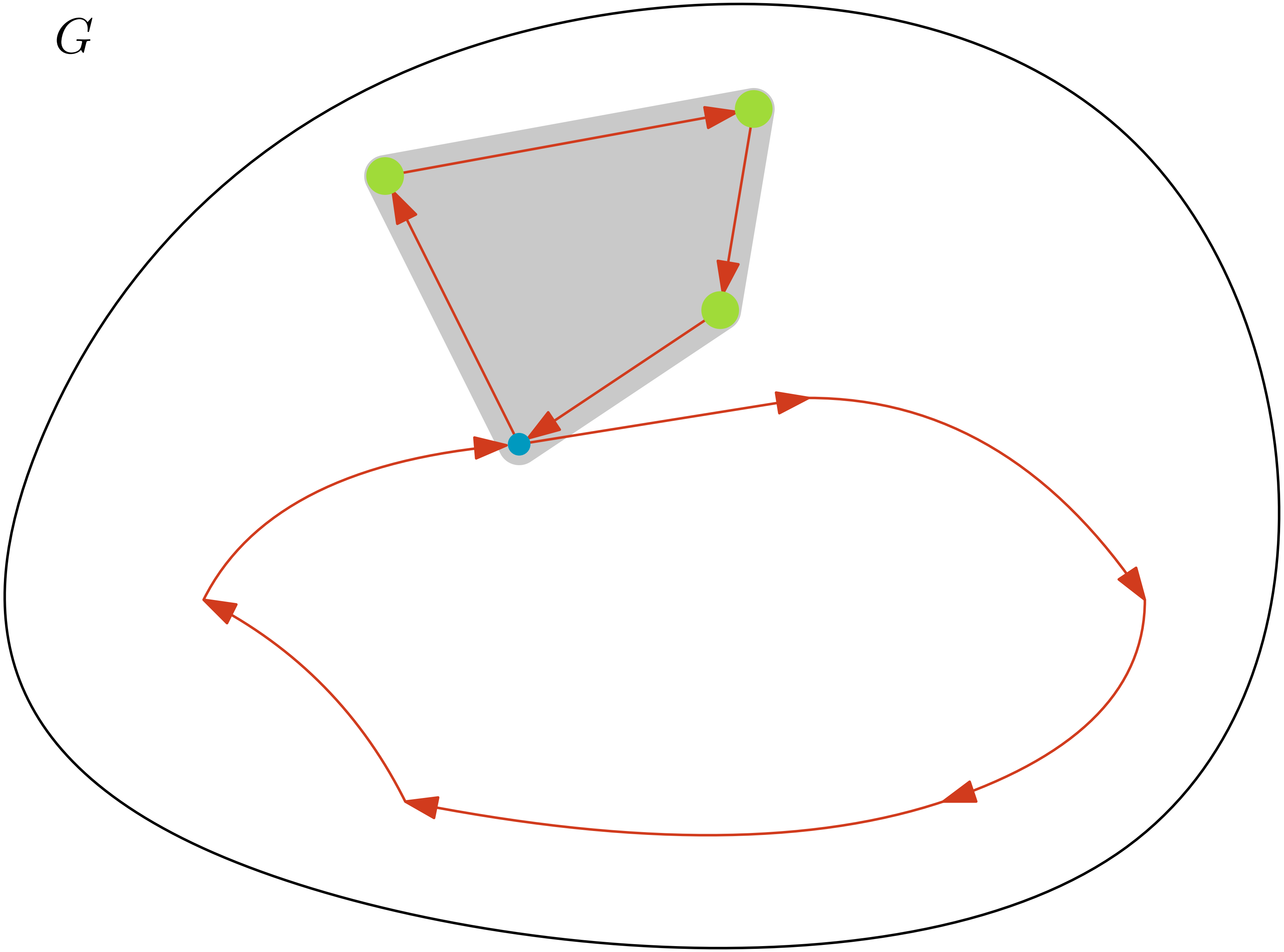

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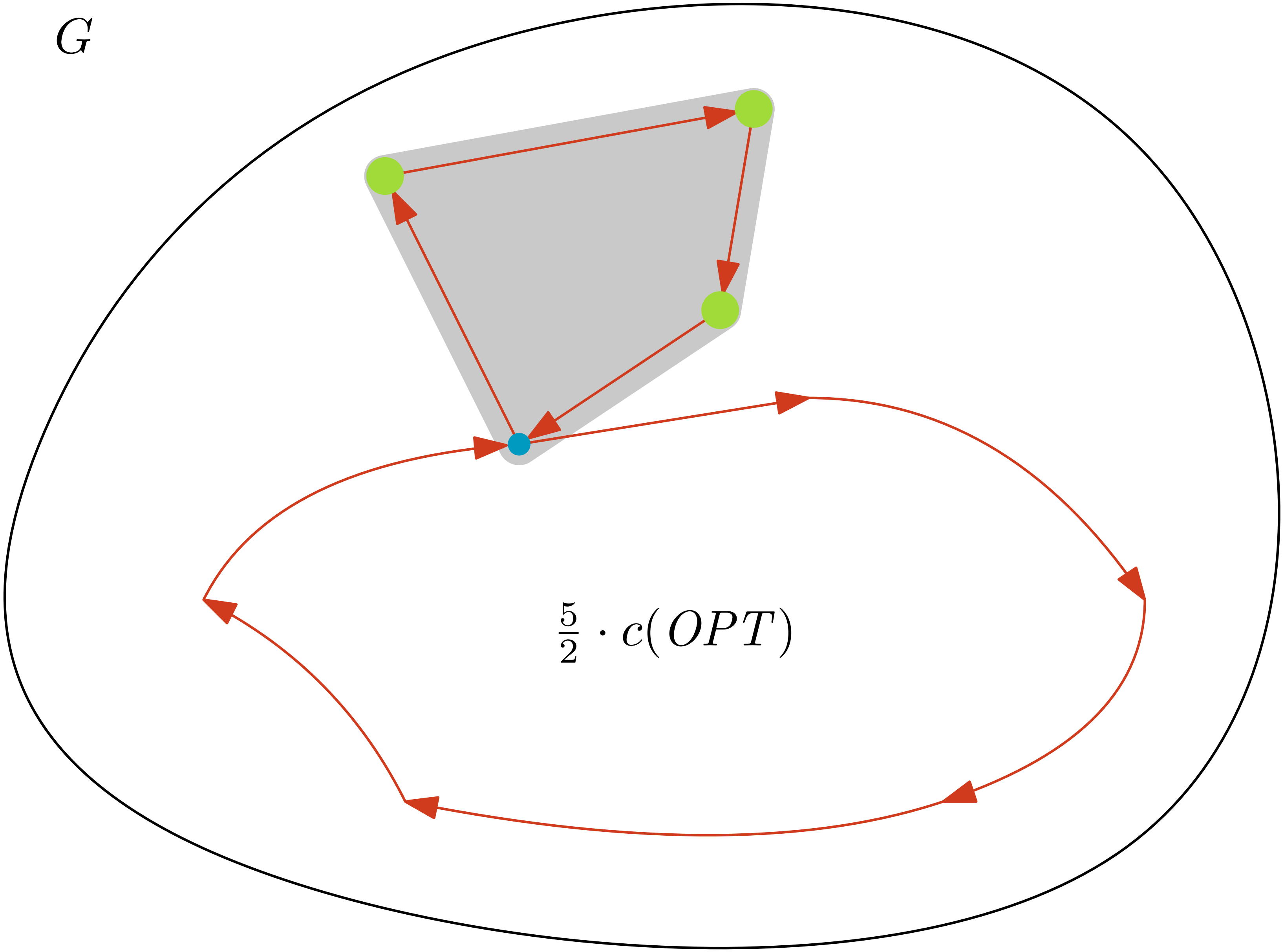

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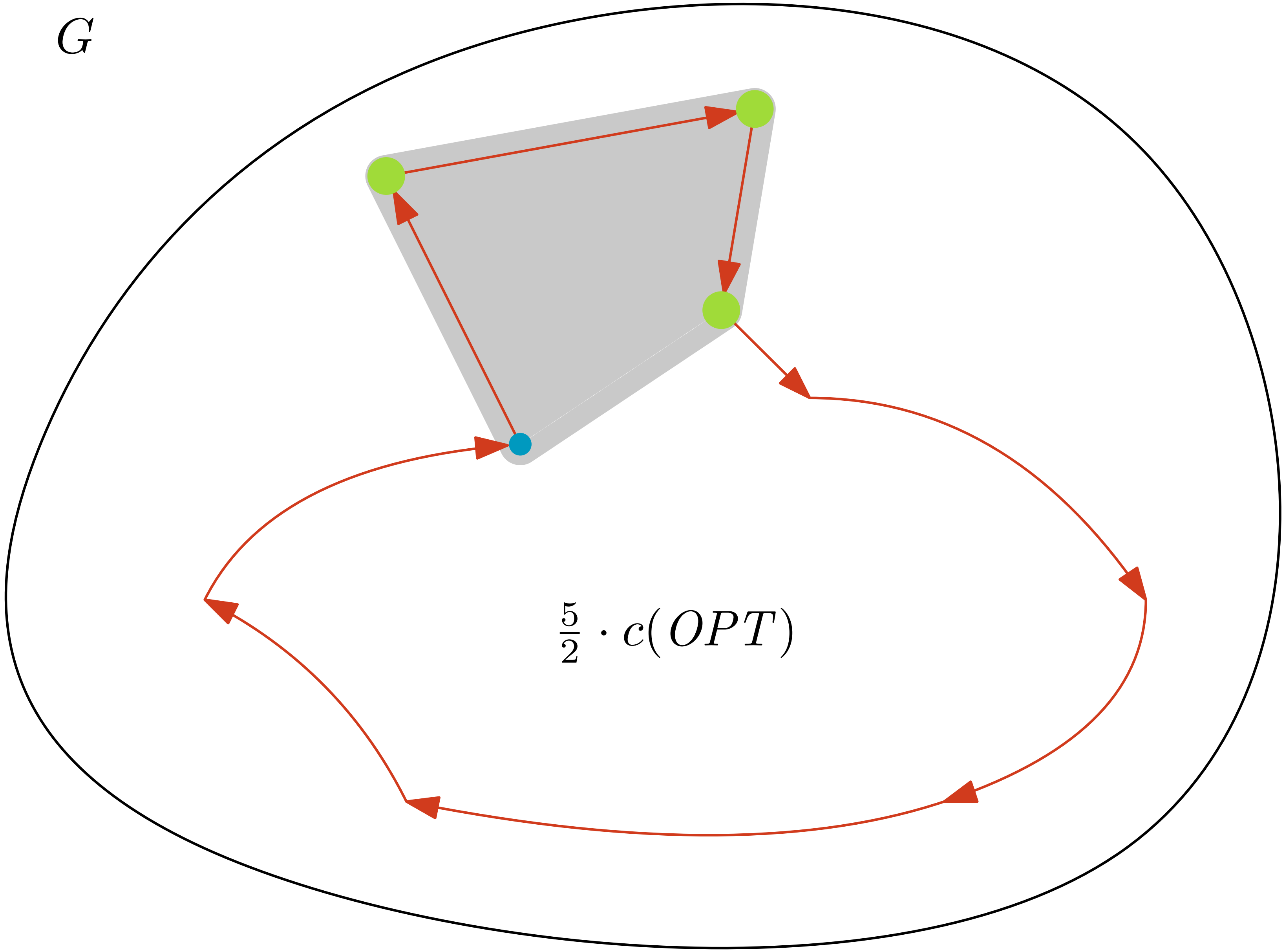

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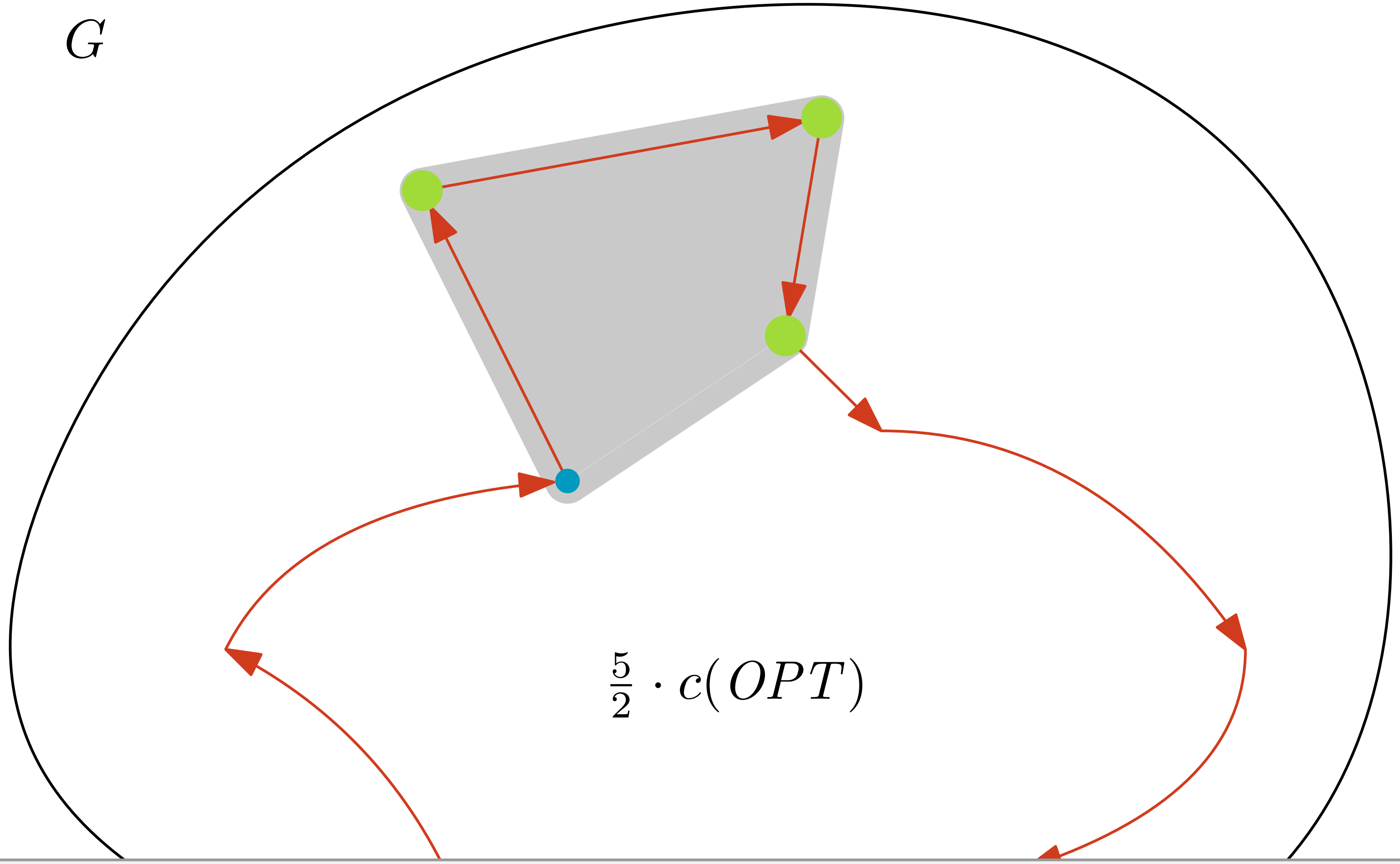

Theorem: The algorithm computes a 2.5-approx. for ATSP in $\mathcal{O}(2^k k^2 + n^3)$, where k is the size of a minimum vertex cover on the graph induced by the asymmetric links.

Going polynomial

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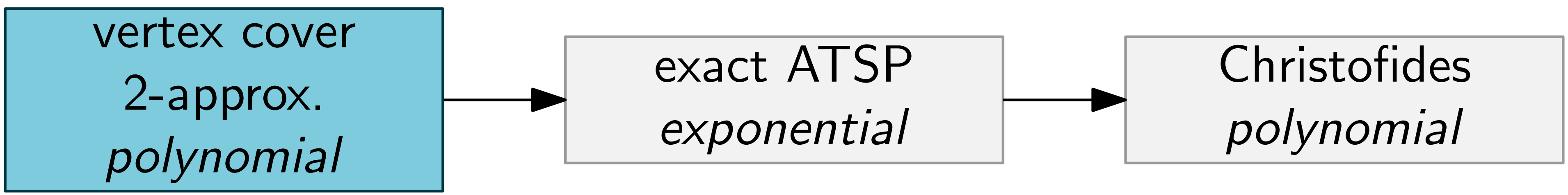

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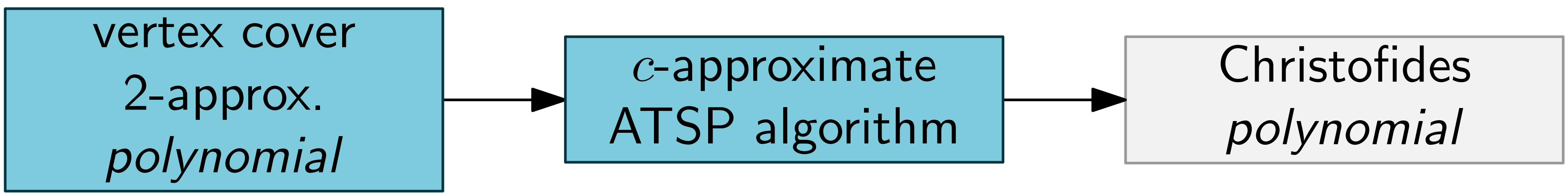

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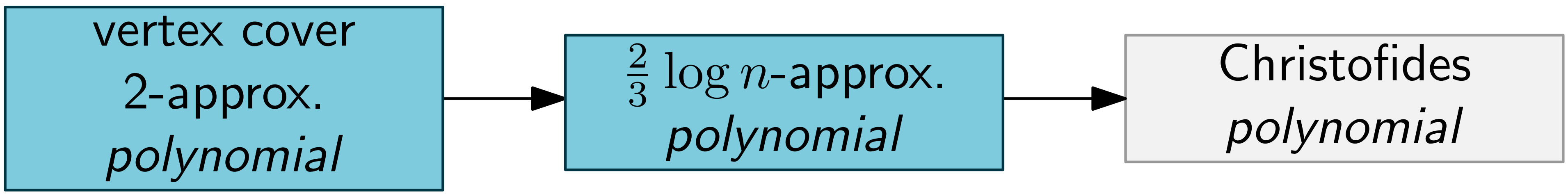

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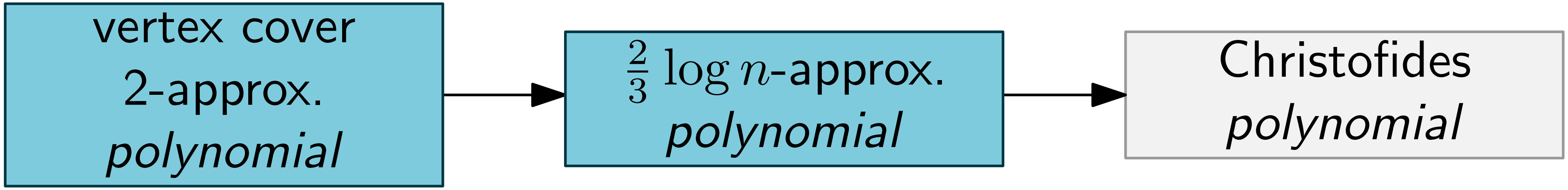

Approximation factor $\frac{2}{3}\log n + 1.5$?

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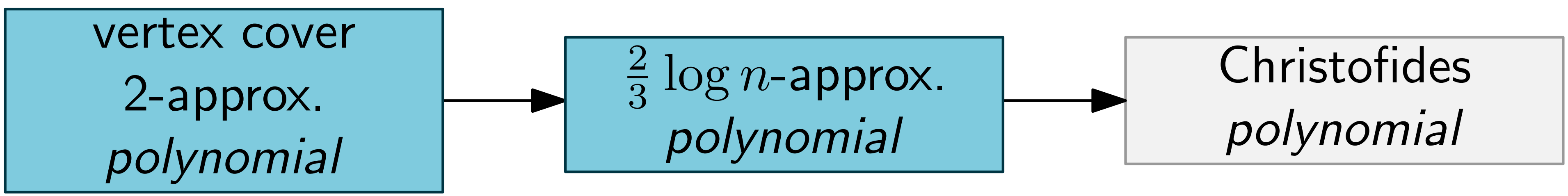

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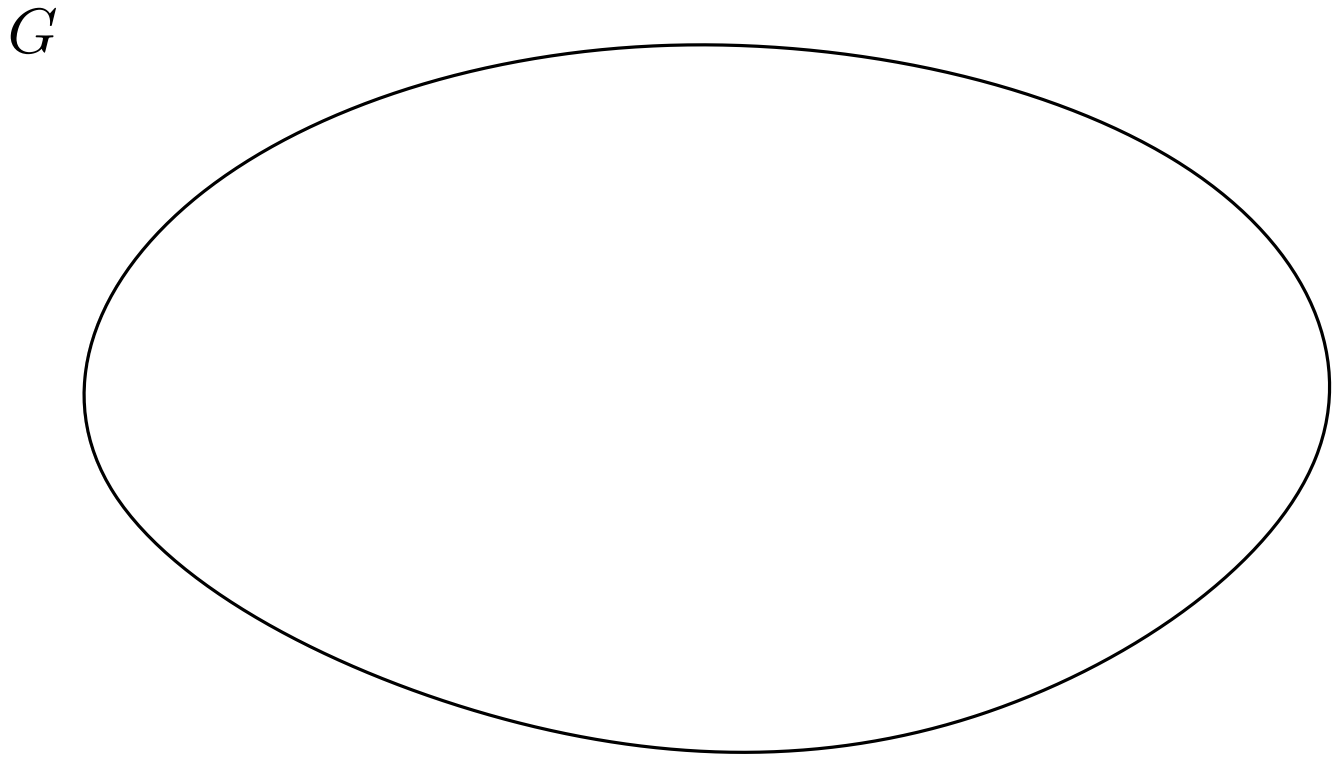
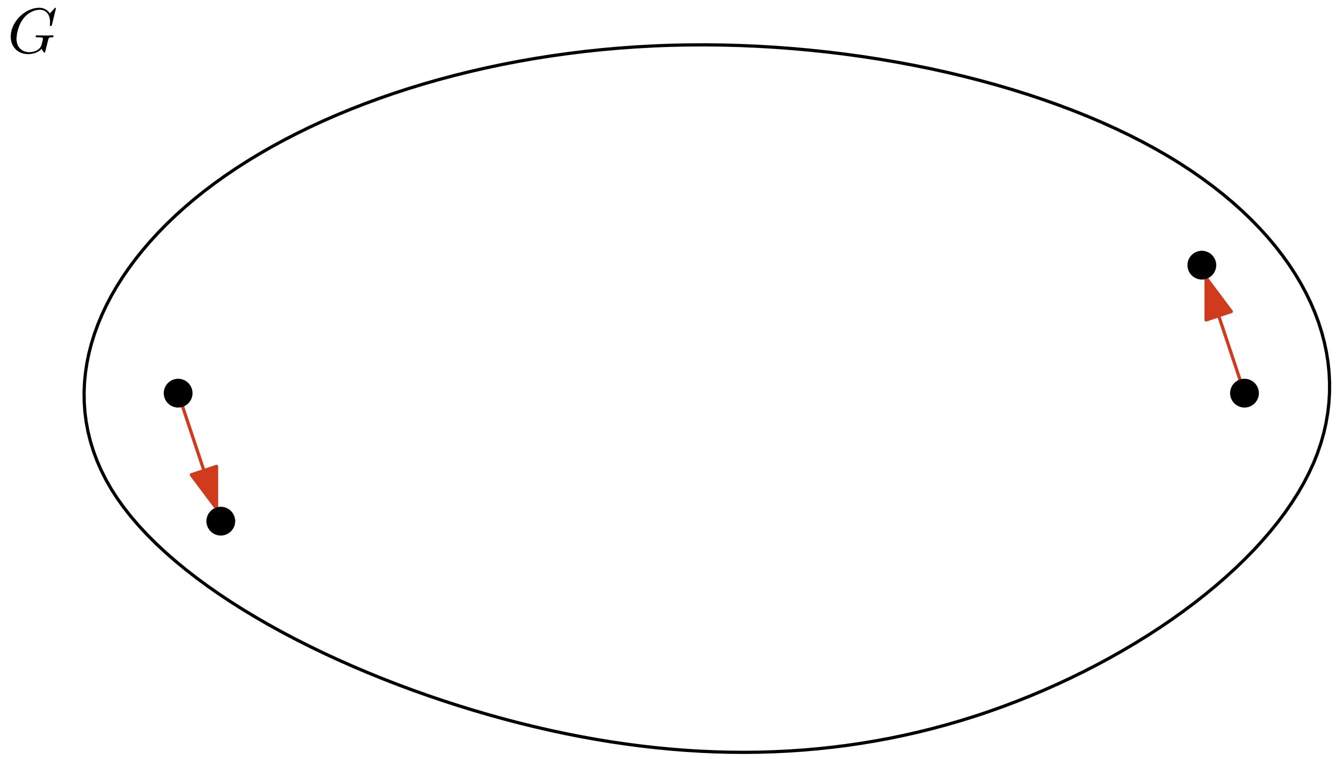
$\frac{2}{3}\log k + 1.5$!

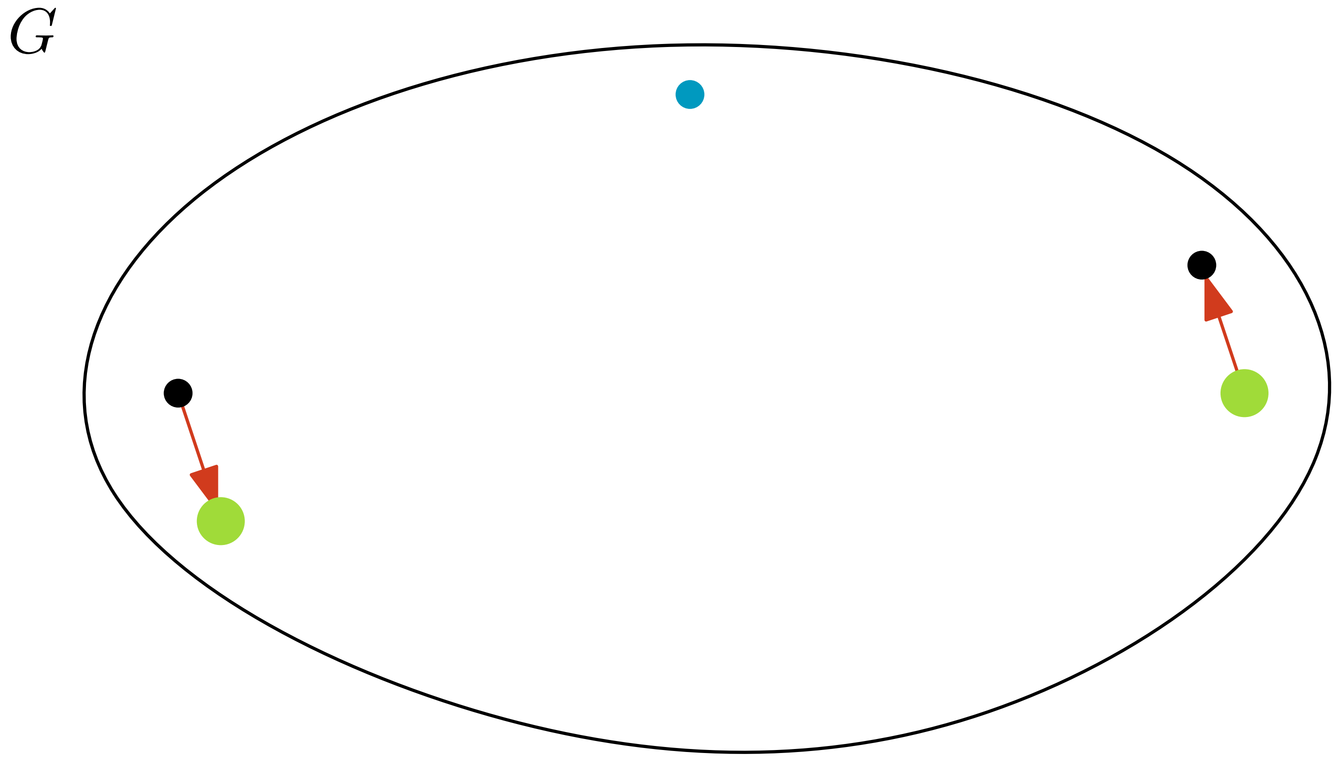
Unfavourable edge-case?



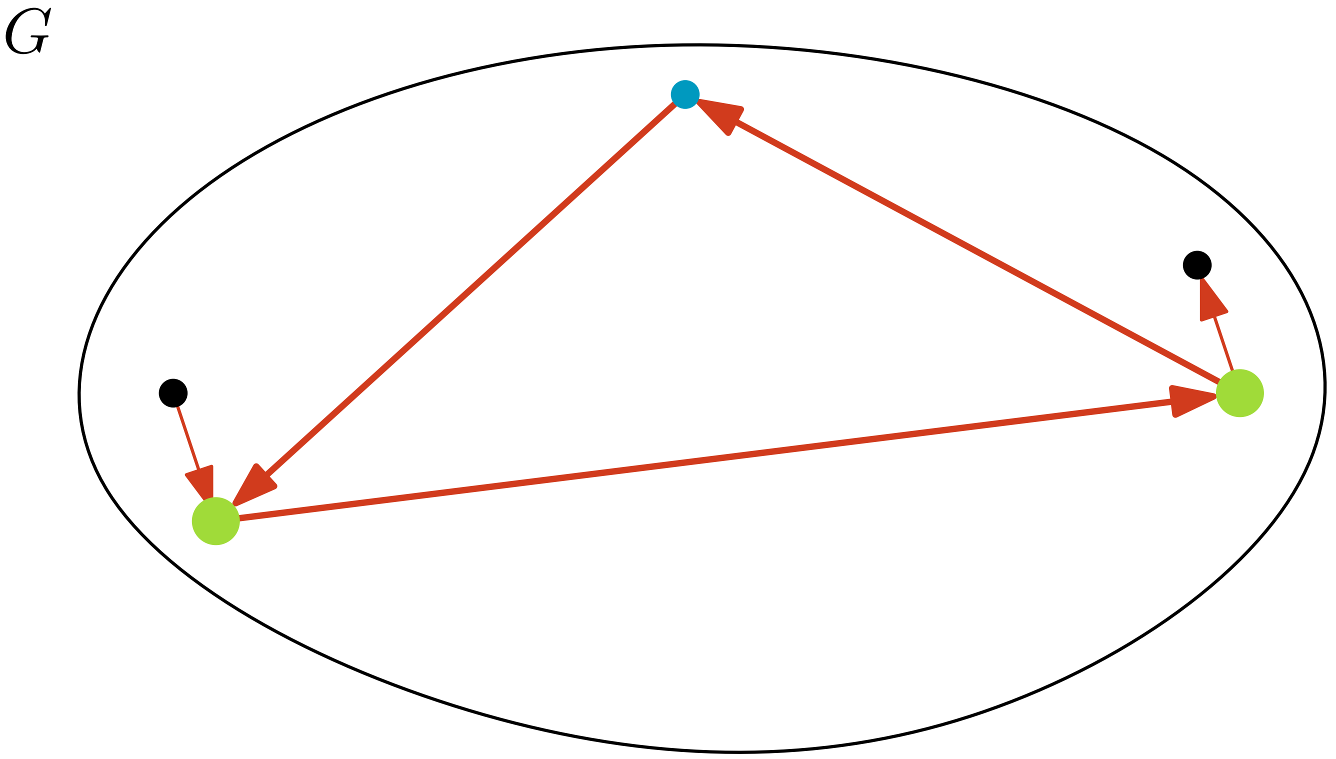
Unfavourable edge-case?



Unfavourable edge-case?



Unfavourable edge-case?



Unfavourable edge-case?